Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

MANFRED BRANDT, ANDREAS BRANDT[1]

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[1]Institut für Operations Research, Humboldt-Universität zu Berlin, Germany

ZIB-Report 11-28 (June 2011)

On sojourn times for an infinite-server system in random environment and its application to processor sharing systems

Manfred Brandt
Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB), Takustr. 7, D-14195 Berlin, Germany
e-mail: brandt@zib.de

Andreas Brandt
Institut für Operations Research, Humboldt-Universität zu Berlin, Spandauer Str. 1, D-10178 Berlin, Germany
e-mail: brandt@wiwi.hu-berlin.de

Abstract

We deal with an infinite-server system where the service speed is governed by a stationary and ergodic process with countably many states. Applying a random time transformation such that the service speed becomes one, the sojourn time of a class of virtual requests with given required service time is equal in distribution to an additive functional defined via a stationary version of the time-changed process. Thus bounds for the expectation of functions of additive functionals yield bounds for the expectation of functions of virtual sojourn times, in particular bounds for fractional moments and the distribution function. Interpreting the $GI(n)/GI(n)/\infty$ system or equivalently the $GI(n)/GI$ system under state-dependent processor sharing as an infinite-server system with random states given by the number n of requests in the system provides results for sojourn times of virtual requests. In case of $M(n)/GI(n)/\infty$, the sojourn times of arriving and added requests are equal in distribution to sojourn times of virtual requests in modified systems, which yields many results for the sojourn times of arriving and added requests. In case of integer moments, the bounds generalize earlier results for $M/GI(n)/\infty$. In particular, the mean sojourn times of arriving and added requests in $M(n)/GI(n)/\infty$ are proportional to the required service time, generalizing Cohen's famous result for $M/GI(n)/\infty$.

Mathematics Subject Classification (MSC 2000): 60K25, 68M20, 90B22, 60E15, 60G10.

Keywords: infinite-server; random environment; time transformation; additive functionals; sojourn times; moments; distribution; bounds; $GI(n)/GI(n)/\infty$; $M(n)/GI(n)/\infty$; state-dependent processor sharing.

1 Introduction

We deal with an infinite-server system, where the service speed at time t depends on a random state $N(t) \in \mathbb{Z}_+ := \{0, 1, \ldots\}$. More precisely, the service speed of the server in state $N(t) = n$, $n \in \mathbb{Z}_+$, equals $\varphi(n) > 0$, and we assume that the process $N(t)$, $t \in \mathbb{R}$, is stationary and ergodic. We analyze the sojourn time of virtual requests with required service time τ (τ-requests) by applying a random time transformation to the infinite-server system such that the service speed becomes one. In Section 2.1 we construct the distribution of a stationary and ergodic version $\tilde{N}(t)$, $t \in \mathbb{R}$, of the time-changed process of $N(t)$, $t \in \mathbb{R}$, by using the Palm distribution, which provides a representation of the sojourn time of a class of virtual τ-requests by a smooth additive functional. In Section 2.2 we analyze the expectation of non negative convex and concave functions of additive functionals at a given time instant τ, in particular, we analyze fractional moments and the distribution function of additive functionals.

In Section 3, the results of Section 2.2 are applied to the representation of the sojourn time of a class of virtual τ-requests by an additive functional given in Section 2.1, which yields bounds for the expectation of non negative convex and concave functions of virtual sojourn times, in particular bounds for fractional moments and the distribution function of virtual sojourn times.

In Section 4 we deal with the $GI(n)/GI(n)/\infty$ system or equivalently with the $GI(n)/GI$ system under State-Dependent Processor Sharing, i.e. with the $GI(n)/GI/SDPS$ system. Note that we have the single-server processor sharing system $GI(n)/GI/1-PS$ in the special case of $\varphi(n) := 1/n$, $n \in \mathbb{N} := \mathbb{Z}_+ \setminus \{0\}$, for other special cases see e.g. [BB5]. Processor sharing systems have been widely used in the last decades for modeling and analyzing computer and communication systems, cf. e.g. [CMT], [Ram], [BP], [BBJ], [GRZ], [HHM], [YY], [BB1]–[BB5], [ZLK], [LSZ], and the references therein. For an application of a random time transformation to processor sharing systems see [Tol], [Kit], [YY], and the references therein. We interpret the $GI(n)/GI(n)/\infty$ system as an infinite-server system in random environment, where the state $N(t)$ of the infinite-server system is given by the number of requests in $GI(n)/GI(n)/\infty$ at time t. Thus we obtain results for sojourn times of virtual τ-requests.

For the $M(n)/GI(n)/\infty$ system or equivalently the $M(n)/GI/SDPS$ system we show that the sojourn time of an arbitrary arriving τ-request equals in distribution the sojourn time of a class of virtual τ-requests in the modified system with one permanent request and that the sojourn time of an added τ-request equals in distribution the sojourn time of this class of virtual τ-requests in another modified system. Thus we obtain bounds for the expectation of non negative convex and concave functions of the sojourn times of arriving as well as of added τ-requests in $M(n)/GI(n)/\infty$, in particular bounds for all fractional moments and the distribution functions of the sojourn times. Note that these bounds are given in terms of the well-known stationary occupancy distribution in $M(n)/GI(n)/\infty$, cf. [Coh], being insensitive with respect to the service time distribution given its mean. The lower and upper bounds for the fractional moments are asymptotically tight. In case of non negative integer moments, the bounds generalize corresponding results for the $M/GI/SDPS$ system given in [BB3], for the $M/M/SDPS$ system given in [BB2], and for the $M/GI/1-PS$ system given in [CVB], to $M(n)/GI/SDPS$. Moreover, for fixed $k \in [1,\infty)$ ($k \in (-\infty,1] \setminus \{0\}$) it follows that the kth root of the kth moment of the sojourn times of arriving as well as of added τ-requests in $M(n)/GI/SDPS$ are subadditive (superadditive) functions of $\tau \in (0,\infty)$, generalizing Cohen's famous proportional result for the expectation of the sojourn time of τ-requests in $M/GI/SDPS$ in several directions, cf. [Coh].

2 Preliminary results

We consider a stationary and ergodic process $N = (N(t),\, t \in \mathbb{R})$[1], where $N(t)$ takes values in $\mathbb{Z}_+$ and the sample paths of N are P-a.s. in the set $D(\mathbb{R}, \mathbb{Z}_+)$ of all piecewise constant, right-continuous functions having a finite number of jumps in any finite interval. Let

$$p(n) := P(N(0)=n), \quad n \in \mathbb{Z}_+, \tag{2.1}$$

be the marginal distribution of N and $\mathbb{Z}'_+ := \{m \in \mathbb{Z}_+ : p(m) > 0\}$ the support of $N(0)$. Further we assume that

$$0 < \lambda := E[\#\{t : 0<t\leq 1,\, N(t-)\neq N(t)\}] < \infty, \tag{2.2}$$

where $\#A$ denotes the number of elements of a set A, i.e., the intensity λ of jumps is positive and finite. The process N describes a random environment of an infinite-server system – system for short – where requests are served with speed $\varphi(n) > 0$ at time t if $N(t) = n$. We assume that

$$g := E[\varphi(N(0))] = \sum_{n\in\mathbb{Z}'_+} \varphi(n)p(n) < \infty, \tag{2.3}$$

[1] If convenient, we consider the process later also only on $\mathbb{R}_+$. Due to Kolmogorov's extension theorem, the distribution of the stationary ergodic process $(N(t),\, t \in \mathbb{R}_+)$ can be uniquely extended to the whole axis $\mathbb{R}$, which is again stationary and ergodic.

and hence by the ergodicity of N it follows

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(N(t))\mathrm{d}t = g \quad \text{a.s.} \tag{2.4}$$

Summarizing, we make the following assumption:

(A1) Assume that $N = (N(t),\, t \in \mathbb{R})$, is a stationary and ergodic process with values in $\mathbb{Z}_+$, whose trajectories are P-a.s. piecewise constant and right-continuous and which satisfies (2.2), (2.3).

Below we define two classes of sojourn times of virtual τ-requests used in this paper, where virtual means that the request does not interact with the infinite-server system. Thus a virtual request may be considered as a real (non virtual) request if the process N is independent of the arrival process and the required service times of the real requests. Further, we consider the service received by a virtual request. In the following basic properties and relations between the sojourn times are given and outlined, respectively.

1. *Sojourn time of a virtual request*: The sojourn time $V_v(t,\tau)$ of a virtual τ-request arriving at time t at the system is the time until the virtual τ-request has received its required service time $\tau \in \mathbb{R}_+$, i.e.

$$V_v(t,\tau) = \inf\Big\{v\in\mathbb{R}_+ : \int_t^{t+v}\varphi(N(u))\mathrm{d}u \geq \tau\Big\}. \tag{2.5}$$

Let $V_v(\tau) := V_v(0,\tau)$, and for $n \in \mathbb{Z}'_+$ let $V_v(\tau \,|\, n)$ be the corresponding sojourn time of a virtual τ-request arriving at time $t = 0$ conditioned that $N(0) = n$, i.e.

$$P(V_v(\tau) \leq x) = \sum_{n\in\mathbb{Z}'_+} P(V_v(\tau \,|\, n) \leq x)p(n). \tag{2.6}$$

For other concepts of sojourn times of virtual requests in special cases see Remark 4.3 in Section 4.

2. *Sojourn time of a synchronized virtual request*: For giving another interpretation of the sojourn time of a virtual τ-request, we send a state-dependent Poisson process of virtual τ-requests to the system whose arrival intensity is $\alpha(n) > 0$ at time t if $N(t) = n$, i.e., the arrival process of the virtual τ-requests is a Cox process, driven by the random intensity $\alpha(N(t))$, $t \in \mathbb{R}$. Let $\Phi_s = \{T^s_\ell,\, \ell \in \mathbb{Z}\}$ be the point process of arrival times of the Cox process with $\ldots < T^s_{-1} < T^s_0 \leq 0 < T^s_1 < \ldots$. The stationarity and

ergodicity of N implies the stationarity and ergodicity of Φ_s. We assume that the intensity λ_s of Φ_s is finite, i.e.

$$\lambda_s = \sum_{n \in \mathbb{Z}_+} \alpha(n)p(n) < \infty. \tag{2.7}$$

Let $\tau \in \mathbb{R}_+$ be fixed in the following and $X := (X(t), t \in \mathbb{R})$, where $X(t) := (N(t), V_v(t,\tau))$, $t \in \mathbb{R}$. Then the stationarity and ergodicity of N implies that (Φ_s, X) is jointly stationary and ergodic, too, because the construction (2.5) of $V_v(t,\tau)$ is a measurable mapping of N compatible to the shift operator θ_t and Φ_s is a Cox process driven by $\alpha(N(t))$, $t \in \mathbb{R}$, cf. e.g. [DV]. Consider the canonical form of (Φ_s, X), i.e., the basic probability space is the set of all realizations of (Φ_s, X), endowed with the appropriate Borel σ-field, and P is the distribution of (Φ_s, X), cf. e.g. [BB]. The Palm distribution P_s^0 of (Φ_s, X) is uniquely defined, and for all measurable functions f it holds

$$\lambda_s \int f(\varphi_s, x) P_s^0(\mathrm{d}(\varphi_s, x)) = E\Big[\int_0^1 f(\theta_t \Phi_s, \theta_t X)\Phi_s(\mathrm{d}t)\Big] \tag{2.8}$$

and $P_s^0(T_0^s = 0) = 1$, cf. e.g. [Kal]. Now let $D(\mathbb{R}, \mathbb{Z}_+ \times \mathbb{R}_+)$ be the set of all $\mathbb{Z}_+ \times \mathbb{R}_+$ valued functions on $\mathbb{R}$ which are right-continuous with left-hand limits and with a finite number of discontinuities in any finite interval. Since Φ_s is a Cox process driven by the random measure $\xi(\mathrm{d}t) := \alpha(N(t))\mathrm{d}t$, we have the following well-known result.

Lemma 2.1 *For measurable functions $h : D(\mathbb{R}, \mathbb{Z}_+ \times \mathbb{R}_+) \to \mathbb{R}_+$ it holds*

$$\lambda_s \int h(x) P_s^0(\mathrm{d}(\varphi_s, x)) = \sum_{n \in \mathbb{Z}_+} \alpha(n) E[\,\mathbb{I}\{N(0)=n\}h(X)]. \tag{2.9}$$

Proof Consider the canonical form of (Φ_s, ξ, X). Let $P_{(\xi,X)}^0$ be the Palm distribution of (ξ, X) and Π_γ the distribution of a Poisson process with intensity measure γ. Since the function h only depends on x and Φ_s is a Cox process driven by the random measure $\xi(\mathrm{d}t) = \alpha(N(t))\mathrm{d}t$, we obtain

$$\begin{aligned}&\int h(x) P_s^0(\mathrm{d}(\varphi_s, x)) \\ &\quad = \iint h(x)\Pi_\gamma(\mathrm{d}\mu) P_{(\xi,X)}^0(\mathrm{d}(\gamma, x)) = \int h(x) P_{(\xi,X)}^0(\mathrm{d}(\gamma, x)).\end{aligned}$$

Using the definition of the Palm measure for $P^0_{(\xi,X)}$ and the stationarity of the process N, i.e., $\theta_t N = (\theta_t N(s),\, s \in \mathbb{R})$ has the same distribution as N for $t \in \mathbb{R}$, we can continue

$$\begin{aligned}
&\int h(x) P^0_s(\mathrm{d}(\varphi_s, x)) \\
&\quad = \lambda_*^{-1} E\Big[\int_0^1 h(\theta_t X)\xi(\mathrm{d}t)\Big] = \lambda_*^{-1} E\Big[\int_0^1 h(\theta_t X)\alpha(N(t))\mathrm{d}t\Big] \\
&\quad = \lambda_*^{-1} \int_0^1 E[\, h(\theta_t X)\alpha(N(t))]\mathrm{d}t = \lambda_*^{-1} \int_0^1 E[\, h(X)\alpha(N(0))]\mathrm{d}t \\
&\quad = \lambda_*^{-1} \sum_{n\in\mathbb{Z}_+} \alpha(n) E[\mathbb{I}\{N(0)=n\} h(X)].
\end{aligned}$$

Choosing $h(x) \equiv 1$, we obtain in particular $\lambda_* = \lambda_s$ finishing the proof. $\square$

Applying now Lemma 2.1 to the function $h(X) := \mathbb{I}\{N(0) = n, V_v(0,\tau) > x\}$ for $n \in \mathbb{Z}_+$, $x \in \mathbb{R}$, we find

$$\lambda_s P^0_s(V_v(0,\tau)>x, N(0)=n) = \alpha(n) P(V_v(0,\tau)>x, N(0)=n). \tag{2.10}$$

In particular, for $x < 0$ it follows that the probability $\mathring{p}_s(n)$ that an arriving virtual τ-request finds the system in state n is given by

$$\mathring{p}_s(n) = \frac{\alpha(n)}{\lambda_s} p(n), \quad n \in \mathbb{Z}_+. \tag{2.11}$$

Further, dividing (2.10) by $\alpha(n)p(n)$, $n \in \mathbb{Z}'_+$, and taking into account (2.11), it follows

$$P^0_s(V_v(0,\tau)>x \,|\, N(0)=n) = P(V_v(0,\tau)>x \,|\, N(0)=n),$$

i.e., the sojourn time $\mathring{V}_s(\tau \,|\, n)$ of an arriving virtual τ-request finding the system in state n has the same distribution as $V_v(\tau \,|\, n)$:

$$\mathring{V}_s(\tau \,|\, n) \overset{\mathcal{D}}{=} V_v(\tau \,|\, n), \quad \tau \in \mathbb{R}_+,\ n \in \mathbb{Z}'_+, \tag{2.12}$$

where $\overset{\mathcal{D}}{=}$ means equality in distribution. For the sojourn time $\mathring{V}_s(\tau)$ of an arbitrary arriving virtual τ-request hence we obtain

$$P(\mathring{V}_s(\tau)\le x) = \sum_{n\in\mathbb{Z}'_+} P(V_v(\tau \,|\, n)\le x)\mathring{p}_s(n). \tag{2.13}$$

Choosing $\alpha(n) = \alpha$, $n \in \mathbb{Z}_+$, for some $\alpha > 0$, (2.7) implies $\lambda_s = \alpha$, and (2.11) yields $\mathring{p}_s(n) = p(n)$, $n \in \mathbb{Z}_+$. Because of (2.6) and (2.13), thus the sojourn time of arriving virtual τ-requests equals in distribution $V_v(\tau)$ in this case. Note that the clock governing the arrival process of the virtual τ-requests is asynchronous to the clock governing the service process in this case, in general. Choosing $\alpha(n) = \alpha\varphi(n)$, $n \in \mathbb{Z}_+$, for some $\alpha > 0$, which we will always assume in the following, the clock governing the arrival process of the virtual τ-requests is synchronous to the clock governing the service process. Therefore we denote the arriving virtual τ-requests as synchronized virtual τ-requests. Note that (2.7), $\alpha(n) = \alpha\varphi(n)$, $n \in \mathbb{Z}_+$, and (2.3) imply $\lambda_s = \alpha g$, and $\lambda_s < \infty$ is equivalent to (2.3). From (2.11) it follows that the probability $\mathring{p}_s(n)$ that an arriving synchronized virtual τ-request finds the system in state n is given by

$$\mathring{p}_s(n) = \frac{1}{g}\,\varphi(n)p(n), \quad n \in \mathbb{Z}_+. \tag{2.14}$$

3. *Service received by a virtual request*: Let a virtual request with infinite required service time (permanent virtual request) arrive at time 0, let $U(t)$ be the service received by the virtual request from time 0 until time t, and for $n \in \mathbb{Z}'_+$ let $U(t\,|\,n)$ be the service received by the virtual request from time 0 until time t conditioned that $N(0) = n$. Obviously, it holds

$$U(t) = \int_0^t \varphi(N(u))\mathrm{d}u, \quad t \in \mathbb{R}_+, \tag{2.15}$$

$$U(t\,|\,n) = \int_0^t \varphi(N(u\,|\,n))\mathrm{d}u, \quad t \in \mathbb{R}_+,\ n \in \mathbb{Z}'_+. \tag{2.16}$$

Analogously to (2.6) we find

$$P(U(t) \le x) = \sum_{n \in \mathbb{Z}'_+} P(U(t\,|\,n) \le x)p(n). \tag{2.17}$$

Note that in view of $\varphi(m) > 0$, $m \in \mathbb{Z}_+$, the processes $U(t)$ and $U(t\,|\,n)$, $n \in \mathbb{Z}'_+$, are strictly increasing in t. Thus from the definitions of $V_v(\tau)$, $U(t)$ and $V_v(\tau\,|\,n)$, $U(t\,|\,n)$ for $\tau, t \in \mathbb{R}_+$ it follows that $V_v(\tau) = t$ is equivalent to $U(t) = \tau$ and that $V_v(\tau\,|\,n) = t$ is equivalent to $U(t\,|\,n) = \tau$ for $n \in \mathbb{Z}'_+$. Thus Fubini's theorem, (2.15), the stationarity of $N(t)$, $t \in \mathbb{R}_+$, and (2.3) yield

$$\int_{\mathbb{R}_+} P(V_v(\tau) \le t)\mathrm{d}\tau = \int_{\mathbb{R}_+} E[\mathbb{I}\{\tau \le U(t)\}]\mathrm{d}\tau = E\Big[\int_{\mathbb{R}_+} \mathbb{I}\{\tau \le U(t)\}\mathrm{d}\tau\Big]$$

$$= EU(t) = E\Big[\int_0^t \varphi(N(u))\mathrm{d}u\Big] = \int_0^t E[\varphi(N(0))]\mathrm{d}u$$
$$= gt, \quad t \in \mathbb{R}_+. \tag{2.18}$$

Moreover, Fubini's theorem and (2.18) provide

$$\int_{\mathbb{R}_+} E[e^{-sV_v(\tau)}]\mathrm{d}\tau = E\Big[\int_{\mathbb{R}_+}\int_{V_v(\tau)}^{\infty} se^{-st}\mathrm{d}t\mathrm{d}\tau\Big]$$
$$= E\Big[\int_{\mathbb{R}_+^2} \mathbb{I}\{V_v(\tau)\le t\}se^{-st}\mathrm{d}t\mathrm{d}\tau\Big] = \int_{\mathbb{R}_+^2} P(V_v(\tau)\le t)se^{-st}\mathrm{d}\tau\mathrm{d}t$$
$$= \int_{\mathbb{R}_+} gtse^{-st}\mathrm{d}t = \frac{g}{s}, \quad s \in (0,\infty). \tag{2.19}$$

Note that (2.19) implies

$$\lim_{\tau\to\infty} V_v(\tau) = \lim_{\tau\to\infty} V_v(\tau\,|\,n) = \infty \quad \text{a.s.}, \quad n \in \mathbb{Z}'_+. \tag{2.20}$$

As $\varphi(n)$ is positive, from (2.15), (2.4), and (2.3) we find

$$\lim_{t\to\infty} U(t)/t = g = E[\varphi(N(0))] > 0 \quad \text{a.s.}, \tag{2.21}$$

which yields

$$\lim_{t\to\infty} U(t) = \lim_{t\to\infty} U(t\,|\,n) = \infty \quad \text{a.s.}, \quad n \in \mathbb{Z}'_+. \tag{2.22}$$

Because of (2.20), (2.22), finally we find that $V_v(\cdot)$ is a.s. the inverse function of $U(\cdot)$ and that $V_v(\cdot\,|\,n)$ is a.s. the inverse function of $U(\cdot\,|\,n)$ for $n \in \mathbb{Z}'_+$. In view of (2.22) and (2.21), therefore the substitution $\tau = U(t)$ provides

$$\lim_{\tau\to\infty} V_v(\tau)/\tau = \lim_{t\to\infty} t/U(t) = 1/g \quad \text{a.s.}, \tag{2.23}$$

which, in view of (2.12), implies

$$\lim_{\tau\to\infty} \mathring{V}_s(\tau)/\tau = 1/g \quad \text{a.s.} \tag{2.24}$$

2.1 A random time transformation

Note that $\int_0^t \varphi(N(u))\mathrm{d}u$, $t \in \mathbb{R}$, defines an additive functional generated by the process N. The associated random time transformation is given a.s. by

$$\vartheta(\tau) := \inf\Big\{t\in\mathbb{R} : \int_0^t \varphi(N(u))\mathrm{d}u \ge \tau\Big\}, \quad \tau \in \mathbb{R}. \tag{2.25}$$

As $V_v(\cdot)$ is a.s. the inverse function of $U(\cdot)$, from (2.25) and (2.15) it follows

$$\vartheta(\tau) = V_v(\tau) \quad \text{a.s.}, \quad \tau \in \mathbb{R}_+. \tag{2.26}$$

Let $\hat{N} := (\hat{N}(t),\, t \in \mathbb{R})$, where

$$\hat{N}(t) := N(\vartheta(t)), \quad t \in \mathbb{R}, \tag{2.27}$$

be the time-changed process of N. Remember that if the system is in state n then the clock governing the service process runs with speed $\varphi(n)$. The time transformation (2.25), (2.27) implies that the service clock is speeded up by the factor $1/\varphi(n)$, and hence the service clock runs with speed 1 under the time-changed dynamics.

Also, in view of (2.25) and (2.27), there is a one-to-one correspondence between the sample paths of N and $\hat{N}$, and we have the following.

Lemma 2.2 *For each trajectory and $\tau \in \mathbb{R}$ it holds*

$$\vartheta(\tau) = \int_0^\tau \frac{1}{\varphi(\hat{N}(u))}\, \mathrm{d}u. \tag{2.28}$$

Proof From (2.25) and (2.27) it follows

$$\vartheta(\tau) = \int_0^\tau \vartheta'(u)\mathrm{d}u = \int_0^\tau \frac{1}{\varphi(N(\vartheta(u)))}\, \mathrm{d}u = \int_0^\tau \frac{1}{\varphi(\hat{N}(u))}\, \mathrm{d}u.$$

□

Note that the time-changed process $\hat{N}$ is not a stationary process in general, although N is a stationary one. However, we will construct the distribution of a stationary process with the time-changed dynamics.

Let T_ℓ, $\ell \in \mathbb{Z}$, be the jump epochs of N, i.e. $N(T_\ell -) \neq N(T_\ell)$, ordered such that $\ldots < T_{-1} < T_0 \leq 0 < T_1 < \ldots$, and $K_\ell := N(T_\ell)$ be the state of the system at T_ℓ. Note that

$$N(t) = \sum_{\ell \in \mathbb{Z}} \mathbb{I}\{T_\ell \leq t < T_{\ell+1}\} K_\ell, \quad t \in \mathbb{R}, \tag{2.29}$$

since the sample paths of N are in $D(\mathbb{R}, \mathbb{Z}_+)$. The marked point process (MPP) $\Psi := \{[T_\ell, K_\ell], \ell \in \mathbb{Z}\}$ is stationary and ergodic, too, since N is stationary and ergodic and has the finite intensity λ, cf. (2.2). Note that Ψ can be considered as the natural embedded MPP of N. Also, Ψ determines N uniquely, cf. (2.29). Consider the canonical representation of Ψ with distribution P, cf. [BB]. More precisely, $(M_K, \mathcal{M}_K, P)$ is the probability space and Ψ the identical mapping $\Psi : M_K \to M_K$. The set M_K is the set of all simple counting measures $\psi(\cdot) = \sum_{\ell\in\mathbb{Z}} \delta_{[t_\ell,k_\ell]}(\cdot)$ on $\mathbb{R}\times\mathbb{Z}_+$ endowed with the appropriate Borel σ-field $\mathcal{M}_K$ such that $\ldots < t_{-1} < t_0 \leq 0 < t_1 < \ldots$ and $\lim_{\ell\to\pm\infty} t_\ell = \pm\infty$. Note that $\psi \in M_K$ can be represented by the sequence $\psi = \{[t_\ell, k_\ell],\ \ell \in \mathbb{Z}\}$, which we will use in the following. Further, the T_ℓ and K_ℓ correspond to the mappings $T_\ell(\psi) = t_\ell$ and $K_\ell(\psi) = k_\ell$ for $\psi \in M_K$. The shift operator θ_t applied to the measure $\psi \in M_K$ is defined by $\theta_t\psi = \sum_{\ell\in\mathbb{Z}} \delta_{[T_\ell(\psi)-t,K_\ell(\psi)]}$. Note that $T_\ell(\theta_t\psi) = T_{\ell+c(t)}(\psi) - t$, $K_\ell(\theta_t\psi) = K_{\ell+c(t)}(\psi)$, where $c(t)$ is the number of points of ψ in $(0,t]$ if $t > 0$ and the negative number of points in $(t,0]$ if $t \leq 0$. Since (Ψ, P) is stationary, it holds $P(A) = P(\theta_t A)$, $t \in \mathbb{R}$, $A \in \mathcal{M}_K$. Let P^0 be the Palm distribution of (Ψ, P). It holds $P^0(T_0 = 0) = 1$, i.e., P^0 is concentrated on $M_K^0 := \{\psi \in M_K : t_0 = 0\}$, $\mathcal{M}_K^0 := \{A \subseteq M_K^0 : A \in \mathcal{M}_K\}$, and P^0 is invariant with respect to the mapping θ defined by $\theta\psi := \theta_{T_1(\psi)}\psi$ for $\psi \in M_K$, i.e.

$$P^0(A) = P^0(\theta^{-1}A), \quad A \in \mathcal{M}_K^0, \tag{2.30}$$

$$E_{P^0}[T_1] = \lambda^{-1}. \tag{2.31}$$

Also, P^0 is ergodic since P is ergodic. The invariance property (2.30) and the ergodicity of P^0 imply that the sequence $\{[A_\ell, K_\ell],\ \ell \in \mathbb{Z}\}$ of the spacings A_ℓ, where $A_\ell(\psi) := T_{\ell+1}(\psi) - T_\ell(\psi)$, and marks K_ℓ is a stationary and ergodic sequence with respect to P^0. Also, because of $P^0(T_0 = 0) = 1$, there is a one-to-one correspondence between the distribution of $\{[A_\ell, K_\ell],\ \ell \in \mathbb{Z}\}$ and the Palm distribution of the MPP Ψ, cf. e.g. [BB], [BFL], and [FKAS].

Consider now the time-changed MPP $\hat{\Psi} := \{[\hat{T}_\ell, \hat{K}_\ell],\ \ell \in \mathbb{Z}\}$, where $\hat{K}_\ell = K_\ell$, $\ell \in \mathbb{Z}$, and, in view of (2.25) and (2.29),

$$\hat{T}_\ell := \int_0^{T_\ell} \varphi(N(u))\mathrm{d}u = \begin{cases} \varphi(K_0)T_0 - \sum\limits_{j=\ell}^{-1} \varphi(K_j)(T_{j+1}-T_j), & \ell \leq 0, \\ \varphi(K_0)T_1 + \sum\limits_{j=1}^{\ell-1} \varphi(K_j)(T_{j+1}-T_j), & \ell > 0. \end{cases} \tag{2.32}$$

Note that (2.32) defines a one-to-one mapping $\hat{h} : M_K \to M_K$ and that $\hat{\Psi} = \hat{h}(\Psi)$. Because of the construction, it holds $\hat{h}(\theta\Psi) = \theta\hat{h}(\Psi)$, and from (2.32) we obtain that the distribution $\hat{P}^0(A) := P^0(\hat{\Psi} \in A) = P^0(\hat{h}^{-1}(A))$, $A \in \mathcal{M}_K^0$, of $\hat{\Psi}$ on $(M_K^0, \mathcal{M}_K^0)$ is invariant with respect to θ and ergodic and that $P^0(\hat{T}_0 = 0) = 1$. Note that for $A \in \mathcal{M}_K^0$ it holds $\hat{h}(A) \in \mathcal{M}_K^0$ and vice versa and that $\{[\hat{A}_\ell, \hat{K}_\ell],\, \ell \in \mathbb{Z}\}$, where $\hat{A}_\ell := \hat{T}_{\ell+1} - \hat{T}_\ell = \varphi(K_\ell)A_\ell$, $\hat{K}_\ell = K_\ell$, $\ell \in \mathbb{Z}$, is a stationary and ergodic sequence with respect to P^0 as basic probability measure on $(M_k^0, \mathcal{M}_k^0)$ in view of

$$P^0(\psi : A_{\ell+1}(\hat{h}(\psi)) = A_\ell(\hat{h}(\theta\psi))) = 1.$$

From the inversion formula of Ryll-Nardzewski and Slivnyak, cf. e.g. [BB] (1.2.25), or the inversion formula for embedded MPPs, cf. e.g. [FKAS] (1.5.2), and (2.32), (2.3) it follows

$$E_{P^0}[\hat{T}_1] = E_{P^0}\Big[\int_0^{T_1} \varphi(N(t))\mathrm{d}t\Big] = \lambda^{-1}E[\varphi(N(0))] = \lambda^{-1}g. \tag{2.33}$$

Let $\tilde{P}$ be the uniquely determined θ_t-invariant distribution on $(M_K, \mathcal{M}_K)$ given by the inversion formula from the Palm distribution $\hat{P}^0$

$$\tilde{P}(A) = \tilde{\lambda}\int_0^\infty P^0(\hat{T}_1 > t, \theta_t\hat{\Psi} \in A)\mathrm{d}t, \quad A \in \mathcal{M}_K, \tag{2.34}$$

whose intensity $\tilde{\lambda} = 1/E_{P^0}[\hat{T}_1]$ is finite and given by

$$\tilde{\lambda} = \lambda/g, \tag{2.35}$$

cf. (2.31), (2.33). Taking into account $\hat{T}_1 = \varphi(K_0)T_1$, $\hat{\Psi} = h(\Psi)$, using Fubini's Theorem, applying the substitution $s := t/\varphi(K_0)$, and using the fact that for $s \in \mathbb{R}_+$

$$\mathbb{I}\{t_1 > s, \theta_{s\varphi(k_0)}\hat{h}(\psi) \in A\} = \mathbb{I}\{t_1 > s, \hat{h}(\theta_s\psi) \in A\}, \quad \psi \in M_K, \tag{2.36}$$

from (2.34) for $A \in \mathcal{M}_K$ we obtain

$$\begin{aligned}
\tilde{P}(A) &= \tilde{\lambda}\int_0^\infty E_{P^0}[\mathbb{I}\{\varphi(K_0)T_1 > t,\, \theta_t\hat{h}(\Psi) \in A\}]\mathrm{d}t \\
&= \tilde{\lambda}E_{P^0}\Big[\int_0^\infty \mathbb{I}\{\varphi(K_0)T_1 > t,\, \theta_t\hat{h}(\Psi) \in A\}\mathrm{d}t\Big] \\
&= \tilde{\lambda}E_{P^0}\Big[\int_0^\infty \mathbb{I}\{T_1 > s,\, \theta_{s\varphi(K_0)}\hat{h}(\Psi) \in A\}\varphi(K_0)\mathrm{d}s\Big] \\
&= \tilde{\lambda}E_{P^0}\Big[\int_0^\infty \mathbb{I}\{T_1 > s,\, \hat{h}(\theta_s\Psi) \in A\}\varphi(K_0)\mathrm{d}s\Big] \\
&= \tilde{\lambda}\int_0^\infty E_{P^0}[\mathbb{I}\{T_1 > s,\, \hat{h}(\theta_s\Psi) \in A\}\varphi(K_0)]\mathrm{d}s
\end{aligned}$$

$$= \tilde{\lambda} \int_0^\infty \sum_{n\in\mathbb{Z}_+} \varphi(n) E_{P^0}[\mathbb{I}\{T_1 > s,\, \hat{h}(\theta_s\Psi) \in A, K_0 = n\}]\mathrm{d}s$$

$$= \tilde{\lambda} \sum_{n\in\mathbb{Z}_+} \varphi(n) \int_0^\infty P^0(T_1 > s,\, \theta_s\Psi \in \hat{h}^{-1}(A), K_0 = n)\mathrm{d}s.$$

Applying now the inversion formula to P^0 and taking into account (2.35), we obtain

$$\tilde{P}(A) = \frac{1}{g} \sum_{n\in\mathbb{Z}_+} \varphi(n) P(\Psi \in \hat{h}^{-1}(A), K_0 = n), \quad A \in \mathcal{M}_K. \tag{2.37}$$

Since $\hat{P}^0$ is ergodic with respect to θ, $\tilde{P}$ is an ergodic distribution with respect to θ_t, $t \in \mathbb{R}$, too. Let $\tilde{\Psi} = \{[\tilde{T}_\ell, \tilde{K}_\ell],\, \ell \in \mathbb{Z}\}$ be an MPP with distribution $\tilde{P}$. Then

$$\tilde{N}(t) := \sum_{\ell\in\mathbb{Z}} \mathbb{I}\{\tilde{T}_\ell \le t < \tilde{T}_{\ell+1}\}\tilde{K}_\ell, \quad t \in \mathbb{R}, \tag{2.38}$$

provides a stationary and ergodic process $\tilde{N} := (\tilde{N}(t),\, t \in \mathbb{R})$, corresponding to a time-stationary version of the time-changed process $\hat{N} = (\hat{N}(t),\, t \in \mathbb{R})$.

Summarizing, we have the following theorem.

Theorem 2.1 *Assume that the process $N = (N(t),\, t \in \mathbb{R})$ satisfies (A1). Let $\Psi = \{[T_\ell, K_\ell],\, \ell \in \mathbb{Z}\}$ be the given embedded MPP of jump epochs T_ℓ and marks $K_\ell = N(T_\ell)$, $\ell \in \mathbb{Z}$, where $\ldots < T_0 \le 0 < T_1 < \ldots$.*

Then (2.37) *defines the distribution of a stationary and ergodic MPP $\tilde{\Psi} = \{[\tilde{T}_\ell, \tilde{K}_\ell],\, \ell \in \mathbb{Z}\}$ and via* (2.38) *a related stationary and ergodic process $\tilde{N} = (\tilde{N}(t),\, t \in \mathbb{R})$ being a time-stationary and ergodic version of the time-changed process $\hat{N} = (\hat{N}(t),\, t \in \mathbb{R})$ with a finite intensity of jumps.*

For $n \in \mathbb{Z}'_+$ let $\hat{N}_n := (\hat{N}(t\,|\,n),\, t \in \mathbb{R})$ and $\tilde{N}_n := (\tilde{N}(t\,|\,n),\, t \in \mathbb{R})$ be processes with distributions $P((\hat{N}(t),\, t \in \mathbb{R}) \in (\cdot) \,|\, \hat{N}(0) = n)$, cf. (2.27), and $P((\tilde{N}(t),\, t \in \mathbb{R}) \in (\cdot) \,|\, \tilde{N}(0) = n)$, respectively. Further, let us denote by $\tilde{p}(n) := P(\tilde{N}(0) = n)$, $n \in \mathbb{Z}_+$, the marginal probabilities of $\tilde{N}$.

Lemma 2.3 *Assume that (A1) is fulfilled. Then it holds*

$$\tilde{p}(n) = \mathring{p}_s(n), \quad n \in \mathbb{Z}_+, \tag{2.39}$$

$$\tilde{N}_n \stackrel{\mathcal{D}}{=} \hat{N}_n, \quad n \in \mathbb{Z}'_+, \tag{2.40}$$

$$E\Big[\frac{1}{\varphi(\tilde N(0))}\Big] = 1/g, \tag{2.41}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{1}{\varphi(\tilde N(\tau))}\,\mathrm{d}\tau = 1/g \quad a.s. \tag{2.42}$$

Proof Applying (2.37) to $A := \{K_0 = n\} = \{N(0) = n\}$ and taking into account $\hat h^{-1}(A) = \{K_0 = n\}$, we obtain

$$\tilde p(n) = \frac{1}{g}\,\varphi(n)p(n), \tag{2.43}$$

and thus (2.39) in view of (2.14). From Theorem 2.1, (2.37), (2.43), and because of $\hat N(0) = N(0)$, we find that

$$\begin{aligned}
P(\tilde N_n \in A) &= \frac{1}{\tilde p(n)}\,P(\tilde N_n \in A, \tilde N(0){=}n)\\
&= \frac{1}{\tilde p(n)}\,\frac{1}{g}\,\varphi(n)P((\hat N(t),\, t{\in}\mathbb{R}){\in} A, N(0){=}n)\\
&= \frac{1}{p(n)}\,P((\hat N(t),\, t{\in}\mathbb{R}){\in} A, N(0){=}n)\\
&= P((\hat N(t),\, t{\in}\mathbb{R}){\in} A \,|\, \hat N(0){=}n),
\end{aligned}$$

finishing the proof of (2.40). Assertion (2.41) follows directly from (2.43). Since $\tilde N$ is an ergodic process, assertion (2.42) follows immediately due to Birkhoff's ergodic theorem and (2.41). $\square$

The following lemma provides the key representations for the sojourn times $V_v(\tau)$ and $\mathring{V}_s(\tau)$ in terms of the stationary process $\tilde N(t)$, $t \in \mathbb{R}_+$.

Lemma 2.4 *Assume that (A1) is fulfilled. Then it holds*

$$V_v(\tau\,|\,n) \stackrel{\mathcal{D}}{=} \int_0^\tau \frac{1}{\varphi(\tilde N(u\,|\,n))}\,\mathrm{d}u, \quad \tau \in \mathbb{R}_+,\ n \in \mathbb{Z}'_+, \tag{2.44}$$

$$\mathring{V}_s(\tau) \stackrel{\mathcal{D}}{=} \int_0^\tau \frac{1}{\varphi(\tilde N(u))}\,\mathrm{d}u, \quad \tau \in \mathbb{R}_+, \tag{2.45}$$

and for any measurable non negative function $f(x)$, $x \in (0,\infty)$, it holds

$$E[f(V_v(\tau)/\tau)] = gE\Big[f\Big(\frac{1}{\tau}\int_0^\tau \frac{1}{\varphi(\tilde N(u))}\,\mathrm{d}u\Big)\frac{1}{\varphi(\tilde N(0))}\Big], \quad \tau \in (0,\infty). \tag{2.46}$$

Proof For $n \in \mathbb{Z}'_+$ from (2.26), (2.28), and (2.40) we obtain

$$V_v(\tau \,|\, n) = \int_0^\tau \frac{1}{\varphi(\hat{N}(u \,|\, n))}\,\mathrm{d}u \overset{\mathcal{D}}{=} \int_0^\tau \frac{1}{\varphi(\tilde{N}(u \,|\, n))}\,\mathrm{d}u,$$

and from (2.13), (2.44), and (2.39) it follows that

$$P(\mathring{V}_s(\tau) \le x) = \sum_{n \in \mathbb{Z}'_+} P(V_v(\tau \,|\, n) \le x)\mathring{p}_s(n)$$

$$= \sum_{n \in \mathbb{Z}'_+} P\Big(\int_0^\tau \frac{1}{\varphi(\tilde{N}(u \,|\, n))}\,\mathrm{d}u \le x\Big)\tilde{p}(n)$$

$$= P\Big(\int_0^\tau \frac{1}{\varphi(\tilde{N}(u))}\,\mathrm{d}u \le x\Big).$$

Taking into account (2.14), (2.44), and (2.39), we find

$$E[f(V_v(\tau)/\tau)] = \sum_{n \in \mathbb{Z}'_+} E[f(V_v(\tau \,|\, n)/\tau)]p(n)$$

$$= g \sum_{n \in \mathbb{Z}'_+} E[f(V_v(\tau \,|\, n)/\tau)]\,\frac{1}{\varphi(n)}\,\mathring{p}_s(n)$$

$$= g \sum_{n \in \mathbb{Z}'_+} E\Big[f\Big(\frac{1}{\tau}\int_0^\tau \frac{1}{\varphi(\tilde{N}(u \,|\, n))}\,\mathrm{d}u\Big)\frac{1}{\varphi(\tilde{N}(0 \,|\, n))}\Big]\tilde{p}(n),$$

which provides (2.46). □

2.2 Expectation of functions of additive functionals

In order to exploit the representations (2.44)–(2.46) and (2.15), we will derive some results on the expectation of functions of smooth additive functionals in the following. Let $Y(t)$, $t \in \mathbb{R}_+$, be a stationary càdlàg process with values in $(0, \infty)$, and assume that $EY(0) < \infty$. Further, let

$$Z(\tau) := \int_0^\tau Y(t)\mathrm{d}t, \quad \tau \in \mathbb{R}_+. \tag{2.47}$$

Note that $Z(\tau)$, $\tau \in \mathbb{R}_+$, is an additive functional with density $Y(t)$, $t \in \mathbb{R}_+$. Because of the stationarity of the process $Y(t)$, $t \in \mathbb{R}_+$, from (2.47) it follows immediately, cf. [Hor],

$$EZ(\tau) = \int_0^\tau E[Y(t)]\mathrm{d}t = \tau EZ(1) = \tau EY(0), \quad \tau \in (0, \infty). \tag{2.48}$$

Note that due to Birkhoff's ergodic theorem

$$Y_0 := \lim_{\tau\to\infty} Z(\tau)/\tau \tag{2.49}$$

exists a.s. and that

$$EY_0 = EY(0). \tag{2.50}$$

Theorem 2.2 *Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $\tau_1, \tau_2 \in (0,\infty)$ it holds*

$$\begin{aligned} &E[(\tau_1+\tau_2) f(Z(\tau_1+\tau_2)/(\tau_1+\tau_2))] \\ &\qquad \le E[\tau_1 f(Z(\tau_1)/\tau_1)] + E[\tau_2 f(Z(\tau_2)/\tau_2)], \end{aligned} \tag{2.51}$$

i.e., $E[\tau f(Z(\tau)/\tau)]$ is subadditive for $\tau \in (0,\infty)$.

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $\tau_1, \tau_2 \in (0,\infty)$ it holds

$$\begin{aligned} &E[(\tau_1+\tau_2) f(Z(\tau_1+\tau_2)/(\tau_1+\tau_2))] \\ &\qquad \ge E[\tau_1 f(Z(\tau_1)/\tau_1)] + E[\tau_2 f(Z(\tau_2)/\tau_2)], \end{aligned} \tag{2.52}$$

i.e., $E[\tau f(Z(\tau)/\tau)]$ is superadditive for $\tau \in (0,\infty)$.

Proof We will give the proof of (2.51), the proof of (2.52) runs analogously. As the function $f(x)$, $x \in (0,\infty)$, is convex, from (2.47) for $\tau_1, \tau_2 \in (0,\infty)$ it follows that

$$\begin{aligned} &E[f(Z(\tau_1+\tau_2)/(\tau_1+\tau_2))] \\ &\qquad = E\Big[f\Big(\frac{\tau_1}{\tau_1+\tau_2}\,\frac{1}{\tau_1}\int_0^{\tau_1} Y(t)\mathrm{d}t + \frac{\tau_2}{\tau_1+\tau_2}\,\frac{1}{\tau_2}\int_{\tau_1}^{\tau_1+\tau_2} Y(t)\mathrm{d}t\Big)\Big] \\ &\qquad \le \frac{\tau_1}{\tau_1+\tau_2}\, E\Big[f\Big(\frac{1}{\tau_1}\int_0^{\tau_1} Y(t)\mathrm{d}t\Big)\Big] + \frac{\tau_2}{\tau_1+\tau_2}\, E\Big[f\Big(\frac{1}{\tau_2}\int_{\tau_1}^{\tau_1+\tau_2} Y(t)\mathrm{d}t\Big)\Big] \\ &\qquad = (E[\tau_1 f(Z(\tau_1)/\tau_1)] + E[\tau_2 f(Z(\tau_2)/\tau_2)])/(\tau_1+\tau_2), \end{aligned}$$

where the last equality follows due to the stationarity of the process $Y(t)$, $t \in \mathbb{R}_+$. $\square$

Theorem 2.3 *Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $\tau \in (0,\infty)$ it holds*

$$\begin{aligned} f(E[Y(0)]) &\le E[f(Y_0)] \le \liminf_{t\to\infty} E[f(Z(t)/t)] \\ &\le E[f(Z(\tau)/\tau)] \le \lim_{t\downarrow 0} E[f(Z(t)/t)] = E[f(Y(0))]. \end{aligned} \tag{2.53}$$

If additionally $E[f(Z(\tau)/\tau)] < \infty$ for some $\tau \in (0,\infty)$, then

$$E[f(Y_0)] = \liminf_{t\to\infty} E[f(Z(t)/t)], \tag{2.54}$$

and if $E[f(Z(t)/t)]$ is bounded in a neighborhood of some $\tau \in (0,\infty)$, then

$$E[f(Y_0)] = \lim_{t\to\infty} E[f(Z(t)/t)]. \tag{2.55}$$

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $\tau \in (0,\infty)$ it holds

$$\begin{aligned} E[f(Y(0))] &= \lim_{t\downarrow 0} E[f(Z(t)/t)] \le E[f(Z(\tau)/\tau)] \\ &\le \lim_{t\to\infty} E[f(Z(t)/t)] = E[f(Y_0)] \le f(E[Y(0)]). \end{aligned} \tag{2.56}$$

Proof (i) Let $f(x)$, $x \in (0,\infty)$, non negative and convex. We use the subadditivity of $E[\tau f(Z(\tau)/\tau)]$, $\tau \in (0,\infty)$, in the following, cf. [BO]. By induction on n from (2.51) we find that

$$E[f(Z(n\tau)/(n\tau))] \le E[f(Z(\tau)/\tau)], \quad \tau \in (0,\infty),\ n \in \mathbb{N}, \tag{2.57}$$

which implies

$$\liminf_{t\to\infty} E[f(Z(t)/t)] \le E[f(Z(\tau)/\tau)] \le \limsup_{t\downarrow 0} E[f(Z(t)/t)]$$

for $\tau \in (0,\infty)$. Further, Jensen's inequality and Fubini's theorem provide

$$\begin{aligned} E[f(Z(\tau)/\tau)] &= E\Big[f\Big(\frac{1}{\tau}\int_0^\tau Y(t)\mathrm{d}t\Big)\Big] \le E\Big[\frac{1}{\tau}\int_0^\tau f(Y(t))\mathrm{d}t\Big] \\ &= \frac{1}{\tau}\int_0^\tau E[f(Y(t))]\mathrm{d}t = E[f(Y(0))], \end{aligned}$$

where the last equality follows from the stationarity of the process $Y(t)$, $t \in \mathbb{R}_+$. Thus it holds

$$\limsup_{t\downarrow 0} E[f(Z(t)/t)] \le E[f(Y(0))].$$

On the other hand, Fatou's lemma provides

$$\begin{aligned}\liminf_{t\downarrow 0} E[f(Z(t)/t)] &\geq E[\liminf_{t\downarrow 0} f(Z(t)/t)]\\ &= E[f(\liminf_{t\downarrow 0} Z(t)/t)] = E[f(Y(0))]\end{aligned}$$

as well as

$$\begin{aligned}\liminf_{t\to\infty} E[f(Z(t)/t)] &\geq E[\liminf_{t\to\infty} f(Z(t)/t)]\\ &= E[f(\liminf_{t\to\infty} Z(t)/t)] = E[f(Y_0)] \geq f(E[Y_0]) = f(E[Y(0)])\end{aligned}$$

due to (2.49), Jensen's inequality, and (2.50). Summarizing, we have proved (2.53).

(ii) Assume that $E[f(Z(\tau)/\tau)] < \infty$ for some $\tau \in (0, \infty)$. Then for $m \in \mathbb{N}$ we find

$$\begin{aligned}&\liminf_{t\to\infty} E[f(Z(t)/t)]\\ &\qquad = \liminf_{t\to\infty}(E[\min(f(Z(t)/t), m)] + E[\max(f(Z(t)/t)-m, 0)])\\ &\qquad = E[\lim_{t\to\infty} \min(f(Z(t)/t), m)] + \liminf_{t\to\infty} E[\max(f(Z(t)/t)-m, 0)],\end{aligned}$$

where the last equality follows from dominated convergence of the first summand on the r.h.s. Taking into account (2.49) and applying (2.57) to $g(\xi) := \max(f(\xi) - m, 0)$ instead of $f(\xi)$, we obtain

$$\begin{aligned}&\liminf_{t\to\infty} E[f(Z(t)/t)]\\ &\qquad \leq E[\min(f(Y_0), m)] + \liminf_{n\to\infty} E[\max(f(Z(n\tau)/(n\tau))-m, 0)]\\ &\qquad \leq E[f(Y_0)] + E[\max(f(Z(\tau)/\tau)-m, 0)].\end{aligned}$$

In view of (2.53), taking the limit $m \to \infty$ provides (2.54) due to dominated convergence.

(iii) Assume that $E[f(Z(t)/t)]$ is bounded in a neighborhood of some $\tau \in (0, \infty)$. Then there exist $m \in \mathbb{N}$ and $M \in \mathbb{R}_+$ such that

$$E[f(Z(t)/t)] \leq M, \quad t \in [\tau, \tau+\tau/m).$$

Let $t \in [m\tau, \infty)$. Then there exists $n \in \mathbb{N}$ such that $t \in [n\tau, (n+1)\tau)$ and $n \geq m$, which implies $t/n \in [\tau, \tau + \tau/n) \subseteq [\tau, \tau + \tau/m)$. Thus from (2.57) we find

$$E[f(Z(t)/t)] \leq M, \quad t \in [m\tau, \infty). \tag{2.58}$$

Let $t \in [m\tau, \infty)$, and let $t' \in [2t, \infty)$. Then there exists $n \in \mathbb{N}$ such that $t' \in [(n+1)t, (n+2)t)$, and from (2.51) it follows that

$$\begin{aligned} &E[t' f(Z(t')/t')] \\ &\qquad \leq E[(t'-nt) f(Z(t'-nt)/(t'-nt))] + E[nt f(Z(nt)/(nt))], \end{aligned}$$

which implies

$$E[f(Z(t')/t')] \leq (2t/t') E[f(Z(t'-nt)/(t'-nt))] + E[f(Z(nt)/(nt))].$$

Because of $t' - nt \geq t \geq m\tau$, thus from (2.58) and (2.57) we obtain

$$E[f(Z(t')/t')] \leq (2t/t')M + E[f(Z(t)/t)].$$

Letting now $t' \to \infty$ and then $t \to \infty$ provides

$$\limsup_{t' \to \infty} E[f(Z(t')/t')] \leq \liminf_{t \to \infty} E[f(Z(t)/t)],$$

which yields (2.55) in view of (2.54).

(iv) Let $f(x)$, $x \in (0, \infty)$, non negative and concave. By induction on n from (2.52) it follows

$$E[f(Z(n\tau)/(n\tau))] \geq E[f(Z(\tau)/\tau)], \quad \tau \in (0, \infty),\ n \in \mathbb{N},$$

which implies

$$\liminf_{t \downarrow 0} E[f(Z(t)/t)] \leq E[f(Z(\tau)/\tau)] \leq \limsup_{t \to \infty} E[f(Z(t)/t)]$$

for $\tau \in (0, \infty)$. Again, Fatou's lemma provides

$$\begin{aligned} \liminf_{t \downarrow 0} E[f(Z(t)/t)] &\geq E[\liminf_{t \downarrow 0} f(Z(t)/t)] \\ &= E[f(\liminf_{t \downarrow 0} Z(t)/t)] = E[f(Y(0))] \end{aligned}$$

as well as

$$\begin{aligned} \liminf_{t \to \infty} E[f(Z(t)/t)] &\geq E[\liminf_{t \to \infty} f(Z(t)/t)] \\ &= E[f(\liminf_{t \to \infty} Z(t)/t)] = E[f(Y_0)] \end{aligned}$$

due to (2.49). As $f(x)$, $x \in (0, \infty)$, is a concave function, there exists an affine function $g(x)$ such that $f(x) \leq g(x)$, $x \in (0, \infty)$. From (2.48) for $t \in (0, \infty)$ it follows

$$E[g(Z(t)/t)] = g(E[Z(t)/t]) = g(E[Y(0)]).$$

Thus Fatou's lemma provides

$$\begin{aligned}
& g(E[Y(0)]) - \limsup_{t\downarrow 0} E[f(Z(t)/t)] \\
& \qquad = \liminf_{t\downarrow 0} E[g(Z(t)/t) - f(Z(t)/t)] \\
& \qquad \geq E[\liminf_{t\downarrow 0}(g(Z(t)/t) - f(Z(t)/t))] \\
& \qquad = E[g(Y(0)) - f(Y(0))] = g(E[Y(0)]) - E[f(Y(0))],
\end{aligned}$$

which implies

$$\limsup_{t\downarrow 0} E[f(Z(t)/t)] \leq E[f(Y(0))].$$

Analogously, in view of (2.49) and (2.50), we find

$$\begin{aligned}
& g(E[Y(0)]) - \limsup_{t\to\infty} E[f(Z(t)/t)] \\
& \qquad = \liminf_{t\to\infty} E[g(Z(t)/t) - f(Z(t)/t)] \\
& \qquad \geq E[\liminf_{t\to\infty}(g(Z(t)/t) - f(Z(t)/t))] \\
& \qquad = E[g(Y_0) - f(Y_0)] = g(E[Y(0)]) - E[f(Y_0)],
\end{aligned}$$

which implies

$$\limsup_{t\to\infty} E[f(Z(t)/t)] \leq E[f(Y_0)] \leq f(E[Y_0]) = f(E[Y(0)])$$

due to Jensen's inequality and (2.50). Summarizing, we have proved (2.56). □

Note that (2.55) holds if $E[f(Y(0))] < \infty$. The function $f(x) := x^k$, $x \in (0,\infty)$, is convex for $k \in \mathbb{R} \setminus (0,1)$ and concave for $k \in [0,1]$. Thus Theorem 2.2 and 2.3 provide results on the moments of $Z(\tau)$ in particular. However, for the moments of $Z(\tau)$ slightly stronger statements can be proved.

Corollary 2.1 *For $\tau_1, \tau_2 \in (0,\infty)$ it holds*

$$(E[Z^k(\tau_1+\tau_2)])^{1/k} \geq (E[Z^k(\tau_1)])^{1/k} + (E[Z^k(\tau_2)])^{1/k}, \qquad k \in (-\infty, 1] \setminus \{0\}, \tag{2.59}$$

$$(E[Z^k(\tau_1+\tau_2)])^{1/k} \le (E[Z^k(\tau_1)])^{1/k} + (E[Z^k(\tau_2)])^{1/k},$$
$$k \in [1,\infty), \tag{2.60}$$

i.e., $(E[Z^k(\tau)])^{1/k}$ is for fixed $k \in (-\infty,1]\setminus\{0\}$ a superadditive and for fixed $k \in [1,\infty)$ a subadditive function of $\tau \in (0,\infty)$.

Further, $(E[Z^k(\tau)])^{1/k}$ is for fixed $\tau \in (0,\infty)$ a non decreasing function of $k \in \mathbb{R}\setminus\{0\}$.

Proof We will give the proof of (2.60), the proof of (2.59) runs analogously. Assume that $E[Z^k(\tau_1)]$ and $E[Z^k(\tau_2)]$ are finite. As the function $f(x) := x^k$, $x \in (0,\infty)$, is convex for $k \in [1,\infty)$, for $\tau_1, \tau_2 \in (0,\infty)$ and $\xi \in (0,1)$ from (2.47) it follows

$$\begin{aligned}
E[Z^k(\tau_1+\tau_2)] &= E\Big[\Big(\int_0^{\tau_1+\tau_2} Y(t)\mathrm{d}t\Big)^k\Big] \\
&= E\Big[\Big(\xi\,\frac{1}{\xi}\int_0^{\tau_1} Y(t)\mathrm{d}t + (1-\xi)\,\frac{1}{1-\xi}\int_{\tau_1}^{\tau_1+\tau_2} Y(t)\mathrm{d}t\Big)^k\Big] \\
&\le \xi E\Big[\Big(\frac{1}{\xi}\int_0^{\tau_1} Y(t)\mathrm{d}t\Big)^k\Big] + (1-\xi)\,E\Big[\Big(\frac{1}{1-\xi}\int_{\tau_1}^{\tau_1+\tau_2} Y(t)\mathrm{d}t\Big)^k\Big] \\
&= \xi^{1-k}E[Z^k(\tau_1)] + (1-\xi)^{1-k}E[Z^k(\tau_2)],
\end{aligned}$$

where the last equality follows due to the stationarity of the process $Y(t)$, $t \in \mathbb{R}_+$. Choosing

$$\xi := (E[Z^k(\tau_1)])^{1/k}/((E[Z^k(\tau_1)])^{1/k}+(E[Z^k(\tau_2)])^{1/k}),$$

which minimizes the r.h.s., provides (2.60). Further, the monotonicity of $(E[Z^k(\tau)])^{1/k}$ with respect to k follows from Hölder's inequality. □

Corollary 2.2 *For $\tau \in (0,\infty)$ it holds*

$$\begin{aligned}
(E[Y(0)])^k &\le E[Y_0^k] \le \lim_{t\to\infty} E[(Z(t)/t)^k] \le E[(Z(\tau)/\tau)^k] \\
&\le \lim_{t\downarrow 0} E[(Z(t)/t)^k] = E[(Y(0))^k], \quad k \in \mathbb{R}\setminus(0,1),
\end{aligned} \tag{2.61}$$

$$\begin{aligned}
E[(Y(0))^k] &= \lim_{t\downarrow 0} E[(Z(t)/t)^k] \le E[(Z(\tau)/\tau)^k] \\
&\le \lim_{t\to\infty} E[(Z(t)/t)^k] = E[Y_0^k] \le (E[Y(0)])^k, \quad k \in [0,1].
\end{aligned} \tag{2.62}$$

For fixed $k \in \mathbb{R}$ it holds

$$\lim_{t\to\infty} E[(Z(t)/t)^k] = E[Y_0^k] \quad or \quad \lim_{t\to\infty} E[(Z(t)/t)^k] = \infty. \tag{2.63}$$

Proof Note that only (2.63) for $k \in \mathbb{R} \setminus (0,1)$ remains to be proved. If $E[(Z(\tau)/\tau)^k] = \infty$ for all $\tau \in (0,\infty)$, then $\lim_{t\to\infty} E[(Z(t)/t)^k] = \infty$. If $E[(Z(\tau)/\tau)^k] < \infty$ for some $\tau \in (0,\infty)$, then it holds $E[(Z(t)/t)^k] < \infty$ for $t \in (0,\tau]$ or $t \in [\tau,\infty)$ due to the monotonicity of $E[(Z(t))^k]$ with respect to $t \in (0,\infty)$, which implies $\lim_{t\to\infty} E[(Z(t)/t)^k] = E[Y_0^k]$ because of (2.55).

□

Apart from the moments, there are also other interesting applications.

Corollary 2.3 *For any $a \in \mathbb{R}_+$ it holds*

$$E[\min(a(x-Y(0)),1)] \le P(Z(\tau)/\tau \le x) \le E[\max(a(x-Y(0))+1,0)],$$

$$\tau \in (0,\infty),\ x \in (0,\infty). \tag{2.64}$$

Proof Let $a \in \mathbb{R}_+$ and $x \in (0,\infty)$ be fixed. As $f(\xi) := \max(a(x-\xi)+1,0)$, $\xi \in (0,\infty)$, is non negative and convex, from (2.53) it follows

$$\begin{aligned} P(Z(\tau)/\tau \le x) &= E[\mathbb{I}\{Z(\tau)/\tau \le x\}] \\ &\le E[\max(a(x-Z(\tau)/\tau)+1,0)] \le E[\max(a(x-Y(0))+1,0)]. \end{aligned}$$

As also $f(\xi) := \max(-a(x-\xi)+1,0)$, $\xi \in (0,\infty)$, is non negative and convex, from (2.53) we obtain

$$\begin{aligned} P(Z(\tau)/\tau \le x) &= 1 - E[\mathbb{I}\{Z(\tau)/\tau > x\}] \\ &\ge 1 - E[\max(-a(x-Z(\tau)/\tau)+1,0)] \\ &\ge 1 - E[\max(-a(x-Y(0))+1,0)] = E[\min(a(x-Y(0)),1)]. \end{aligned}$$

□

3 Sojourn times under random service speed

We consider an infinite-server system where the service speed at time t depends on the random state $N(t) \in \mathbb{Z}_+$ of the infinite-server system at time t and is given by $\varphi(N(t)) > 0$. With respect to the process $(N(t),\ t \in \mathbb{R})$ we assume that (A1) is fulfilled.

First we will prove general relations between the Laplace-Stieltjes transforms (LSTs) and moments of the sojourn time $V_v(\tau)$ of a virtual τ-request and of the sojourn time $\mathring{V}_s(\tau)$ of an arriving synchronized virtual τ-request.

Theorem 3.1 *Assume that (A1) is fulfilled. Then it holds*

$$E[e^{-s\mathring{V}_s(\tau)}] = 1 - \frac{s}{g}\int_0^\tau E[e^{-sV_v(t)}]\mathrm{d}t \tag{3.1}$$

$$= \frac{s}{g}\int_\tau^\infty E[e^{-sV_v(t)}]\mathrm{d}t, \quad s \in (0,\infty),\ \tau \in \mathbb{R}_+, \tag{3.2}$$

and for $k \in (-\infty, 0)$ *it holds*

$$E[\mathring{V}_s^k(\tau)] = \frac{-k}{g}\int_\tau^\infty E[V_v^{k-1}(t)]\mathrm{d}t, \quad \tau \in (0,\infty), \tag{3.3}$$

for $k \in (0,\infty)$ *it holds*

$$E[\mathring{V}_s^k(\tau)] = \frac{k}{g}\int_0^\tau E[V_v^{k-1}(t)]\mathrm{d}t, \quad \tau \in \mathbb{R}_+. \tag{3.4}$$

Proof Using Lemma 2.4, the abbreviation $Y(t) := 1/\varphi(\tilde{N}(t))$, $t \in \mathbb{R}_+$, Fubini's theorem, and the stationarity of the process $Y(t)$, $t \in \mathbb{R}_+$, we obtain

$$\begin{aligned}
1 - E[e^{-s\mathring{V}_s(\tau)}] &= E\Big[1 - e^{-s\int_0^\tau Y(u)\mathrm{d}u}\Big] \\
&= E\Big[\int_0^\tau sY(\tau - t)e^{-s\int_0^t Y(v+\tau-t)\mathrm{d}v}\mathrm{d}t\Big] \\
&= s\int_0^\tau E\Big[Y(\tau - t)e^{-s\int_0^t Y(v+\tau-t)\mathrm{d}v}\Big]\mathrm{d}t \\
&= s\int_0^\tau E\Big[Y(0)e^{-s\int_0^t Y(v)\mathrm{d}v}\Big]\mathrm{d}t \\
&= s\int_0^\tau \Big(\sum_{n\in\mathbb{Z}'_+}\frac{1}{\varphi(n)}\,E\Big[e^{-s\int_0^t 1/\varphi(\tilde{N}(v\,|\,n))\mathrm{d}v}\Big]\tilde{p}(n)\Big)\mathrm{d}t.
\end{aligned}$$

Therefore from Lemma 2.4, (2.39), and (2.14) we find

$$\begin{aligned}
1 - E[e^{-s\mathring{V}_s(\tau)}] &= s\int_0^\tau \Big(\sum_{n\in\mathbb{Z}'_+}\frac{1}{\varphi(n)}\,E[e^{-sV_v(t\,|\,n)}]\,\tilde{p}(n)\Big)\mathrm{d}t \\
&= \frac{s}{g}\int_0^\tau \Big(\sum_{n\in\mathbb{Z}'_+}E[e^{-sV_v(t\,|\,n)}]\,p(n)\Big)\mathrm{d}t = \frac{s}{g}\int_0^\tau E[e^{-sV_v(t)}]\mathrm{d}t,
\end{aligned}$$

which is equivalent to (3.1). Note that (3.2) follows from (3.1) and (2.19).

Let $k \in (-\infty, 0)$ and $\tau \in (0, \infty)$. Replacing s by $s + \xi$ in (3.2), integrating on $\xi \in \mathbb{R}_+$ with weight ξ^{-k-1}, applying Fubini's theorem, using the substitution $x = \xi \mathring{V}_s(\tau)$ and $x = \xi V_v(t)$, respectively, and taking into account that $\Gamma(-k+1) = (-k)\Gamma(-k)$ yield

$$E[\mathring{V}_s^k(\tau) e^{-s\mathring{V}_s(\tau)}] = \frac{-k}{g} \int_\tau^\infty E[V_v^{k-1}(t) e^{-sV_v(t)}] \mathrm{d}t$$

$$+ \frac{1}{g} \int_\tau^\infty E[sV_v^k(t) e^{-sV_v(t)}] \mathrm{d}t.$$

In view of $xe^{-x} < 1$, $x \in \mathbb{R}_+$, it holds $sV_v^k(t)e^{-sV_v(t)} \le V_v^{k-1}(t)$, and thus we obtain (3.3) by taking the limit $s \downarrow 0$ due to monotone convergence and dominated convergence of the second summand on the r.h.s. if the limit of the first summand is finite.

Let $k \in (0, \infty)$ and $\tau \in \mathbb{R}_+$. Analogously to the first part of the proof we find

$$\begin{aligned}
E[\mathring{V}_s^k(\tau)] &= E\Big[\Big(\int_0^\tau Y(u)\mathrm{d}u\Big)^k\Big] \\
&= E\Big[\int_0^\tau kY(\tau-t)\Big(\int_0^t Y(v+\tau-t)\mathrm{d}v\Big)^{k-1}\mathrm{d}t\Big] \\
&= k\int_0^\tau E\Big[Y(\tau-t)\Big(\int_0^t Y(v+\tau-t)\mathrm{d}v\Big)^{k-1}\Big]\mathrm{d}t \\
&= k\int_0^\tau E\Big[Y(0)\Big(\int_0^t Y(v)\mathrm{d}v\Big)^{k-1}\Big]\mathrm{d}t \\
&= k\int_0^\tau \Big(\sum_{n\in\mathbb{Z}'_+} \frac{1}{\varphi(n)}\, E\Big[\Big(\int_0^t \frac{1}{\varphi(\tilde{N}(v\,|\,n))}\,\mathrm{d}v\Big)^{k-1}\Big]\tilde{p}(n)\Big)\mathrm{d}t \\
&= k\int_0^\tau \Big(\sum_{n\in\mathbb{Z}'_+} \frac{1}{\varphi(n)}\, E[V_v^{k-1}(t\,|\,n)]\,\tilde{p}(n)\Big)\mathrm{d}t \\
&= \frac{k}{g}\int_0^\tau \Big(\sum_{n\in\mathbb{Z}'_+} E[V_v^{k-1}(t\,|\,n)]\,p(n)\Big)\mathrm{d}t = \frac{k}{g}\int_0^\tau E[V_v^{k-1}(t)]\mathrm{d}t.
\end{aligned}$$

Thus (3.4) is proved. □

Choosing $k = 1$ in (3.4) provides the following.

Corollary 3.1 *Assume that (A1) is fulfilled. Then it holds*

$$E\mathring{V}_s(\tau) = \frac{\tau}{g}, \quad \tau \in \mathbb{R}_+. \tag{3.5}$$

Remark 3.1 *Note that due to* (3.4)*, the variance of the sojourn time $\mathring{V}_s(\cdot)$ of an arriving synchronized virtual request is given by the expectation of the sojourn time $V_v(\cdot)$ of a virtual request.*

3.1 Sojourn time of a synchronized virtual request

In this section we will prove some results on the sojourn time $\mathring{V}_s(\tau)$ of an arriving synchronized virtual τ-request. We assume again that $N(t)$, $t \in \mathbb{R}_+$, satisfies (A1).

Theorem 3.2 *Assume that (A1) is fulfilled.*

Let $f(x)$, $x \in (0, \infty)$, non negative and convex. Then for $\tau_1, \tau_2 \in (0, \infty)$ it holds

$$\begin{aligned} &E[(\tau_1+\tau_2) f(\mathring{V}_s(\tau_1+\tau_2)/(\tau_1+\tau_2))] \\ &\qquad \le E[\tau_1 f(\mathring{V}_s(\tau_1)/\tau_1)] + E[\tau_2 f(\mathring{V}_s(\tau_2)/\tau_2)], \end{aligned} \tag{3.6}$$

i.e., $E[\tau f(\mathring{V}_s(\tau)/\tau)]$ is subadditive for $\tau \in (0, \infty)$.

Let $f(x)$, $x \in (0, \infty)$, non negative and concave. Then for $\tau_1, \tau_2 \in (0, \infty)$ it holds

$$\begin{aligned} &E[(\tau_1+\tau_2) f(\mathring{V}_s(\tau_1+\tau_2)/(\tau_1+\tau_2))] \\ &\qquad \ge E[\tau_1 f(\mathring{V}_s(\tau_1)/\tau_1)] + E[\tau_2 f(\mathring{V}_s(\tau_2)/\tau_2)], \end{aligned} \tag{3.7}$$

i.e., $E[\tau f(\mathring{V}_s(\tau)/\tau)]$ is superadditive for $\tau \in (0, \infty)$.

Proof Note that $Y(t) := 1/\varphi(\tilde{N}(t))$, $t \in \mathbb{R}_+$, is a stationary process with values in $(0, \infty)$. Moreover, it holds $EY(0) < \infty$ due to (2.41), (2.3). In view of (2.45) and (2.47), thus we obtain Theorem 3.2 from Theorem 2.2. □

Theorem 3.3 *Assume that (A1) is fulfilled.*

Let $f(x)$, $x \in (0, \infty)$, non negative and convex. Then for $\tau \in (0, \infty)$ it holds

$$\begin{aligned} f(1/g) &\le \liminf_{t\to\infty} E[f(\mathring{V}_s(t)/t)] \le E[f(\mathring{V}_s(\tau)/\tau)] \\ &\le \lim_{t\downarrow 0} E[f(\mathring{V}_s(t)/t)] = \frac{1}{g}\sum_{n=0}^{\infty} f(1/\varphi(n))\varphi(n)p(n). \end{aligned} \tag{3.8}$$

If additionally $E[f(\mathring{V}_s(\tau)/\tau)] < \infty$ *for some* $\tau \in (0,\infty)$*, then*

$$f(1/g) = \liminf_{t\to\infty} E[f(\mathring{V}_s(t)/t)], \tag{3.9}$$

and if $E[f(\mathring{V}_s(t)/t)]$ *is bounded in a neighborhood of some* $\tau \in (0,\infty)$*, then*

$$f(1/g) = \lim_{t\to\infty} E[f(\mathring{V}_s(t)/t)]. \tag{3.10}$$

Let $f(x)$*,* $x \in (0,\infty)$*, non negative and concave. Then for* $\tau \in (0,\infty)$ *it holds*

$$\begin{aligned} \frac{1}{g}\sum_{n=0}^{\infty} f(1/\varphi(n))\varphi(n)p(n) &= \lim_{t\downarrow 0} E[f(\mathring{V}_s(t)/t)] \\ &\le E[f(\mathring{V}_s(\tau)/\tau)] \le \lim_{t\to\infty} E[f(\mathring{V}_s(t)/t)] = f(1/g). \end{aligned} \tag{3.11}$$

Proof Remember that $Y(t) := 1/\varphi(\tilde{N}(t))$, $t \in \mathbb{R}_+$, is a stationary process with values in $(0,\infty)$ and that $EY(0) < \infty$. In view of (2.45), (2.47), (2.49), (2.24), (2.39), and (2.14), thus we obtain Theorem 3.3 from Theorem 2.3. □

Note that choosing $f(x) := x^k$, $x \in (0,\infty)$, for fixed $k \in \mathbb{R}$ in Theorem 3.2 and 3.3 provides results for the kth moment of $\mathring{V}_s(\tau)$. However, Corollary 2.1 and 2.2 yield slightly stronger results.

Corollary 3.2 *Assume that (A1) is fulfilled. For* $\tau_1, \tau_2 \in (0,\infty)$ *it holds*

$$(E[\mathring{V}_s^k(\tau_1+\tau_2)])^{1/k} \ge (E[\mathring{V}_s^k(\tau_1)])^{1/k} + (E[\mathring{V}_s^k(\tau_2)])^{1/k}, \quad k \in (-\infty,1]\setminus\{0\}, \tag{3.12}$$

$$(E[\mathring{V}_s^k(\tau_1+\tau_2)])^{1/k} \le (E[\mathring{V}_s^k(\tau_1)])^{1/k} + (E[\mathring{V}_s^k(\tau_2)])^{1/k}, \quad k \in [1,\infty), \tag{3.13}$$

i.e., $(E[\mathring{V}_s^k(\tau)])^{1/k}$ *is for fixed* $k \in (-\infty,1]\setminus\{0\}$ *a superadditive and for fixed* $k \in [1,\infty)$ *a subadditive function of* $\tau \in (0,\infty)$*.*

Further, $(E[\mathring{V}_s^k(\tau)])^{1/k}$ *is for fixed* $\tau \in (0,\infty)$ *a non decreasing function of* $k \in \mathbb{R}\setminus\{0\}$*.*

Proof Remember that $Y(t) := 1/\varphi(\tilde{N}(t))$, $t \in \mathbb{R}_+$, is a stationary process with values in $(0,\infty)$ and that $EY(0) < \infty$. In view of (2.45) and (2.47), thus we obtain Corollary 3.2 from Corollary 2.1. □

Corollary 3.3 *Assume that (A1) is fulfilled. For $\tau \in (0,\infty)$ it holds*

$$g^{-k} \le \lim_{t\to\infty} E[(\mathring{V}_s(t)/t)^k\,] \le E[(\mathring{V}_s(\tau)/\tau)^k\,]$$

$$\le \lim_{t\downarrow 0} E[(\mathring{V}_s(t)/t)^k\,] = \frac{1}{g}\sum_{n=0}^{\infty} \varphi^{1-k}(n)p(n), \quad k \in \mathbb{R}\setminus(0,1), \tag{3.14}$$

$$\frac{1}{g}\sum_{n=0}^{\infty} \varphi^{1-k}(n)p(n) = \lim_{t\downarrow 0} E[(\mathring{V}_s(t)/t)^k\,] \le E[(\mathring{V}_s(\tau)/\tau)^k\,]$$

$$\le \lim_{t\to\infty} E[(\mathring{V}_s(t)/t)^k\,] = g^{-k}, \quad k \in [0,1]. \tag{3.15}$$

For fixed $k \in \mathbb{R}$ it holds

$$\lim_{t\to\infty} E[(\mathring{V}_s(t)/t)^k\,] = g^{-k} \quad \textit{or} \quad \lim_{t\to\infty} E[(\mathring{V}_s(t)/t)^k\,] = \infty. \tag{3.16}$$

Proof Remember that $Y(t) := 1/\varphi(\tilde{N}(t))$, $t \in \mathbb{R}_+$, is a stationary process with values in $(0,\infty)$ and that $EY(0) < \infty$. In view of (2.45), (2.47), (2.49), (2.24), (2.39), and (2.14), thus we obtain Corollary 3.3 from Corollary 2.2. □

In view of (2.45), (2.47), (2.39), and (2.14), from Corollary 2.3 we obtain the following estimate for the distribution of $\mathring{V}_s(\tau)$.

Corollary 3.4 *Assume that (A1) is fulfilled. Then for $a \in \mathbb{R}_+$ it holds*

$$\frac{1}{g}\sum_{n=0}^{\infty} \min(a(x\varphi(n)-1), \varphi(n))p(n) \le P(\mathring{V}_s(\tau)/\tau \le x)$$

$$\le \frac{1}{g}\sum_{n=0}^{\infty} \max(a(x\varphi(n)-1)+\varphi(n), 0)p(n),$$

$$\tau \in (0,\infty),\ x \in (0,\infty). \tag{3.17}$$

3.2 Service received by a virtual request

We assume again that $N(t)$, $t \in \mathbb{R}_+$, satisfies (A1). The service $U(t)$ received by a virtual request during the time interval $[0,t]$ is given by (2.15). Choosing $Y(t) := \varphi(N(t))$, $t \in \mathbb{R}_+$, from (2.15), (2.47), (2.48), and (2.3) we find

$$EU(t) = tE[\varphi(N(0))] = gt, \quad t \in \mathbb{R}_+. \tag{3.18}$$

Theorem 2.2 and 2.3 provide the following results.

Theorem 3.4 *Assume that (A1) is fulfilled.*

Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $t_1, t_2 \in (0,\infty)$ it holds

$$E[(t_1{+}t_2)f(U(t_1{+}t_2)/(t{+}t_2))] \\ \leq E[t_1 f(U(t_1)/t_1)] + E[t_2 f(U(t_2)/t_2)], \tag{3.19}$$

i.e., $E[tf(U(t)/t)]$ is subadditive for $t \in (0,\infty)$.

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $t_1, t_2 \in (0,\infty)$ it holds

$$E[(t_1{+}t_2)f(U(t_1{+}t_2)/(t_1{+}t_2))] \\ \geq E[t_1 f(U(t_1)/t_1)] + E[t_2 f(U(t_2)/t_2)], \tag{3.20}$$

i.e., $E[tf(U(t)/t)]$ is superadditive for $t \in (0,\infty)$.

Proof Note that $Y(t) := \varphi(N(t))$, $t \in \mathbb{R}_+$, is a stationary process with values in $(0,\infty)$, and that it holds $EY(0) < \infty$ due to (2.3). In view of (2.15) and (2.47), thus we obtain Theorem 3.4 from Theorem 2.2. $\square$

Theorem 3.5 *Assume that (A1) is fulfilled.*

Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $t \in (0,\infty)$ it holds

$$f(g) \leq \liminf_{\tau\to\infty} E[f(U(\tau)/\tau)] \leq E[f(U(t)/t)] \\ \leq \lim_{\tau\downarrow 0} E[f(U(\tau)/\tau)] = \sum_{n=0}^{\infty} f(\varphi(n))p(n). \tag{3.21}$$

If additionally $E[f(U(t)/t)] < \infty$ for some $t \in (0,\infty)$, then

$$f(g) = \liminf_{\tau\to\infty} E[f(U(\tau)/\tau)], \tag{3.22}$$

and if $E[f(U(\tau)/\tau)]$ is bounded in a neighborhood of some $t \in (0,\infty)$, then

$$f(g) = \lim_{\tau\to\infty} E[f(U(\tau)/\tau)]. \tag{3.23}$$

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $t \in (0,\infty)$ it holds

$$\sum_{n=0}^{\infty} f(\varphi(n))p(n) = \lim_{\tau\downarrow 0} E[f(U(\tau)/\tau)] \\ \leq E[f(U(t)/t)] \leq \lim_{\tau\to\infty} E[f(U(\tau)/\tau)] = f(g). \tag{3.24}$$

Proof Remember that $Y(t) := \varphi(N(t))$, $t \in \mathbb{R}_+$, is a stationary process with values in $(0,\infty)$ and that $EY(0) < \infty$. In view of (2.15), (2.47), (2.49), and (2.21), thus we obtain Theorem 3.5 from Theorem 2.3. $\square$

Note that choosing $f(x) := x^k$, $x \in (0,\infty)$, for fixed $k \in \mathbb{R}$ in Theorem 3.4 and 3.5 provides results for the kth moment of $U(t)$. However, Corollary 2.1 and 2.2 yield slightly stronger results.

In view of (2.15) and (2.47), from Corollary 2.3 we obtain the following estimate for the distribution of $U(t)$.

Corollary 3.5 *Assume that (A1) is fulfilled. Then for $a \in \mathbb{R}_+$ it holds*

$$\sum_{n=0}^{\infty} \min(a(x-\varphi(n)),1)p(n) \le P(U(t)/t \le x)$$
$$\le \sum_{n=0}^{\infty} \max(a(x-\varphi(n))+1,0)p(n), \quad t \in (0,\infty),\ x \in (0,\infty). \tag{3.25}$$

3.3 Sojourn time of a virtual request

In this section we will prove some results on the sojourn time $V_v(\tau)$ of a virtual τ-request. Note that $V_v(\tau)$ is not given by an additive functional. We assume again that $N(t)$, $t \in \mathbb{R}_+$, satisfies (A1).

Theorem 3.6 *Assume that (A1) is fulfilled. Then for $k \in \mathbb{R}_+$, $\tau \in (0,\infty)$ it holds*

$$\frac{g^{-k}}{k+1} \le E[(V_v(\tau)/\tau)^k] \le \lim_{t \downarrow 0} E[(V_v(t)/t)^k] = \sum_{n=0}^{\infty} \varphi^{-k}(n)p(n). \tag{3.26}$$

Further, for any $a \in \mathbb{R}_+$ it holds

$$\sum_{n=0}^{\infty} \min(a(x\varphi(n)-1),1)p(n) \le P(V_v(\tau)/\tau \le x)$$
$$\le \sum_{n=0}^{\infty} \max(a(x\varphi(n)-1)+1,0)p(n),$$
$$\tau \in (0,\infty),\ x \in (0,\infty). \tag{3.27}$$

Proof (i) Obviously, (3.26) holds for $k=0$. Thus, let $k\in(0,\infty)$ be fixed. If $E[V_v^k(\tau)]$ is finite for some $\tau\in(0,\infty)$, then, in view of the monotonicity of $V_v(t)$, (3.4), and (3.14), we obtain

$$\begin{aligned}E[(V_v(\tau)/\tau)^k\,] &\ge \frac{1}{\tau^{k+1}}\int_0^\tau E[V_v^k(t)]\mathrm{d}t\\ &= \frac{g}{k+1}\,E[(\mathring{V}_s(\tau)/\tau)^{k+1}] \ge \frac{g^{-k}}{k+1}\,,\end{aligned}$$

which is the bound on the l.h.s. of (3.26). Using $Y(t):=1/\varphi(\tilde{N}(t))$, $t\in\mathbb{R}_+$, for $f(x):=x^k$ from (2.46) it follows that

$$E[(V_v(\tau)/\tau)^k\,] = gE\Big[\Big(\frac{1}{\tau}\int_0^\tau Y(u)\,\mathrm{d}u\Big)^k Y(0)\Big],\quad \tau\in(0,\infty). \tag{3.28}$$

In view of Hölder's inequality, (2.45), and (2.39), from (3.28) we obtain

$$\begin{aligned}&E[(V_v(\tau)/\tau)^k\,]\\ &\quad\le gE\Big[\Big(\frac{1}{\tau}\int_0^\tau Y(u)\,\mathrm{d}u\Big)^{k+1}\Big]^{k/(k+1)} E[Y^{k+1}(0)]^{1/(k+1)}\\ &\quad= \Big(gE[(\mathring{V}_s(\tau)/\tau)^{k+1}]\Big)^{k/(k+1)}\Big(g\sum_{n=0}^\infty \varphi^{-(k+1)}(n)\mathring{p}_s(n)\Big)^{1/(k+1)}.\end{aligned}$$

Thus (3.14) and (2.14) provide

$$E[(V_v(\tau)/\tau)^k\,] \le \sum_{n=0}^\infty \varphi^{-k}(n)p(n), \tag{3.29}$$

which is the bound on the r.h.s. of (3.26). From (3.28), Fatou's lemma, (2.39), and (2.14) we find

$$\begin{aligned}\liminf_{t\downarrow 0} E[(V_v(t)/t)^k\,] &\ge gE\Big[\liminf_{t\downarrow 0}\Big(\frac{1}{t}\int_0^t Y(u)\,\mathrm{d}u\Big)^k Y(0)\Big]\\ &= gE[Y^{k+1}(0)] = g\sum_{n=0}^\infty \varphi^{-(k+1)}(n)\mathring{p}_s(n) = \sum_{n=0}^\infty \varphi^{-k}(n)p(n).\end{aligned}$$

On the other hand, from (3.29) it follows that

$$\limsup_{t\downarrow 0} E[(V_v(t)/t)^k\,] \le \sum_{n=0}^\infty \varphi^{-k}(n)p(n),$$

and thus it holds

$$\lim_{t\downarrow 0} E[(V_v(t)/t)^k\,] = \sum_{n=0}^{\infty} \varphi^{-k}(n)p(n).$$

(ii) As $V_v(\cdot)$ is a.s. the inverse function of $U(\cdot)$, for $a \in \mathbb{R}_+$, $t \in (0,\infty)$, $x \in (0,\infty)$ from (3.25) we find

$$\sum_{n=0}^{\infty} \min(a(x-\varphi(n)),1)p(n) \le P(t \le V_v(xt))$$
$$\le \sum_{n=0}^{\infty} \max(a(x-\varphi(n))+1,0)p(n).$$

Choosing $t := \tau/x$ and replacing then x by $1/x$ and then a by ax provides

$$\sum_{n=0}^{\infty} \min(a(1-x\varphi(n)),1)p(n) \le P(V_v(\tau)/\tau \ge x)$$
$$\le \sum_{n=0}^{\infty} \max(a(1-x\varphi(n))+1,0)p(n), \quad \tau \in (0,\infty),\ x \in (0,\infty),$$

which is equivalent to (3.27) as the bounds are continuous with respect to $x \in (0,\infty)$ due to (2.3). $\square$

Remark 3.2 *Note that for fixed $k \in \mathbb{R}_+$, the r.h.s. of* (3.26) *is the asymptotically tight upper bound for $E[(V_v(\tau)/\tau)^k\,]$.*

4 Sojourn times in $GI(n)/GI(n)/\infty$

As an application, in this section we deal with the $GI(n)/GI(n)/\infty$ system. At an infinite-server system requests arrive, where the inter-arrival times are i.i.d. and have a general distribution function $A(x)$ with finite positive mean m_A, but the speed of the clock governing the arrival process depends on the number of requests in the system and runs with speed $\psi(n) \ge 0$ if there are $n \in \mathbb{Z}_+$ requests in the system, where we assume that $\psi(0) > 0$ and $\psi(m) = 0$ implies $\psi(m+1) = 0$, $m \in \mathbb{N}$. The required service times are i.i.d., sampled independently of the arrival process, and have a general distribution function $B(x)$ with finite positive mean m_B, but the speed of the clock governing the service process depends also on the number of requests in the system and runs with speed $\varphi(n) > 0$ if there are $n \in \mathbb{Z}_+$ requests in the system. Within this paper, $A(0) > 0$ and $B(0) > 0$ are allowed, corresponding to batch arrivals and zero service times, respectively.

Remark 4.1 *Note that the dynamics of the $GI(n)/GI(n)/\infty$ system correspond to the dynamics of a $GI(n)/GI$ system under state-dependent processor sharing, i.e. of the $GI(n)/GI/SDPS$ system, where each request in the system receives a service capacity $\varphi(n)$ if there are n requests in the system. Thus the results of this section are also results for systems under state-dependent processor sharing.*

We may interpret the $GI(n)/GI(n)/\infty$ system as an infinite-server system in random environment, where the state $N(t)$ of the infinite-server system is given by the number of requests in $GI(n)/GI(n)/\infty$ at time t. We assume that the process $(N(t),\ t \in \mathbb{R})$, where $N(t)$ is the number of requests in the $GI(n)/GI(n)/\infty$ system at time t, satisfies (A1), being an assumption for the $GI(n)/GI(n)/\infty$ system.

Corresponding to the interpretation of the $GI(n)/GI(n)/\infty$ system as an infinite-server system in random environment, we obtain immediately results for the sojourn time of a virtual request, the sojourn time of a synchronized virtual request, and the service received by a virtual request for the $GI(n)/GI(n)/\infty$ system by applying the results of Section 3, where $N(t)$ is the number of requests in the $GI(n)/GI(n)/\infty$ system at time t and $p(n) = P(N(0) = n)$, $n \in \mathbb{Z}_+$, cf. (2.1), is the distribution of the number of requests in the $GI(n)/GI(n)/\infty$ system.

4.1 Sojourn times in $M(n)/GI(n)/\infty$

For giving results also for other sojourn times we restrict ourselves to the special case of an $M(n)/GI(n)/\infty$ system or equivalently an $M(n)/GI/SDPS$ system, cf. Remark 4.1, in the following, i.e., we assume that $A(x)$ is an exponential distribution. At the $M(n)/GI(n)/\infty$ system the requests arrive according to a Poisson process with the state-dependent intensity

$$\lambda(n) := \lambda\psi(n), \quad n \in \mathbb{Z}_+, \tag{4.1}$$

where $\lambda > 0$ is the parameter of the exponential distribution $A(x)$. Note that $\psi(n) := \mathbb{I}\{n < m\}$, $n \in \mathbb{Z}_+$, models the $M/GI(n)/m/0$ loss system. In case of an $M(n)/GI(n)/\infty$ system, we consider additionally the following sojourn times.

1. *Arrival stationary sojourn time*: Consider the $M(n)/GI(n)/\infty$ system in steady state, and assume that

$$E[\psi(N(0))] < \infty. \tag{4.2}$$

Note that the effective arrival rate $\mathring{\lambda}$ is given by

$$\mathring{\lambda} = \sum_{n\in\mathbb{Z}_+} \lambda(n)p(n) = \lambda E[\psi(N(0))]. \tag{4.3}$$

Let $\mathring{V}(\tau)$ be the sojourn time of an arbitrary arriving request with required service time τ. For $n \in \mathbb{Z}''_+ := \{m \in \mathbb{Z}_+ : \lambda(m)p(m) > 0\}$ we denote by $\mathring{V}(\tau\,|\,n)$ the sojourn time of an arriving request conditioned that its service time is τ and that additionally there are n requests in the system immediately before its arrival. Because of (4.2), it holds

$$P(\mathring{V}(\tau) \leq x) = \sum_{n\in\mathbb{Z}''_+} P(\mathring{V}(\tau\,|\,n) \leq x)\mathring{p}(n), \tag{4.4}$$

where the probability $\mathring{p}(n)$ that an arriving request finds n requests in the system is given by

$$\mathring{p}(n) = \frac{1}{\mathring{\lambda}}\,\lambda(n)p(n), \quad n \in \mathbb{Z}_+. \tag{4.5}$$

Remark 4.2 *For general $\tau \in \mathbb{R}_+$, in particular if τ is not in* $\mathrm{supp}(B)$*, one can proceed similarly to [Coh] Theorem 5.4 and its proof by changing the distribution function $B(x)$ such that τ belongs to the support and then taking the limit in distribution and applying arguments of continuity.*

2. *Sojourn time of an added request*: Consider the $M(n)/GI(n)/\infty$ system in steady state, i.e., at $t = 0$ it has the stationary state distribution, and let us add a request at $t = 0$. Concerning the added request we assume that it is counted in the number of requests governing the service process, but it is not counted in the number of requests governing the arrival process. If there are $n \in \mathbb{Z}_+$ original requests and the added request in the node then the clocks governing the arrival and service process run with speed $\psi(n)$ and $\varphi(n+1)$, respectively. Let $V(\tau)$ be the sojourn time of an added τ-request at $t = 0$, and for $n \in \mathbb{Z}'_+$ let $V(\tau\,|\,n)$ be the sojourn time of an added τ-request at $t = 0$ conditioned that there are n original requests in the system at $t = 0$. Obviously, it holds

$$P(V(\tau)\leq x) = \sum_{n\in\mathbb{Z}'_+} P(V(\tau\,|\,n)\leq x)p(n). \tag{4.6}$$

Remark 4.3 *Note that the sojourn time of a virtual request considered in this section in connection with the interpretation of the $M(n)/GI/SDPS$ system as an infinite-server system in random environment is different from the sojourn time of virtual requests used in [HHM], [HM], and [YY].*

In [HHM], [HM] the sojourn time of a virtual request is considered in the framework of a single-server egalitarian processor sharing system. The service rate of the virtual request, which does not interact with the system, too, is $\varphi(n) = 1/(n+1)$ if n real requests are in the single-server system; the sojourn time of the virtual request is defined via (2.5) *with $N(t)$ being the process of the number of real requests in the system, i.e., the virtual request is counted with respect to the sharing of the single-server resource, but the real requests receive the larger service capacity $1/n$.*

In [YY] the sojourn time of a virtual τ-request in a single-server egalitarian processor sharing system with permanent requests corresponds to the sojourn time of an added τ-request in our context, cf. [YY] p. 1669.

It turns out that our concept of a virtual request is more general and provides new relations for sojourn time characteristics even in these special cases.

For the $M(n)/GI(n)/\infty$ system or equivalently the $M(n)/GI/SDPS$ system some basic results are known, which we will use and therefore shortly review in the following. Let $R(t) := (R_1(t), \ldots, R_{N(t)}(t))$ the vector of the randomly ordered residual service times $R_1(t), \ldots, R_{N(t)}(t)$ of the $N(t)$ requests in the $M(n)/GI(n)/\infty$ system at time t. Note that the vector process $X(t) := (N(t); R(t))$, $t \in \mathbb{R}$, is a Markov process. There exists a unique stationary and ergodic process $X(t)$, $t \in \mathbb{R}$, where the process $N(t)$, $t \in \mathbb{R}$, satisfies (A1), if

$$\sum_{n=0}^{\infty} \theta(n) < \infty, \quad \sum_{n=0}^{\infty} \lambda(n)\theta(n) < \infty, \tag{4.7}$$

where

$$\theta(n) := \prod_{\ell=1}^{n} \frac{\lambda(\ell-1)m_B}{\ell\varphi(\ell)}, \quad n \in \mathbb{Z}_+, \tag{4.8}$$

cf. [Zac] Theorem 1. We assume in the following that (4.7) is fulfilled and that $X(t)$, $t \in \mathbb{R}$, is the stationary and ergodic Markov process. Then the stationary occupancy distribution $p(n) := P(N(t) = n)$, $n \in \mathbb{Z}_+$, and the stationary distribution

$$P(n; r_1, \ldots, r_n) := P(N(t)=n; R_1(t)\le r_1, \ldots, R_n(t)\le r_n),$$
$$n \in \mathbb{Z}_+,\ r_1, \ldots, r_n \in \mathbb{R}_+,$$

of $X(t)$ on $\{N(t)=n\}$ are given by

$$p(n) = \Big(\sum_{m=0}^{\infty}\theta(m)\Big)^{-1}\theta(n) = p(0)\theta(n), \tag{4.9}$$

$$P(n; r_1, \dots, r_n) = p(n)\prod_{\ell=1}^{n} B_R(r_\ell), \tag{4.10}$$

respectively, where

$$B_R(x) := \frac{1}{m_B}\int_0^x (1-B(u))\mathrm{d}u, \quad x \in \mathbb{R}_+, \tag{4.11}$$

denotes the stationary residual service time distribution, cf. [Zac] Theorem 1. Note that (4.1), (4.7), and (4.9) imply (4.2).

We obtain immediately results for the sojourn time of a virtual request, the sojourn time of a synchronized virtual request, and the service received by a virtual request in case of the $M(n)/GI(n)/\infty$ system by using the results of Section 3. In the theorems and corollaries of Section 3 we only have to replace assumption (A1) by (4.7). Note that $p(n)$, $n \in \mathbb{Z}_+$, is explicitly given by (4.9), (4.8) and that it is insensitive with respect to the service time distribution $B(x)$ given its mean m_B.

For deriving results for the sojourn time $\mathring{V}(\tau)$ of an arriving request we consider a modified system. Let

$$\psi_+(n) := \psi(n+1), \quad \varphi_+(n) := \varphi(n+1), \quad n \in \mathbb{Z}_+. \tag{4.12}$$

By replacing $\psi(\cdot)$ by $\psi_+(\cdot)$ and $\varphi(\cdot)$ by $\varphi_+(\cdot)$ in the $M(n)/GI(n)/\infty$ system we obtain the modified system $M_+(n)/GI_+(n)/\infty$. Note that this system may be considered as the original system with one permanent request. The corresponding quantities related to this system will be indexed by $_{++}$. From (4.1), (4.8), and (4.12) we find

$$\theta_{++}(n) = \frac{\lambda(n)}{\lambda(0)}\,\frac{\varphi(1)}{\varphi(n+1)}\,\theta(n) = \frac{\varphi(1)}{\lambda(0)m_B}\,(n+1)\theta(n+1). \tag{4.13}$$

In view of (4.1), (4.7), (4.12), and (4.13), we make here the following assumption:

$$\text{(A2)} \qquad \sum_{n=0}^{\infty} n\theta(n) < \infty, \quad \sum_{n=0}^{\infty} n\lambda(n)\theta(n) < \infty. \tag{4.14}$$

Note that (A2) implies (A1) for the $M(n)/GI(n)/\infty$ system and for the $M_+(n)/GI_+(n)/\infty$ system. Because of (4.9) and (4.13), the stationary occupancy distribution $p_{++}(n) := P(N_{++}(t) = n)$, $n \in \mathbb{Z}_+$, is given by

$$p_{++}(n) = \frac{1}{EN(0)}\,(n+1)p(n+1), \quad n \in \mathbb{Z}_+. \tag{4.15}$$

From (2.3), (4.12), and (4.15) it follows

$$g_{++} = E[\varphi_+(N_{++}(0))] = \frac{E[N(0)\varphi(N(0))]}{EN(0)}\,. \tag{4.16}$$

Theorem 4.1 *Assume that (A2) is fulfilled. Then*

$$\mathring{V}(\tau\,|\,n) \stackrel{\mathcal{D}}{=} V_{v,++}(\tau\,|\,n), \quad \tau \in \mathbb{R}_+,\ n \in \mathbb{Z}''_+, \tag{4.17}$$

$$\mathring{V}(\tau) \stackrel{\mathcal{D}}{=} \mathring{V}_{s,++}(\tau), \quad \tau \in \mathbb{R}_+. \tag{4.18}$$

Proof Let the $M(n)/GI(n)/\infty$ system and the $M_+(n)/GI_+(n)/\infty$ system in steady state. Note that $\lambda(n)p(n) > 0$ if and only if $p_{++}(n) > 0$. The product form solution (4.10) yields that conditioned that there are n requests at $t = 0$ in the $M(n)/GI(n)/\infty$ system the n residual service times $R_i(0)$, $i = 1, \dots, n$, are stochastically independent and have the distribution function $B_R(\cdot)$. Analogously, it follows that conditioned that there are n requests at $t = 0$ in the $M_+(n)/GI_+(n)/\infty$ system the n residual service times $R_{i,++}(0)$, $i = 1, \dots, n$, are stochastically independent and have the distribution function $B_R(\cdot)$. As an arriving request at the $M(n)/GI(n)/\infty$ system finding n requests sees the time-stationary distribution of the n residual service times due to conditional PASTA, the definitions of $\mathring{V}(\tau\,|\,n)$ and $V_{v,++}(\tau\,|\,n)$ as well as the dynamics of both systems thus imply (4.17). Further, from (4.4), (4.17), (4.5), (4.8), (4.9), (4.15), (4.12), (4.16), (2.14), and (2.13) we find

$$\begin{aligned}
P(\mathring{V}(\tau) \le x) &= \sum_{n \in \mathbb{Z}''_+} P(\mathring{V}(\tau\,|\,n) \le x)\mathring{p}(n) \\
&= \sum_{n \in \mathbb{Z}''_+} P(V_{v,++}(\tau\,|\,n) \le x)\,\frac{\lambda(n)p(n)}{\sum_{m=0}^{\infty}\lambda(m)p(m)} \\
&= \sum_{n \in \mathbb{Z}''_+} P(V_{v,++}(\tau\,|\,n) \le x)\,\frac{\varphi(n+1)(n+1)p(n+1)}{\sum_{m=0}^{\infty}\varphi(m+1)(m+1)p(m+1)}
\end{aligned}$$

$$= \sum_{n \in \mathbb{Z}''_+} P(V_{v,++}(\tau \,|\, n) \le x) \frac{\varphi_+(n) p_{++}(n)}{\sum_{m=0}^{\infty} \varphi_+(m) p_{++}(m)}$$

$$= \sum_{n \in \mathbb{Z}''_+} P(V_{v,++}(\tau \,|\, n) \le x) \, \mathring{p}_{s,++}(n) = P(\mathring{V}_{s,++}(\tau) \le x),$$

which yields (4.18). □

In view of Theorem 4.1, we obtain results for the sojourn time of an arbitrary arriving request in $M(n)/GI(n)/\infty$ by using the results for the sojourn time of a synchronized virtual request in an infinite-server system in random environment, where the state of the infinite-server system at time t is given by $N_{++}(t)$ corresponding to the interpretation of the $M_+(n)/GI_+(n)/\infty$ system as an infinite-server system in random environment. As (A2) implies (A1) for this infinite-server system, we obtain immediately results for the sojourn time of an arbitrary arriving request by using the results of Section 3 for the sojourn time of a synchronized virtual request. Note that (A1) has to be replaced by (A2), $\mathring{V}_s(\cdot)$ by $\mathring{V}(\cdot)$, $\varphi(\cdot)$ by $\varphi_+(\cdot)$, $p(\cdot)$ by $p_{++}(\cdot)$, and g by g_{++}.

In view of (4.18) and (4.16), Corollary 3.1 provides the following.

Corollary 4.1 *Assume that (A2) is fulfilled. Then for the mean sojourn time of an arriving request in $M(n)/GI(n)/\infty$ ($M(n)/GI/SDPS$) it holds*

$$E\mathring{V}(\tau) = \frac{EN(0)}{E[N(0)\varphi(N(0))]} \, \tau, \quad \tau \in \mathbb{R}_+. \tag{4.19}$$

Remark 4.4 *(i) Note that* (4.19) *is a generalization of Cohen's famous proportional result for the mean sojourn time in $M/GI/SDPS$, cf. [Coh], to $M(n)/GI/SDPS$. From a higher point of view,* (3.5) *may also be considered as a generalization of Cohen's result.*

(ii) Note that $E[N(0)\varphi(N(0))] = \mathring{\lambda} m_B$ because of (4.3), (4.8), *and* (4.9). *Thus Little's law for $M(n)/GI(n)/\infty$ is recovered by integration.*

(iii) Assume that (A2) is fulfilled. Due to Theorem 4.1 and Theorem 3.1, the LST and the moments of $\mathring{V}(\cdot)$ are given by the LST and the moments of $V_{v,++}(\cdot)$, respectively. In particular, the variance of $\mathring{V}(\cdot)$ is given by the expectation of $V_{v,++}(\cdot)$.

From Theorem 3.2 we obtain immediately the following.

Theorem 4.2 *Assume that (A2) is fulfilled.*

Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $\tau_1, \tau_2 \in (0,\infty)$ it holds

$$E[(\tau_1+\tau_2) f(\mathring{V}(\tau_1+\tau_2)/(\tau_1+\tau_2))]$$
$$\leq E[\tau_1 f(\mathring{V}(\tau_1)/\tau_1)] + E[\tau_2 f(\mathring{V}(\tau_2)/\tau_2)], \tag{4.20}$$

i.e., $E[\tau f(\mathring{V}(\tau)/\tau)]$ is subadditive for $\tau \in (0,\infty)$.

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $\tau_1, \tau_2 \in (0,\infty)$ it holds

$$E[(\tau_1+\tau_2) f(\mathring{V}(\tau_1+\tau_2)/(\tau_1+\tau_2))]$$
$$\geq E[\tau_1 f(\mathring{V}(\tau_1)/\tau_1)] + E[\tau_2 f(\mathring{V}(\tau_2)/\tau_2)], \tag{4.21}$$

i.e., $E[\tau f(\mathring{V}(\tau)/\tau)]$ is superadditive for $\tau \in (0,\infty)$.

Because of (4.18), (4.16), (4.12), and (4.15), from Theorem 3.3 we obtain the following.

Theorem 4.3 *Assume that (A2) is fulfilled.*

Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $\tau \in (0,\infty)$ it holds

$$f\Big(\frac{EN(0)}{E[N(0)\varphi(N(0))]}\Big) \leq \liminf_{t\to\infty} E[f(\mathring{V}(t)/t)] \leq E[f(\mathring{V}(\tau)/\tau)]$$
$$\leq \lim_{t\downarrow 0} E[f(\mathring{V}(t)/t)] = \frac{E[N(0)\varphi(N(0))f(1/\varphi(N(0)))]}{E[N(0)\varphi(N(0))]}\,. \tag{4.22}$$

If additionally $E[f(\mathring{V}(\tau)/\tau)] < \infty$ for some $\tau \in (0,\infty)$, then

$$f\Big(\frac{EN(0)}{E[N(0)\varphi(N(0))]}\Big) = \liminf_{t\to\infty} E[f(\mathring{V}(t)/t)], \tag{4.23}$$

and if $E[f(\mathring{V}(t)/t)]$ is bounded in a neighborhood of some $\tau \in (0,\infty)$, then

$$f\Big(\frac{EN(0)}{E[N(0)\varphi(N(0))]}\Big) = \lim_{t\to\infty} E[f(\mathring{V}(t)/t)]. \tag{4.24}$$

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $\tau \in (0,\infty)$ it holds

$$\frac{E[N(0)\varphi(N(0))f(1/\varphi(N(0)))]}{E[N(0)\varphi(N(0))]} = \lim_{t\downarrow 0} E[f(\mathring{V}(t)/t)] \leq E[f(\mathring{V}(\tau)/\tau)]$$
$$\leq \lim_{t\to\infty} E[f(\mathring{V}(t)/t)] = f\Big(\frac{EN(0)}{E[N(0)\varphi(N(0))]}\Big). \tag{4.25}$$

Note that choosing $f(x) := x^k$, $x \in (0,\infty)$, for fixed $k \in \mathbb{R}$ in Theorem 4.2 and 4.3 provides results for the kth moment of $\mathring{V}(\tau)$. However, Corollary 3.2 and 3.3 yield slightly stronger results. From Corollary 3.2 we obtain immediately the following.

Corollary 4.2 *Assume that (A2) is fulfilled. For $\tau_1, \tau_2 \in (0,\infty)$ it holds*

$$(E[\mathring{V}^k(\tau_1+\tau_2)])^{1/k} \geq (E[\mathring{V}^k(\tau_1)])^{1/k} + (E[\mathring{V}^k(\tau_2)])^{1/k}, \qquad k \in (-\infty,1] \setminus \{0\}, \tag{4.26}$$

$$(E[\mathring{V}^k(\tau_1+\tau_2)])^{1/k} \leq (E[\mathring{V}^k(\tau_1)])^{1/k} + (E[\mathring{V}^k(\tau_2)])^{1/k}, \qquad k \in [1,\infty), \tag{4.27}$$

i.e., $(E[\mathring{V}^k(\tau)])^{1/k}$ is for fixed $k \in (-\infty,1]\setminus\{0\}$ a superadditive and for fixed $k \in [1,\infty)$ a subadditive function of $\tau \in (0,\infty)$.

Further, $(E[\mathring{V}^k(\tau)])^{1/k}$ is for fixed $\tau \in (0,\infty)$ a non decreasing function of $k \in \mathbb{R} \setminus \{0\}$.

Because of (4.18), (4.16), (4.12), and (4.15), from Corollary 3.3 we obtain the following.

Corollary 4.3 *Assume that (A2) is fulfilled. For $\tau \in (0,\infty)$ it holds*

$$\Big(\frac{EN(0)}{E[N(0)\varphi(N(0))]}\Big)^k \leq \lim_{t\to\infty} E[(\mathring{V}(t)/t)^k\,] \leq E[(\mathring{V}(\tau)/\tau)^k\,]$$
$$\leq \lim_{t\downarrow 0} E[(\mathring{V}(t)/t)^k\,] = \frac{E[N(0)\varphi^{1-k}(N(0))]}{E[N(0)\varphi(N(0))]}\,, \quad k \in \mathbb{R}\setminus(0,1), \tag{4.28}$$

$$\frac{E[N(0)\varphi^{1-k}(N(0))]}{E[N(0)\varphi(N(0))]} = \lim_{t\downarrow 0} E[(\mathring{V}(t)/t)^k\,] \leq E[(\mathring{V}(\tau)/\tau)^k\,]$$
$$\leq \lim_{t\to\infty} E[(\mathring{V}(t)/t)^k\,] = \Big(\frac{EN(0)}{E[N(0)\varphi(N(0))]}\Big)^k, \quad k \in [0,1]. \tag{4.29}$$

For fixed $k \in \mathbb{R}$ it holds

$$\lim_{t\to\infty} E[(\mathring{V}(t)/t)^k\,] = \Big(\frac{EN(0)}{E[N(0)\varphi(N(0))]}\Big)^k \tag{4.30}$$

or it holds $\lim_{t\to\infty} E[(\mathring{V}(t)/t)^k\,] = \infty$.

Remark 4.5 *Theorem 4.2 and 4.3 as well as Corollary 4.2 and 4.3 seem to be new even in case of $M/M/1-PS$.*

For $M/GI/1-PS$ and $M/M/SDPS$, the lower and upper bounds for $E[(\overset{\circ}{V}(\tau)/\tau)^k]$ in case of $k \in \mathbb{N}$, cf. (4.28)*, are given in [CVB] and [BB2], respectively.*

For $M/GI/SDPS$, the upper bound for $E[(\overset{\circ}{V}(\tau)/\tau)^k]$ in case of $k \in \mathbb{N}$, cf. (4.28)*, is given in [BB3] based on analyzing corresponding Kolmogorov equations. The lower bound follows immediately from Cohen's proportional result for $E\overset{\circ}{V}(\tau)$ in $M/GI/SDPS$ and Hölder's inequality.*

Note that (4.22) *and* (4.25) *for $f(x)=x$, $x \in (0,\infty)$, as well as* (4.28) *and* (4.29) *for $k=1$, all provide again the proportional result* (4.19) *for $E\overset{\circ}{V}(\tau)$ in $M(n)/GI/SDPS$.*

In view of (4.18), (4.16), (4.12), and (4.15), from Corollary 3.4 we obtain the following estimate for the distribution of $\overset{\circ}{V}(\tau)$.

Corollary 4.4 *Assume that (A2) is fulfilled. Then for $a \in \mathbb{R}_+$ it holds*

$$\begin{aligned} \frac{E[N(0)\min(a(x\varphi(N(0))-1),\varphi(N(0)))]}{E[N(0)\varphi(N(0))]} &\le P(\overset{\circ}{V}(\tau)/\tau \le x) \\ \le \frac{E[N(0)\max(a(x\varphi(N(0))-1)+\varphi(N(0)),0)]}{E[N(0)\varphi(N(0))]}&, \\ \tau \in (0,\infty),\ x \in (0,\infty). & \quad (4.31) \end{aligned}$$

For deriving results for the sojourn time $V(\tau)$ of an added request we consider another modified system. By replacing $\varphi(\cdot)$ by $\varphi_+(\cdot)$, cf. (4.12), in the $M(n)/GI(n)/\infty$ system we obtain the modified system $M(n)/GI_+(n)/\infty$. The corresponding quantities related to this system will be indexed by $_+$. From (4.8) and (4.12) we find

$$\theta_+(n) = \frac{\varphi(1)}{\varphi(n+1)}\,\theta(n). \tag{4.32}$$

In view of (4.7), (4.32), and (4.8), we make here the following assumption:

(A3) $$\sum_{n=0}^{\infty} n\theta(n) < \infty, \quad \sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)}\,\theta(n) < \infty, \quad \sum_{n=0}^{\infty} \lambda(n)\theta(n) < \infty. \tag{4.33}$$

Note that (A3) implies (A1) for the $M(n)/GI(n)/\infty$ system and for the $M(n)/GI_+(n)/\infty$ system. Because of (4.9) and (4.32), the stationary occupancy distribution $p_+(n) := P(N_+(t) = n)$, $n \in \mathbb{Z}_+$, is given by

$$p_+(n) = \frac{1}{E[1/\varphi(N(0)+1)]} \, \frac{1}{\varphi(n+1)} \, p(n), \quad n \in \mathbb{Z}_+. \tag{4.34}$$

From (2.3), (4.12), and (4.34) it follows

$$g_+ = E[\varphi_+(N_+(0))] = \frac{1}{E[1/\varphi(N(0)+1)]} \,. \tag{4.35}$$

Theorem 4.4 *Assume that (A3) is fulfilled. Then*

$$V(\tau \,|\, n) \stackrel{\mathcal{D}}{=} V_{v,+}(\tau \,|\, n), \quad \tau \in \mathbb{R}_+, \; n \in \mathbb{Z}'_+, \tag{4.36}$$

$$V(\tau) \stackrel{\mathcal{D}}{=} \mathring{V}_{s,+}(\tau), \quad \tau \in \mathbb{R}_+. \tag{4.37}$$

Proof Let the $M(n)/GI(n)/\infty$ system and the $M(n)/GI_+(n)/\infty$ system in steady state. Note that $p(n) > 0$ if and only if $p_+(n) > 0$. The product form solution (4.10) yields that conditioned that there are n requests at $t = 0$ in the $M(n)/GI(n)/\infty$ system the n residual service times $R_i(0)$, $i = 1, \ldots, n$, are stochastically independent and have the distribution function $B_R(\cdot)$. Analogously, it follows that conditioned that there are n requests at $t = 0$ in the $M(n)/GI_+(n)/\infty$ system the n residual service times $R_{i,+}(0)$, $i = 1, \ldots, n$, are stochastically independent and have the distribution function $B_R(\cdot)$. The definitions of $V(\tau \,|\, n)$ and $V_{v,+}(\tau \,|\, n)$ as well as the dynamics of both systems thus imply (4.36). Further, from (4.6), (4.36), (4.34), (4.35), (4.12), (2.14) and (2.13) we find

$$\begin{aligned}
P(V(\tau) \le x) &= \sum_{n \in \mathbb{Z}'_+} P(V(\tau \,|\, n) \le x) p(n) \\
&= \sum_{n \in \mathbb{Z}'_+} P(V_{v,+}(\tau \,|\, n) \le x) \, E[1/\varphi(N(0)+1)] \varphi(n+1) p_+(n) \\
&= \sum_{n \in \mathbb{Z}'_+} P(V_{v,+}(\tau \,|\, n) \le x) \, \frac{1}{g_+} \, \varphi_+(n) p_+(n) \\
&= \sum_{n \in \mathbb{Z}'_+} P(V_{v,+}(\tau \,|\, n) \le x) \, \mathring{p}_{s,+}(n) = P(\mathring{V}_{s,+}(\tau) \le x),
\end{aligned}$$

which yields (4.37). $\square$

In case of $M/GI(n)/\infty$, $\mathring{V}(\tau)$ equals in distribution $V(\tau)$. However, from Theorem 4.1 and Theorem 4.4 it follows that $\mathring{V}(\tau)$ in $M(n)/GI(n)/\infty$ equals in distribution $V(\tau)$ in a modified system.

Theorem 4.5 *Assume that*

$$\sum_{n=0}^{\infty} n\theta(n) < \infty, \quad \sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)}\,\theta(n) < \infty, \quad \sum_{n=0}^{\infty} n\lambda(n)\theta(n) < \infty. \tag{4.38}$$

Then the sojourn time $\mathring{V}(\tau)$ of an arriving τ-request in $M(n)/GI(n)/\infty$ equals in distribution the sojourn time of an added τ-request in the modified system $M_+(n)/GI(n)/\infty$ where $\psi(\cdot)$ is replaced by $\psi_+(\cdot)$.

Proof Note that (4.38) implies (A2) and (A3). Thus the assertion follows from (4.18) and (4.37). Moreover, note that $\mathring{p}(n)$, $n \in \mathbb{Z}_+$, cf. (4.5), is the stationary occupancy distribution in $M_+(n)/GI(n)/\infty$, which implies again the assertion due to conditional PASTA and (4.10). $\square$

In view of Theorem 4.4, we obtain results for the sojourn time of an added request in $M(n)/GI(n)/\infty$ by using the results for the sojourn time of a synchronized virtual request in an infinite-server system in random environment, where the state of the infinite-server system at time t is given by $N_+(t)$ corresponding to the interpretation of the $M(n)/GI_+(n)/\infty$ system as an infinite-server system in random environment. As (A3) implies (A1) for this infinite-server system, we obtain immediately results for the sojourn time of an added request by using the results of Section 3 for the sojourn time of a synchronized virtual request. Note that (A1) has to be replaced by (A3), $\mathring{V}_s(\cdot)$ by $V(\cdot)$, $\varphi(\cdot)$ by $\varphi_+(\cdot)$, $p(\cdot)$ by $p_+(\cdot)$, and g by g_+.

In view of (4.37) and (4.35), Corollary 3.1 provides the following.

Corollary 4.5 *Assume that (A3) is fulfilled. Then for the mean sojourn time of an added request in $M(n)/GI(n)/\infty$ ($M(n)/GI/SDPS$) it holds*

$$EV(\tau) = E[1/\varphi(N(0)+1)]\,\tau, \quad \tau \in \mathbb{R}_+. \tag{4.39}$$

Remark 4.6 *(i) Note that* (4.39) *may also be considered as a generalization of Cohen's proportional result for the mean sojourn time in $M/GI/SDPS$, cf. [Coh], to $M(n)/GI/SDPS$, cf. Remark 4.4.*

(ii) Assume that (A3) is fulfilled. Due to Theorem 4.4 and Theorem 3.1, the LST and the moments of $V(\cdot)$ are given by the LST and the moments of $V_{v,+}(\cdot)$, respectively. In particular, the variance of $V(\cdot)$ is given by the expectation of $V_{v,+}(\cdot)$.

From Theorem 3.2 we obtain immediately the following.

Theorem 4.6 *Assume that (A3) is fulfilled.*

Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $\tau_1, \tau_2 \in (0,\infty)$ it holds

$$\begin{aligned} &E[(\tau_1+\tau_2) f(V(\tau_1+\tau_2)/(\tau_1+\tau_2))] \\ &\qquad \le E[\tau_1 f(V(\tau_1)/\tau_1)] + E[\tau_2 f(V(\tau_2)/\tau_2)], \end{aligned} \tag{4.40}$$

i.e., $E[\tau f(V(\tau)/\tau)]$ is subadditive for $\tau \in (0,\infty)$.

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $\tau_1, \tau_2 \in (0,\infty)$ it holds

$$\begin{aligned} &E[(\tau_1+\tau_2) f(V(\tau_1+\tau_2)/(\tau_1+\tau_2))] \\ &\qquad \ge E[\tau_1 f(V(\tau_1)/\tau_1)] + E[\tau_2 f(V(\tau_2)/\tau_2)], \end{aligned} \tag{4.41}$$

i.e., $E[\tau f(V(\tau)/\tau)]$ is superadditive for $\tau \in (0,\infty)$.

Because of (4.37), (4.35), (4.12), and (4.34), from Theorem 3.3 we obtain the following.

Theorem 4.7 *Assume that (A3) is fulfilled.*

Let $f(x)$, $x \in (0,\infty)$, non negative and convex. Then for $\tau \in (0,\infty)$ it holds

$$\begin{aligned} &f(E[1/\varphi(N(0)+1)]) \le \liminf_{t\to\infty} E[f(V(t)/t)] \le E[f(V(\tau)/\tau)] \\ &\qquad \le \lim_{t\downarrow 0} E[f(V(t)/t)] = E[f(1/\varphi(N(0)+1))]. \end{aligned} \tag{4.42}$$

If additionally $E[f(V(\tau)/\tau)] < \infty$ for some $\tau \in (0,\infty)$, then

$$f(E[1/\varphi(N(0)+1)]) = \liminf_{t\to\infty} E[f(V(t)/t)], \tag{4.43}$$

and if $E[f(V(t)/t)]$ is bounded in a neighborhood of some $\tau \in (0,\infty)$, then

$$f(E[1/\varphi(N(0)+1)]) = \lim_{t\to\infty} E[f(V(t)/t)]. \tag{4.44}$$

Let $f(x)$, $x \in (0,\infty)$, non negative and concave. Then for $\tau \in (0,\infty)$ it holds

$$\begin{aligned} E[f(1/\varphi(N(0)+1))] &= \lim_{t\downarrow 0} E[f(V(t)/t)] \le E[f(V(\tau)/\tau)] \\ &\le \lim_{t\to\infty} E[f(V(t)/t)] = f(E[1/\varphi(N(0)+1)]). \end{aligned} \tag{4.45}$$

Note that choosing $f(x) := x^k$, $x \in (0,\infty)$, for fixed $k \in \mathbb{R}$ in Theorem 4.6 and 4.7 provides results for the kth moment of $V(\tau)$. However, Corollary 3.2 and 3.3 yield slightly stronger results. From Corollary 3.2 we obtain immediately the following.

Corollary 4.6 *Assume that (A3) is fulfilled. For $\tau_1, \tau_2 \in (0,\infty)$ it holds*

$$(E[V^k(\tau_1+\tau_2)])^{1/k} \ge (E[V^k(\tau_1)])^{1/k} + (E[V^k(\tau_2)])^{1/k}, \qquad k \in (-\infty,1] \setminus \{0\}, \tag{4.46}$$

$$(E[V^k(\tau_1+\tau_2)])^{1/k} \le (E[V^k(\tau_1)])^{1/k} + (E[V^k(\tau_2)])^{1/k}, \qquad k \in [1,\infty), \tag{4.47}$$

i.e., $(E[V^k(\tau)])^{1/k}$ is for fixed $k \in (-\infty,1]\setminus\{0\}$ a superadditive and for fixed $k \in [1,\infty)$ a subadditive function of $\tau \in (0,\infty)$.

Further, $(E[V^k(\tau)])^{1/k}$ is for fixed $\tau \in (0,\infty)$ a non decreasing function of $k \in \mathbb{R} \setminus \{0\}$.

Because of (4.37), (4.35), (4.12), and (4.34), from Corollary 3.3 we obtain the following.

Corollary 4.7 *Assume that (A3) is fulfilled. For $\tau \in (0,\infty)$ it holds*

$$\begin{aligned} (E[1/\varphi(N(0)+1)])^k &\le \lim_{t\to\infty} E[(V(t)/t)^k] \le E[(V(\tau)/\tau)^k] \\ &\le \lim_{t\downarrow 0} E[(V(t)/t)^k] = E[\varphi^{-k}(N(0)+1)], \quad k \in \mathbb{R} \setminus (0,1), \end{aligned} \tag{4.48}$$

$$\begin{aligned} E[\varphi^{-k}(N(0)+1)] &= \lim_{t\downarrow 0} E[(V(t)/t)^k] \le E[(V(\tau)/\tau)^k] \\ &\le \lim_{t\to\infty} E[(V(t)/t)^k] = (E[1/\varphi(N(0)+1)])^k, \quad k \in [0,1]. \end{aligned} \tag{4.49}$$

For fixed $k \in \mathbb{R}$ it holds

$$\lim_{t\to\infty} E[(V(t)/t)^k] = (E[1/\varphi(N(0)+1)])^k \tag{4.50}$$

or it holds $\lim_{t\to\infty} E[(V(t)/t)^k] = \infty$.

In view of (4.37), (4.35), (4.12), and (4.34), from Corollary 3.4 we obtain the following estimate for the distribution of $V(\tau)$.

Corollary 4.8 *Assume that (A3) is fulfilled. Then for $a \in \mathbb{R}_+$ it holds*

$$E[\min(a(x-1/\varphi(N(0)+1)),1)] \le P(V(\tau)/\tau \le x) \le E[\max(a(x-1/\varphi(N(0)+1))+1,0)], \quad \tau \in (0,\infty),\ x \in (0,\infty). \tag{4.51}$$

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