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Circuit Admissible Triangulations of Oriented Matroids

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CIRCUIT ADMISSIBLE TRIANGULATIONS OF ORIENTED MATROIDS

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ABSTRACT. All triangulations of euclidean oriented matroids are of the same PL-homeomorphism type by a result of Anderson. That means all triangulations of euclidean acyclic oriented matroids are PL-homeomorphic to PL-balls and that all triangulations of totally cyclic oriented matroids are PL-homeomorphic to PL-spheres. For non-euclidean oriented matroids this question is wide open.

One key point in the proof of Anderson is the following fact: for every triangulation of a euclidean oriented matroid the adjacency graph of the set of all simplices "intersecting" a segment $[p_- p_+]$ is a path. We call this graph the $[p_- p_+]$-adjacency graph of the triangulation.

While we cannot solve the problem of the topological type of triangulations of general oriented matroids we show in this note that for every circuit admissible triangulation of an arbitrary oriented matroid the $[p_- p_+]$-adjacency graph is path.

Triangulations of oriented matroids appeared in the literature as natural combinatorial models for triangulations of point configurations [2]. However, since not all oriented matroids model point configurations the notion of a triangulation of an oriented matroid gives rise to additional questions that do not come up in the theory of triangulations of point configurations.

One of these questions is the following: is the abstract simplicial complex defined by a triangulation of an oriented matroid homeomorphic ot a sphere in the totally acyclic case or a ball in the acyclic case? The answer to this question in the realizable case is of course affirmative because in the case of point configurations the triangulation is naturally embedded as a convex set in a euclidean space.

Why care about the general case? An application of triangulations of oriented matroids in their full generality is their appearance in the theory of combinatorial differential manifolds. "Good" topological properties in this context lead to the existence of differentiable strucures on these objects, making the combinatorial model more suitable [1, 4].

But also as an investigation of what weird things might happen in the theory of non-realizable oriented matroids this question has become a challenging open problem in its own right. (An in-depth study of triangulations of oriented matroids is presented in [5], background on oriented matroids can be found in [3].)

For a *euclidean* oriented matroid Anderson has proved that the topological types of all its triangulations are the same. Since for all oriented matroids there are triangulations known that are homeomorphic to a sphere resp. to a ball—the so-called *lifting triangulations*—the answer to the above question is affirmative.

One important building block in the construction of Anderson is the fact that the adjacency graph of the set of simplices in a triangulation "intersecting" an arbitrary segment is always a path. (For exact definitions see below.)

In this note we show that this graph is also a path for general oriented matroids provided the triangulation respects an additional property: it does not contain a so-called *intersection circuit*, a circuit that has positive in one simplex and negative part in another simplex in the triangulation.

We start by defining our main object of study. For simplicity, we call r-subsets of full rank r *simplices*; subsets of rank $r-1$ are called *facets*.

Definition 1 (Circuit Admissible Triangulation)**.** Let $\mathfrak{M} = (E, \mathcal{Z})$ be a rank r oriented matroid on the ground set E given by its set of circuits $\mathcal{Z}$. Its set of facets be denoted by $\mathcal{L}(\mathfrak{M})$.

A non-empty set T of simplices is called a *circuit admissible triangulation* of $\mathfrak{M}$ if it satisfies the following conditions.

(UP) For every $S \in T$ and every facet $R \subset S$ of S either there is a facet $F \in \mathcal{L}(\mathfrak{M})$ of $\mathfrak{M}$ with $R \subseteq F$ or there is another simplex $S' \in T$ different from S with $R \subset S'$.

(IP) For all $S_1, S_2 \in T$ there is no circuit $Z \in \mathcal{Z}$ with $Z^+ \subseteq S_1$ and $Z^- \subseteq S_2$.

If two simplices or facets satisfy (IP) we call them *circuit admissible*. We follow the suggestion of Santos [5] to call a circuit as in (IP) an *intersection circuit* of the simplices involved. In the euclidean case this definition coincides with the definitions of a triangulation of an oriented matroid in the literature.

Condition (UP) makes sure that there are enough simplices to "cover" the whole oriented matroid, (IP) takes care of unwanted intersections between simplices. In the definition of a triangulation of an oriented matroid the condition for proper intersections is—at least locally—different: for every extension that lies in the convex hulls of two simplices it also lies in the convec hull of the intersection. This is called "proper intersection" in the literature. The author has shown earlier (unpublished) an example in rank four based on the non-euclidean 12-point oriented matroid $R(12)$ by Richter-Gebert where two simplices might intersect properly although they contain an intersection circuit.

Meanwhile, Santos has shown that this behaviour already happens in the 8-point non-euclidean oriented matroid $EFM(8)$ [5], the first case in which non-euclidean oriented matroids can appear. This shows a defect in the original definition of triangulations of oriented matroids by Billera and Munson [2]: proper intersection of simplices, although stated locally for two simplices, depends heavily on the rest of the oriented matroid via the existence of certain extensions. For example, the two simplices intersecting properly in $R(12)$ but containing an intersection circuit would no longer intersect properly after suitable deletions, which seems somewhat unnatural. However, by Santos' work [5] we know that there are definitions equivalent to the original one that do not have this unwanted property.

The advantage of circuit admissability is the fact that intersection of two simplices is not affected by elements that are not contained in the union of the two simplices involved. Note that every circuit admissible triangulation is a triangulation and that every triangulation additionally satisfying (IP) is a circuit admissible triangulation.

Here is a formal definition of our target:

Definition 2 (Segment Adjacency Part/Graph)**.** Let T be a triangulation of $\mathfrak{M}$, and let p_- and p_+ be interior extensions in general position.

The $[p_- p_+]$-*adjacency part* of T is the set of all simplices S of T, together with their pairwise common facets, that form a vector with the segment $[p_-, p_+]$ (the restriction of $\mathfrak{M}$ to $\{p_-, p_+\}$) or with one of its end points, i.e., simplices or facets S such that $(S, p_- p_+)$, (S, p_-), or (S, p_+) are vectors in $\mathfrak{M}$.

The $[p_-, p_+]$-*adjacency graph* $G_{[p_-, p_+]}$ of T is the adjacency graph of the $[p_- p_+]$-adjacency part. In other words, it is the following graph:

- The vertex set of $G_{[p_-, p_+]}$ is the set of all simplices S in the $[p_- p_+]$-adjacency part.
- There is an edge between to vertices S_1 and S_2 if the corresponding simplices are adjacent, i.e., share a common facet.

We speak of the *segment adjacency part* resp. *segment adjacency graph* when we do not want to specify the segment explicitely.

Figure 1 depicts an easy example of a triangulation and a segment adjacency graph in the realizable case.

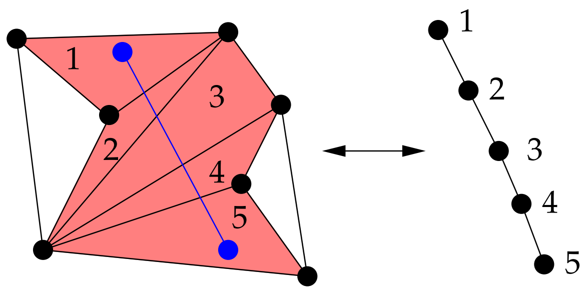

FIGURE 1. The segment adjacency graph.

The following lemma was proved by Anderson for arbitrary triangulations of *euclidean* oriented matroids in a slightly different language.

Lemma 3 (Anderson [1])**.** *Let* T *be a triangulation of a euclidean oriented matroid* $\mathfrak{M}$ *and let* p_- *and* p_+ *be interior extensions in general position. Then the* $[p_-, p_+]$*-adjacency graph of* T *is a path.*

We should remark the following at this point: because the restriction of $\mathfrak{M}$ to a segment plus a simplex is always realizable every node in $G_{[p_-,p_+]}$ has degree one or two. Hence, the graph consists of one path and maybe some cycles. The point is to show that there are no cycles.

Lemma 4 (Main Lemma)**.** *Let* T *be a circuit-admissible triangulation of* $\mathfrak{M}$ *and let* p_- *and* p_+ *be interior extensions in general position. Then the* $[p_-, p_+]$*-adjacency graph of* T *is a path.*

The idea for the proof is based on the following consideration: in the euclidean case we can extend the oriented matroid by the intersection points of facets with the segment $[p_- p_+]$. By looking at their cocircuit signature we can tell a total order on these intersection points, thus on the facets involved. The existence of cycles is incompatible with this observation.

In the non-euclidean case we might not be able to extend $\mathfrak{M}$ by all those intersection points. Thus, we do not have a total order of the facets intersecting a segment at hand. We can, however, replace the total order by a certain weaker relation. This relation will, however, be good enough to prove the Main Lemma. The following definition is the key idea in this note.

Definition 5 (Segment Relation)**.** Let T be a circuit-admissible triangulation of $\mathfrak{M}$. For interior extensions p_- and p_+ in general position, we define a relation on the facets in the $[p_-, p_+]$-adjacency part of T as follows:

$$F \prec_{[p_- p_+]} G :\iff \exists \text{ circuit } X \text{ in } \mathfrak{M}\text{: } p_- \in X^+, X^+ \subseteq p_- \cup G, X^- \subseteq F;$$
$$F \succ_{[p_- p_+]} G :\iff \exists \text{ circuit } X \text{ in } \mathfrak{M}\text{: } p_+ \in X^+, X^+ \subseteq p_+ \cup G, X^- \subseteq F.$$

See Figure 2 for an illustration in rank four.

This relation cannot provide a total order on the facets intersecting a segment because this would contradict the existence of cycling oriented matroid programs, a key property of non-euclidean oriented matroids. The following lemma shows, however, that the segment relation is good enough to tell consistently which one

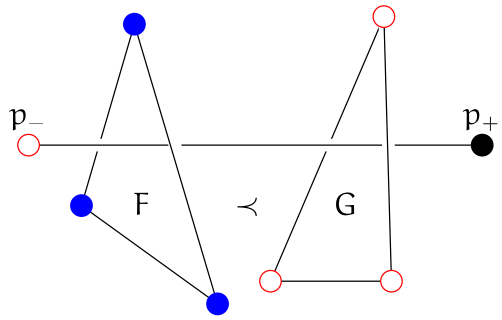

FIGURE 2. The segment relation: $F \prec_{[p_- p_+]} G$ because there is a circuit containing p_- with light positive and dark negative part.

out of two circuit admissible facets is "closer" to p_- and which one is "closer" to p_+. We will use the notation $Z = (Z^+, Z^-)$ for a circuit and $\underline{Z} = Z^+ \cup Z^-$ for its support.

Lemma 6. *Let* T *be a circuit-admissible triangulation of* $\mathfrak{M}$*. Furthermore, let* F *and* G *be facets in the* $[p_-, p_+]$*-adjacency part of* T *for interior extensions* p_- *and* p_+ *in general position. Then the following hold:*

(i) $F \prec_{[p_- p_+]} G$ *if and only if* $G \succ_{[p_- p_+]} F$*;*
(ii) either $F \prec_{[p_- p_+]} G$ *or* $F \succ_{[p_- p_+]} G$*.*

Proof. All assertions follow from easy strong circuit elimination arguments.

To prove (i), let $F \prec_{[p_- p_+]} G$, i.e., there is a circuit X with

$$p_- \in X^+; \quad X^+ \subseteq p_- \cup G; \quad X^- \subseteq F.$$

Using the circuit

$$Y := (G, p_- p_+)$$

eliminate $p_- \in X^+ \cap Y^-$ and introduce $p_+ \in \underline{Y} \setminus \underline{X}$. This yields a circuit Z with

$$\begin{aligned} &p_+ \in Z^-; \\ &Z^+ \subseteq (X^+ \cup Y^+) \setminus p_- \subseteq G; \\ &Z^- \subseteq (X^- \cup Y^-) \setminus p_- \subseteq p_+ F. \end{aligned}$$

Thus, $G \succ_{[p_- p_+]} F$. Since the statement is symmetric, the first claim is proved.

In order to prove (ii), we first show that at most one of the relations $F \prec_{[p_- p_+]} G$ and $F \succ_{[p_- p_+]} G$ holds. To this end, assume—for the sake of contradiction—that $F \prec_{[p_- p_+]} G$ and $F \succ_{[p_- p_+]} G$. Using (i) we also have $G \prec_{[p_- p_+]} F$. By definition, there are circuits X and Y with

$$\begin{aligned} &p_- \in X^+; && X^+ \subseteq p_- \cup G; && X^- \subseteq F; \\ &p_- \in Y^+; && Y^+ \subseteq p_- \cup F; && Y^- \subseteq G. \end{aligned}$$

Elimination of $p_- \in X^+ \cap (-Y)^-$ in X yields a circuit Z with

$$\begin{aligned} &Z^+ \subseteq (X^+ \cup (-Y)^+) \setminus p_- \subseteq G; \\ &Z^- \subseteq (X^- \cup (-Y)^-) \setminus p_- \subseteq F. \end{aligned}$$

In other words, F and G are not circuit admissible. Consequently, $F \prec_{[p_- p_+]} G$ and $F \succ_{[p_- p_+]} G$ cannot hold at the same time.

We finally show that at least one of the relations $F \prec_{[p_- p_+]} G$ or $F \succ_{[p_- p_+]} G$ holds. Since F and G are in the $[p_-, p_+]$-adjacency part, we have two circuits $X :=$

(F, p_-p_+) and $Y := (G, p_-p_+)$. Eliminating p_- yields a circuit Z with

$$Z^+ \subseteq (X^+ \cup (-Y)^+) \setminus p_- \subseteq F \cup p_+;$$
$$Z^- \subseteq (X^- \cup (-Y)^-) \setminus p_- \subseteq G \cup p_+.$$

If $p_+ \notin \underline{Z}$ then F and G are not circuit admissible: contradiction. If $p_+ \in Z^+$ then Z is a circuit showing $G \succ_{[p_-p_+]} F$; if $p_+ \in Z^-$ then Z shows $F \succ_{[p_-p_+]} G$. This proves the claim using (i) one more time. □

We are now in a position to prove our lemma.

Proof of Lemma 4. Since T is a triangulation of $\mathfrak{M}$ and p_- and p_+ are in general position it follows from the work of Santos [5] that there is a unique simplex S_+ "containig" p_+, i.e., such that (S_+, p_+) is a vector, and a unique simplex S_- "containing" p_-, i.e., such that (S_-, p_-) is a vector in $\mathfrak{M}$.

From the fact that $\mathfrak{M}$ restricted to a segment plus one simplex is realizable it follows that S_- and S_+ have degree one in $G_{[p_-,p_+]}$ whereas all other simplices have degree two. Hence, there is a path from S_- to S^+ contained in $G_{[p_-,p_+]}$. We have to show that all simplices in the $[p_-, p_+]$-adjacency part belong to that path.

Let $S_0 = S_-, S_1, \ldots, S_k = S_+$ be the successively adjacent simplices in the path from S_- to S_+. Assume, for the sake of contradiction, that there exists another simplex S in T with $S \neq S_i$ for all $i = 1, \ldots, k$. Define $F_j := S_{j-1} \cap S_j$ for $j = 1, \ldots, k$. Furthermore, let F be a facet of S "pierced" by $[p_-p_+]$, i.e., so that (F, p_-p_+) is a circuit.

By Lemma 6, there are three cases:

- $F \prec_{[p_-p_+]} F_1$,
- $F \succ_{[p_-p_+]} F_k$, or
- $F_i \prec_{[p_-p_+]} F \prec_{[p_-p_+]} F_{i+1}$ for some $i \in \{1, 2, \ldots, k-1\}$.

The case $F \prec_{[p_-p_+]} F_1$. Then there exists a circuit X with

$$p_- \in X^+; \quad X^+ \subseteq p_- \cup F_1; \quad X^- \subseteq F.$$

Eliminate $p_- \in X^+ \cap Y^-$ using the circuit

$$Y := (S_0, p_-).$$

This yields a circuit Z with

$$Z^+ \subseteq (X^+ \cup Y^+) \setminus p_- \subseteq F_1;$$
$$Z^- \subseteq (X^- \cup Y^-) \setminus p_- \subseteq F.$$

Thus, F_1 and F are not circuit admissible, and neither are S_0 and S: contradiction.
The case $F \succ_{[p_-p_+]} F_k$. This case is analogous to the previous one.
The case $F_i \prec_{[p_-p_+]} F \prec_{[p_-p_+]} F_{i+1}$ *for some* $i = 0, \ldots, k$. Then there exist circuits X and Y with

$$\begin{array}{lll} p_- \in X^+, & X^+ \subseteq p_- \cup F, & X^- \subseteq F_i; \\ p_- \in -Y^-, & -Y^- \subseteq p_- \cup F_{i+1}, & -Y^+ \subseteq F; \end{array}$$

Eliminate $p_- \in X^+ \cap -Y^-$. This yields a circuit Z with

$$Z^+ \subseteq (X^+ \cup -Y^+) \setminus p_- \subseteq F;$$
$$Z^- \subseteq (X^- \cup -Y^-) \setminus p_- \subseteq F_i \cup F_{i+1}.$$

However, by construction $F_i \cup F_{i+1} = S_i$, and thus S and S_i are not circuit admissible: contradiction. □

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