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Optimality Conditions for Convex State Constrained Optimal Control Problems with Discontinuous States [1]

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Optimality Conditions for Convex State Constrained Optimal Control Problems with Discontinuous States [†]

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Abstract

We discuss first order optimality conditions for state constrained optimal control problems. Our concern is the treatment of problems, where the solution of the state equation is not known to be continuous, as in the case of boundary control in three space dimensions or optimal control with parabolic partial differential equations. We show existence of measure valued Lagrangian multipliers, which have just enough additional regularity to be applicable to all possibly discontinuous solutions of the state equation.

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1 Introduction

First order optimality conditions for optimal control problems subject to partial differential equations have been studied for a long time, sucessfully for large classes of problems (for an excellent overview we refer to the text book [12], for original papers in the state constrained elliptic case cf. [5, 6, 2]). However, to the knowledge of the author, a particularly hard class of problems is still not fully understood, namely state constrained optimal control problems, where the control to state mapping has poor regularity properties and fails to guarantee continuity of the state. The main obstacle in this respect is the lack of an interior point, which in general only exists with respect to the topology of uniform convergence.

In this work we present a way to overcome this obstacle by careful construction of the spaces in which the optimal control problem is considered. This helps us to make use of additional regularity of the state for regular controls and to derive a full system of first order optimality conditions.

For the sake of clarity we present our ideas in an abstract framework and in a setting as simple as possible. Then we show with an example, how typical optimal control problems, if suitable regularity results are available, fit into this framework.

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2 An Abstract Optimal Control Problem

Consider a convex optimal control problem of the following form:

$$\begin{aligned} \min_{(u,y)\in U\times Y} j(u,y) \quad s.t. \quad & Ay - Bu = 0 \\ & \underline{y} \le y. \end{aligned} \tag{1}$$

The following conditions render this problem well defined. Note that A, which models a differential operator, may be an unbounded operator. Thus it is typically not defined on all of Y. For some remarks on the use of unbounded operators in optimal control refer to the paragraph at the end of this section.

Assumption 2.1 (Assumptions for existence of optimal solutions)**.** Let the following assumptions hold.

(i) Q is a closed, bounded set of $\mathbb{R}^d$, with $d \in \mathbb{N}$.

(ii) U, Y, and R are Banach spaces, U and Y are reflexive, and Y is continuously embedded into $L_1(Q)$.

(iii) The linear operator $A : Y \supset \operatorname{dom} A \to R$ is closed, densely defined and bijective, and the linear operator $B : U \to R$ is continuous.

(iv) The pointwise lower constraint function $\underline{y} : Q \to \mathbb{R}$ is continuous, the inequality is understood to hold almost everywhere in Q and defines a closed subset $G \subset U \times Y$.

(v) The functional $j : U \times Y \to \mathbb{R}$ is lower semi-continuous, convex, coercive on the set $V = \{(u, y) \in U \times \operatorname{dom} A : Ay - Bu = 0\}$ and Gâteaux differentiable.

(vi) $G \cap V$ is non-empty.

Due to our assumptions the operator

$$\begin{aligned} T : U \times Y \supset \operatorname{dom} T \to R \\ (u, y) \mapsto Ay - Bu \end{aligned}$$

is linear, closed and densely defined. In particular, $V = \ker T$ is closed. By our assumptions the main existence theorem of convex optimization (cf. e.g. [7, Proposition II.1.2]) yields existence of a minimizer $x_* := (u_*, y_*) \in U \times Y$ of (1).

With the help of indicator functions we may rewrite (1) as an unconstrained optimal control problem. The indicator function ι_S of a set S is defined by

$$\iota_S(x) = \begin{cases} 0 & : \quad x \in S \\ \infty & : \quad \text{otherwise} \end{cases}.$$

The indicator function ι_S is convex and lower semi-continuous if and only if S is convex and closed. This reformulation yields the problem:

$$\min_{(u,y)\in U\times Y} F(u,y) := j(u,y) + \iota_V(u,y) + \iota_G(u,y), \tag{2}$$

which is equivalent to (1). Here $F : U \times Y \to \mathbb{R} \cup \{+\infty\}$ is an extended real valued function.

The subdifferential $\partial F(x)$ of F at $x = (u, y)$ is the set of all $x^* \in (U \times Y)^*$, for which the relation $\langle x^*, \delta x\rangle \leq F(x + \delta x) - F(x)$ holds for all $\delta x \in U \times Y$. Here and in the following $\langle\cdot,\cdot\rangle$ denotes the dual pairing.

Optimality of $x_* = (u_*, y_*)$ is equivalent to

$$0 \in \partial F(x_*) = \partial\left(j(x_*) + \iota_V(x_*) + \iota_G(x_*)\right). \tag{3}$$

For a detailed introduction to convex analysis and subdifferential calculus we refer to [7, Chapter I]. In the remaining paper we will apply the sum-rule of convex analysis to (3), using carefully constructed spaces in order to obtain first order optimality conditions for the problem (1).

Our aim is to weaken the following assumptions, which are typically present in the analysis of state constrained problems: continuity of the mapping $S := A^{-1}B : U \to Y$ together with the continuous embedding $Y \hookrightarrow C(Q)$. For large classes of optimal control problems these conditions to not hold simultaneously, which impedes a direct application of Lagrange multiplier theorems. Particular examples are elliptic boundary control problems in three space dimensions and parabolic control problems in spacial dimensions higher than one.

Rather we impose the following much weaker assumptions, which capture the fact that most partial differential equations admit more regular solutions, if the data is more regular. The operators A_∞ and B_∞ introduced in the following can be interpreted as *restrictions* of A and B to more regular spaces.

Assumption 2.2 (Assumptions for optimality conditions)**.** Let the following assumptions hold.

(i) U_∞, Y_∞, and R_∞ are Banach spaces. Y_∞ is a closed subspace of $C(Q)$.

(ii) The linear operator $A_\infty : Y_\infty \supset \operatorname{dom} A_\infty \to R_\infty$ is closed, densely defined, and bijective, and the linear operator $B_\infty : U_\infty \to R_\infty$ is continuous.

(iii) There exist continuous embeddings $Y_\infty \hookrightarrow Y$, $U_\infty \hookrightarrow U$, and $R_\infty \hookrightarrow R$. The inclusion $\operatorname{dom} A_\infty \subset \operatorname{dom} A$ holds.

(iv) A_∞ and A coincide on $\operatorname{dom} A_\infty$. B_∞ and B coincide on U_∞.

(v) There is $\varepsilon > 0$ and $(\breve{u}, \breve{y}) \in V \cap G$, such that $\breve{y} + \tilde{y} \in G$ for all $\tilde{y} \in Y_\infty$ with $\|\tilde{y}\|_{Y_\infty} \leq \varepsilon$ (Slater condition).

Under these assumptions we can already derive a weak form of optimality conditions (cf. Theorem 5.1). However, a fully satisfactory theory (cf. Theorem 5.3) can only be derived under the additional crucial assumption that U_∞ is dense in U, and hence *enough regular* solutions are present. In the context of optimal control this is not a severe restriction.

Meaningful choices of Y_∞ are $C(Q)$ and subspaces of functions that vanish on subsets of Q. For example, $C_0(Q)$ (the space of all continuous functions that vanish on the boundary) is useful to accommodate Dirichlet boundary conditions in the elliptic setting. The dual space Y_∞^* can be represented as a subspace of the space of Radon measures, which we denote by $M(Q)$.

Unbounded Operators in Optimal Control. Most authors use the continuous control to state mapping $S := A^{-1}B$ as one of their main tools for the analysis of optimal control problems. In this work the operator A is used directly. Because A is usually a differential operator in applications, we do not assume continuity of A, but only closedness and a dense domain.

We use this approach for several reasons. Usually, optimal control problems and their first order optimality conditions are formulated in terms of differential operators. The use of solution operators S and their adjoints S^* makes two translation steps necessary. In contrast, the adjoints A^* and B^* typically have an immediate interpretation as differential and trace or embedding operators. Moreover, the direct use of differential operators makes the role of the dual space R^*, in which adjoint states live evident and allows a very transparent derivation of optimality conditions by a convex or non-smooth sum rule. Last, but not least, there are settings (such as pure Neumann problems, not treated in this work), where no solution operator exists, but the corresponding optimal control problem is well posed and can be analysed successfully as a system.

For a comprehensive exposition to unbounded operators on normed spaces we refer to [8]. In contrast to continuous operators an unbounded operator

$$A : Y \supset \operatorname{dom} A \to R$$

is typically not defined everywhere in its domain space Y, but possess a *domain* of definition $\operatorname{dom} A$. A is called *closed*, if $\operatorname{dom} A \supset y_k \to y$ and $Ay_k \to r$ imply $Ay \in \operatorname{dom} A$ and $Ay = r$. If an unbounded operator is closed, then this is a sign that its domain has been chosen appropriately. If A is closed and $\operatorname{dom} A = Y$, then A is continuous. Closed, densely defined operators retain many properties of continuous operators, such as an open mapping theorem, existence of an adjoint operator and a closed range theorem. By dropping the assumption of continuity we gain additional flexibility, because the topologies of domain and image space can be varied in a wide range.

If A is a differential operator, then it can usually be defined densely, because a suitable subspace of $C^\infty(Q)$ is dense in most spaces of interest. If additionally the corresponding partial differential equation is uniquely solvable, together with an a-priori estimate, then this operator is closed and vice versa, as asserted in the following lemma.

Lemma 2.3. *For Banach spaces Y and R let $A : Y \supset \operatorname{dom} A \to R$ be a linear operator. A is closed and bijective if and only if A possesses a continuous inverse $A^{-1} : R \to \operatorname{dom} A \subset Y$ in the sense that $A^{-1}A = id_{\operatorname{dom} A}$ and $AA^{-1} = id_R$.*

Proof. Assume first that a continuous inverse A^{-1} exists. Then in particular A is bijective. Let $y_k \to y$ and $r_k = Ay_k \to r$. By surjectivity of A there is $\tilde{y} \in \operatorname{dom} A$: $A\tilde{y} = r$, hence $Ay_k \to A\tilde{y}$. We have to show $y = \tilde{y}$. Because A^{-1} is continuous, we conclude $y_k = A^{-1}Ay_k \to A^{-1}A\tilde{y} = \tilde{y}$, hence $y = \tilde{y}$.

If in converse A is closed and bijective, then existence of a continuous inverse follows from the open mapping theorem (cf. e.g.[13, Satz IV.4.4]), which not only holds for continuous, but also for closed operators. □

Corollary 2.4. *If the Assumptions 2.1 and 2.2 hold, then the control to state mappings $S := A^{-1}B : U \to \operatorname{dom} A \subset Y$ and $S_\infty := A_\infty^{-1}B_\infty : U_\infty \to \operatorname{dom} A_\infty \subset Y_\infty$ are continuous and coincide on U_∞.*

Proof. By our assumptions and Lemma 2.3 A^{-1} and A_∞^{-1} exist and are continuous. By assumption B and B_∞ are also continuous, and our continuity assertion follows. By Assumption 2.2(iv) we have $Bu = B_\infty u$ for $u \in U_\infty$ and thus $y := S_\infty u \in \operatorname{dom} A_\infty$. Hence $A_\infty y = Ay$ and thus also $Ay - Bu = 0$, hence $y = A^{-1}Bu$. □

Let us finally recapitulate the definition of the adjoint of a densely defined operator $A : Y \supset \operatorname{dom} A \to R$, which generalizes the adjoint of a continuous operator. Define

$$\operatorname{dom} A^* := \{r^* \in R^* : \text{ the linear functional } \langle r^*, A\cdot\rangle \text{ is continuous on } \operatorname{dom} A\}.$$

If $r^* \in \operatorname{dom} A^*$, then $\langle r^*, A\cdot\rangle$ can be uniquely and continuously extended to a functional $y^* = A^*r^* \in Y^*$, because it is continuous on the dense subset $\operatorname{dom} A \subset Y$. This yields the definition of $A^* : R^* \supset \operatorname{dom} A^* \to Y^*$, and the relation

$$\langle A^*r^*, y\rangle = \langle r^*, Ay\rangle \quad \forall\, y \in \operatorname{dom} A\, \forall\, r^* \in \operatorname{dom} A^*.$$

In particular, $\operatorname{dom} A^*$ is canonically defined and depends on the topology of Y and R.

3 A Variant of the Sum Rule of Convex Analysis

Our starting point and main tool is the following theorem of convex analysis, which is an existence result, because it asserts equality of sets:

Theorem 3.1. *Let X be a Banach space. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be convex and lower semi-continuous functions. If the regularity condition*

$$0 \in \operatorname{int}(\operatorname{dom} f - \operatorname{dom} g) \tag{4}$$

holds, then

$$\partial(f + g)(x) = \partial f(x) + \partial g(x) \qquad \forall\, x \in X. \tag{5}$$

Proof. This theorem can be found, for example, in [4, Theorem 4.3.3]. Actually, in [4, Theorem 4.3.3] a slightly weaker form of (4) is used: $0 \in \operatorname{core}(\operatorname{dom} f - \operatorname{dom} g)$. For a precise definition of $\operatorname{core}(S)$ of a set $S \subset X$ see [4, Section 4.1.3], where it is also shown that always $\operatorname{int}(S) \subset \operatorname{core}(S)$. $\square$

The condition (4) can be interpreted as follows: there is a ball in X, centered at 0, such that every element of X contained in this ball can be written as the difference of two elements of X which are in $\operatorname{dom} f$ and $\operatorname{dom} g$, respectively.

By our assumptions $j(u, y) < \infty$ for all $(u, y) \in U \times Y$, hence, $\operatorname{dom} j = U \times Y$, and (4) is fulfilled, because $V \cap G$ is non-empty. Thus application of Theorem 3.1 to (3) yields:

$$0 \in \partial j(x_*) + \partial\left(\iota_V(x_*) + \iota_G(x_*)\right) \quad \text{in} \quad (U \times Y)^*. \tag{6}$$

The difficult part of the analysis of state equations is now to show (4) for ι_V and ι_G and thus split the remaining term $\partial\left(\iota_V(x_*) + \iota_G(x_*)\right)$.

We will now consider the case where f is the indicator function of a closed subspace. Let V and W be closed subspaces of X that span X topologically, i.e., for each $x \in X$ there are $x_V \in V$ and $x_W \in W$ such that $x = x_V + x_W$ and there is a constant, independent of x, such that $\|x\| \leq C(\|x_V\| + \|x_W\|)$.

Corollary 3.2. *Let V and W be closed subspaces of the Banach space X that span X topologically. Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be convex and lower semi-continuous on X. Assume that there is $\breve{x} \in V \cap \operatorname{dom} g$ and $\varepsilon > 0$, such that*

$$\breve{x} + \varepsilon B_W \subset \operatorname{dom} g.$$

Then

$$\partial(\iota_V + g)(x) = \partial\iota_V(x) + \partial g(x) \qquad \forall\, x \in X. \tag{7}$$

Proof. Let $x \in X$ be given with $\|x\| = 1$. Then we can write $x = x_V + x_W$ with $x_V \in V$ and $x_W \in W$. Because $\breve{x} + \varepsilon B_W \in \operatorname{dom} g$, there is $\varepsilon > 0$, independent of x, such that $x_g := \breve{x} - \varepsilon x_W \in \operatorname{dom} g$. Moreover, since $\breve{x} \in V$, also $x_{\iota_V} := \breve{x} + \varepsilon x_V \in \operatorname{dom} \iota_V$. Consequently, $\varepsilon x = x_{\iota_V} - x_g = \varepsilon(x_V + x_W)$, and (4) is fulfilled. Hence (5) holds and we conclude (7). $\square$

If V is defined via the kernel of an operator, then its subdifferential can be computed as follows:

Lemma 3.3. *Let X, R be Banach spaces and $T : X \supset \operatorname{dom} T \to R$ a closed, densely defined, linear operator. Denote by $\iota_{\ker T}$ the indicator function of $\ker T$, which is closed. If T has closed range then*

$$\partial\iota_{\ker T}(x) = \operatorname{ran} T^* \quad \forall x \in \ker T. \tag{8}$$

Proof. Since, by definition of the subdifferential, $\partial\iota_{\ker T}(x) = (\ker T)^\perp$, (8) is a consequence of the *closed range theorem*:

$$(\ker T)^\perp = \operatorname{ran} T^*.$$

In standard texts on functional analysis this theorem is usually stated for continuous operators (cf. e.g. [13, Theorem IV.5.1]), but the same result (with the same proof, based on the open mapping theorem) works for closed, densely defined operators on Banach spaces (cf. [8, Theorem IV.1.2]). $\square$

4 An Auxiliary Banach Space and its Properties

In order to apply our abstract results to the optimal control problem (1) we introduce the auxiliary space X by the following definitions. Here and during the whole section the Assumptions 2.1 and 2.2 are assumed to hold.

$$\begin{aligned}
V &:= \left\{(u^V, y^V) \in U \times \operatorname{dom} A : Ay^V - Bu^V = 0 \text{ in } R\right\} \\
W &:= \left\{(u^W, y^W) \in U \times Y : u^W = 0,\, y^W \in Y_\infty\right\} \\
X &:= V + W = \left\{x \in U \times Y : x = x^V + x^W, x^V \in V, x^W \in W\right\} \\
\|v\|_V &:= \left\|u^V\right\|_U, \quad \|w\|_W := \left\|y^W\right\|_{Y_\infty} \\
\|x\|_X &:= \|u^V\|_U + \|y^W\|_{Y_\infty}.
\end{aligned}$$

The space X is continuously embedded into $U \times Y$ via the embedding

$$\begin{aligned}
E : X &\to U \times Y \\
((u^V, y^V), y^W) &\mapsto (u, y) := (u^V, y^V + y^W),
\end{aligned}$$

which is continuous, because by Corollary 2.4

$$\|u\|_U + \|y\|_Y \leq \left\|u^V\right\|_U + \left\|y^V\right\|_Y + \left\|y^W\right\|_Y \leq (1 + \|S\|) \left\|u^V\right\|_U + \left\|y^W\right\|_{Y_\infty}.$$

Hence, the optimal control problem (1) is well defined on X. Moreover, the minimizer of (1) is contained in V, and thus in X. It will turn out that the space X is well suited for the application of the sum-rule via Corollary 3.2.

Proposition 4.1. *X is a Banach space. V and W are closed in X, and they span X topologically. Moreover, $U_\infty \times Y_\infty$ is continuously embedded into X.*

Proof. Let x_k be a Cauchy sequence in X. Then, first of all y_k^W converges in Y_∞ by completeness. Hence, W is closed in X. Moreover, u_k^V converges in U by completeness. Consequently Bu_k^V converges in R by continuity, and thus also Ay_k^V. By our assumptions, and by Lemma 2.3 the control to state mapping $A^{-1}B$ is continuous, and thus y_k^V converges in Y. So $(u_k^V, y_k^V) \to (u^V, y^V)$ in $U \times Y$. By continuity of B and closedness of A also $(u^V, y^V) \in V$, which shows completeness of X and closedness of V. X is spanned topologically by V and W by construction.

For our last assertion define the embedding

$$\begin{aligned}
E_\infty : U_\infty \times Y_\infty &\to X \\
(u, y) &\mapsto ((u^V, y^V), y^W),
\end{aligned} \tag{9}$$

via the definitions $u^V := u$, $y^V := Su$, $y^W := y - y^V$. By Corollary 2.4 we conclude $y^V = Su = S_\infty u$ and thus $y^V \in Y_\infty$, which also yields $y^W \in Y_\infty$. E_∞ is continuous, because

$$\left\|y^W\right\|_{Y_\infty} + \left\|u^V\right\|_U \le \|y\|_{Y_\infty} + \left\|y^V\right\|_{Y_\infty} + \|u\|_{U_\infty} \le \|y\|_{Y_\infty} + (1 + \|S_\infty\|)\,\|u\|_{U_\infty}\,.$$

$\square$

From the continuity of the embedding E_∞ it follows that each linear functional $x^* \in X^*$ has a continuous restriction $r^* \in U_\infty \times Y_\infty$ via $r^* = E_\infty^* x^*$.

Next we study on X the linear operator that defines the state equation. By construction of V it vanishes there.

Lemma 4.2. *Define* $\operatorname{dom} T_\infty := \left\{x \in X : (u^V, y^V) \in V,\, y^W \in \operatorname{dom} A_\infty\right\}$.

(i) The linear operator

$$\begin{aligned} T_\infty : X \supset \operatorname{dom} T_\infty &\to R_\infty \\ x &\mapsto A_\infty y^W. \end{aligned}$$

is densely defined, closed and surjective and $V = \ker T_\infty$.

(ii) Its adjoint operator $T_\infty^* : R_\infty^* \supset \operatorname{dom} T_\infty^* \to X^*$ *is given by* $\operatorname{dom} T_\infty^* = \operatorname{dom} A_\infty^*$ *and*

$$\langle T_\infty^* p, x\rangle = \langle A_\infty^* p, y^W\rangle \quad \forall p \in \operatorname{dom} T_\infty^* \quad \forall x \in X. \tag{10}$$

(iii) The restriction of T_∞ *to* $U_\infty \times Y_\infty$ *is given by*

$$T_\infty x = A_\infty y - B_\infty u \quad \forall\, x = (u, y) \in U_\infty \times \operatorname{dom} A_\infty. \tag{11}$$

Moreover, $T_\infty^* p|_{U_\infty \times Y_\infty} = (-B_\infty^* p, A_\infty^* p)$ *for each* $p \in \operatorname{dom} T_\infty^*$.

Proof. (i): Density of $\operatorname{dom} T_\infty$ in X follows from density of $\operatorname{dom} A_\infty$ in Y_∞. Surjectivity of T_∞ follows from surjectivity of A_∞, and closedness of T_∞ follows from closedness of the space V and the operator A_∞. By injectivity of A_∞ we conclude $V = \ker T_\infty$.

(ii): Let $p \in \operatorname{dom} T_\infty^*$, which means that $T_\infty^* p$ is a continuous linear functional on X and

$$\langle T_\infty^* p, x\rangle = \langle p, A_\infty y^W\rangle \quad \forall\, x \in \operatorname{dom} T_\infty$$

The first equality gives us (10) for all $x \in \operatorname{dom} T_\infty$. In particular $T_\infty^* p$ is continuous on $\operatorname{dom} T_\infty$, if and only if $A_\infty^* p$ is continuous on $\operatorname{dom} A_\infty$, which yields equality of $\operatorname{dom} T_\infty^*$ and $\operatorname{dom} A_\infty^*$ and (10) for all $x \in X$ by unique continuous extension.

(iii): For $(u, y) \in U_\infty \times \operatorname{dom} A_\infty$ let $\hat{x} := E_\infty(u, y)$, defined by (9). As we have seen, $(\hat{u}^V, \hat{y}^V) \in U_\infty \times \operatorname{dom} A_\infty$ and $\hat{y}^W \in \operatorname{dom} A_\infty$. Then, because $Ay = A_\infty y$ on $\operatorname{dom} A_\infty$ and $Bu = B_\infty u$ on U_∞

$$T_\infty \hat{x} = A_\infty \hat{y}^W = A_\infty \hat{y}^W + A_\infty \hat{y}^V - B_\infty \hat{u}^V = A_\infty y - B_\infty u.$$

Further, for $p \in \operatorname{dom} T_\infty^*$ by (11)

$$\langle T_\infty^* p, \hat{x}\rangle = \langle p, A_\infty(\hat{y}^W + \hat{y}^V) - B_\infty \hat{u}^V\rangle = \langle A_\infty^* p, \hat{y}^W + \hat{y}^V\rangle - \langle B_\infty^* p, \hat{u}^V\rangle.$$

Because $p \in \operatorname{dom} A_\infty^*$, this relation extends continuously and uniquely to $\hat{y}^W \in Y_\infty$. Let now $(u, y) \in U_\infty \times Y_\infty$ be arbitrary. Then there is $x \in X$, such that $u = u^V$ and $y = y^V + y^W$, which yields

$$\langle T_\infty^* p, x\rangle = \langle A_\infty^* p, y\rangle - \langle B_\infty^* p, u\rangle.$$

□

Next we study the subdifferential that corresponds to the inequality constraints on X. Note that our inequality constraints read $y^V + y^W \geq \underline{y}$, and thus imply a hidden restriction on u^V via $(u^V, y^V) \in V$, but still define a closed convex set in X.

Lemma 4.3. *Let $m^* \in \partial\iota_G(x)$, where $x \in G$. Let $\delta x \in X$, and $\delta y := \delta y^V + \delta y^W$. Then it follows:*

$$\langle m^*, \delta x\rangle \leq 0 \quad \text{if} \quad \delta y \geq 0 \tag{12}$$

$$\langle m^*, \delta x\rangle = 0 \quad \text{if} \quad \delta y = 0 \tag{13}$$

$$\langle m^*, \delta x\rangle = 0 \quad \text{if} \quad \delta y = \underline{y} - y. \tag{14}$$

The restriction of m^* to $U_\infty \times Y_\infty$ is of the form

$$\langle m^*, \delta x\rangle = \langle m_y^*, \delta y\rangle$$

and $m_y^* \in Y_\infty^*$. If in converse m^* satisfies (12) *and* (14)*, then $m^* \in \partial\iota_G$.*

Proof. By definition $\iota_G(u, y) = 0$, if $y^W + y^V \geq \underline{y}$. Hence, if $y + \delta y \geq \underline{y}$, then $\iota_G(u + \delta u, y + \delta y) = 0$, and $+\infty$ otherwise. Now all asserted equalities and inequalities follow from the definition of the subdifferential:

$$\langle m^*, \delta x\rangle \leq \iota_G(x + \delta x) - \iota_G(x) \tag{15}$$

Representation of the restriction of m^* to $U_\infty \times Y_\infty$ as an element of Y_∞^* follows from (13), if we insert $\delta x = (\delta u, 0)$ with arbitrary $\delta u \in U_\infty$.

For the converse let $\delta x \in X$ be arbitrary. If $x + \delta x \notin G$, then $\iota_G(x + \delta x) = \infty$, and (15) holds trivially. Otherwise $\iota_G(x + \delta x) = 0$ and we have to show $\langle m^*, \delta x\rangle \geq 0$. But this follows from the splitting $\delta y = (\underline{y} - y) + \delta y_+$, where $\delta y_+ = \delta y + y - \underline{y} \geq 0$ and application of (14) and (12). □

5 Abstract First Order Optimality Conditions

In this section we prove our main results. Our first theorem provides us with a weak form of optimality conditions. Under an additional density assumption we can improve this.

Theorem 5.1. *Suppose that the Assumptions 2.1 and 2.2 hold. Let x_* be the minimizer of* (1). *Then the following system of equations has a solution (m^*, p):*

$$\langle j_y(x_*), \delta y\rangle + \langle A^*_\infty p, \delta y\rangle + \langle m^*, \delta y\rangle = 0 \quad \forall\, \delta y \in Y_\infty \tag{16}$$

$$\langle j_u(x_*), \delta u\rangle - \langle B^*_\infty p, \delta u\rangle = 0 \quad \forall\, \delta u \in U_\infty \tag{17}$$

$$\langle m^*, \delta y\rangle \le 0 \quad \forall\, 0 \le \delta y \in Y_\infty \tag{18}$$

$$\langle m^*, (u^* - \delta u, y_* - \underline{y})\rangle = 0 \quad \forall\, \delta u \in U_\infty. \tag{19}$$

Here $p \in \operatorname{dom} A^*_\infty \subset R^*_\infty$ and $m^* \in X^*$, whose restriction to $U_\infty \times Y_\infty$ has a representation as an element of Y^*_∞.

Proof. By Assumption 2.2(v) there is $\breve{x} \in V$ with $\breve{y}^V + \tilde{y} \ge \underline{y}$ for all $\tilde{y} \in Y_\infty$ with $\|\tilde{y}\| \le \varepsilon$. Hence $\breve{x} + \varepsilon B_W \in G = \operatorname{dom} \iota_G$. By Proposition 4.1, X is a Banach space, spanned topologically by its closed subspaces V and W. Hence, we can apply Proposition 3.2 to the function $\iota_V + \iota_G$ and conclude via (6) that

$$0 \in \partial j(x_*) + \partial\iota_V(x_*) + \partial\iota_G(x_*) \quad \text{in } X^*, \tag{20}$$

where $\partial j(x_*) \subset (U \times Y)^* \subset X^*$. Because j is Gâteaux differentiable, its subdifferential is single valued, and we may write $\partial j(x_*) = j' = (j_u, j_y)$. Then (20) asserts that there are $m^* \in \partial\iota_G(x_*)$ and $v^* \in \partial\iota_V(x_*)$, such that

$$0 = \langle j' + v^* + m^*, \delta x\rangle \quad \forall \delta x \in X. \tag{21}$$

By Lemma 3.3 there is $p \in R^*_\infty$, such that $v^* = T^*_\infty p$.

Now let $\delta x = (\delta u, 0)$ with $\delta u \in U_\infty$ we obtain (17) due to Lemma 4.3 (asserting $m^* = 0$ if $\delta y = 0$), and Lemma 4.2*(iii)*. Setting $\delta x = (0, \delta y)$ with $\delta y \in Y_\infty$ we obtain (16) from (21) and Lemma 4.2*(iii)*. By Lemma 4.3 m^* satisfies (18) and (19). □

So far the optimality conditions are not yet complete, because a complementarity condition is not perfectly fulfilled. This is not possible in X, because X does not contain all pairs of functions of the form (y_*, u) with $u \in U$. Moreover, the optimality conditions of Theorem 5.1 are only necessary, but not sufficient, as one should expect in the convex case. For a fully satisfactory result we have to introduce additional spaces. Consider first

$$Y_U := \{y \in \operatorname{dom} A \subset Y : \exists u \in U : Ay - Bu = 0\}.$$

Y_U is the range of $S = A^{-1}B$ and thus the set of all states that can be reached by a control in U. Since we do not assume injectivity of B, u in the above definition may be non-unique, but because S is continuous, the set of all these u forms a closed affine subspace K_y of U. Because U is reflexive and norms are coercive functionals, there is an $u_m(y) \in K_y$ with minimal norm $\|\cdot\|_U$. Hence, Y_U can be equipped with the norm

$$\|y\|_{Y_U} := \|u_m(y)\|_U.$$

By continuity of S, Y_U is continuously embedded into Y.

Finally, for the formulation of our results we define the space $\tilde{Y}$ given by

$$\tilde{Y} = \{y \in Y : \exists y_C \in Y_\infty, y_U \in Y_U : y = y_C + y_U\}, \tag{22}$$

equipped with the norm

$$\|y\|_{\tilde{Y}} = \inf_{y=y_C+y_U} \left(\|y_C\|_{Y_\infty} + \|y_U\|_{Y_U}\right).$$

It is easily verified, that $\|y\|_{\tilde{Y}}$ is indeed a norm. In particular, the triangle inequality follows from the triangle inequalities of the component norms. Moreover, $\tilde{Y}$ is continuously embedded into Y, because for any splitting $y = y_C + y_U$

$$\|y_C + y_U\|_Y \le \|y_C\|_Y + \|y_U\|_Y \le \|y_C\|_{Y_\infty} + \|S\| \, \|y_U\|_{Y_U}.$$

Elements of $\tilde{Y}^*$ are necessarily continuous on Y_∞ and on Y_U. Hence the are the continuous embeddings $\tilde{Y}^* \hookrightarrow Y_\infty^*$ and $\tilde{Y}^* \hookrightarrow Y_U^*$.

Next, consider a redefinition of A_∞ on $\tilde{Y}$, which we call $\tilde{A}$. We define

$$\operatorname{dom} \tilde{A} := \{y \in \tilde{Y} : y_C \in \operatorname{dom} A_\infty, u_m(y_U) \in U_\infty\}.$$

Because $u_m(y_U) \in U_\infty$ we conclude $y_U \in \operatorname{dom} A_\infty$ and thus $y = y_C + y_U \in \operatorname{dom} A_\infty$. Hence, the operator

$$\begin{aligned} \tilde{A} : \tilde{Y} \supset \operatorname{dom} \tilde{A} &\to R_\infty \\ y &\mapsto A_\infty y \end{aligned}$$

is well defined. To be able to define an adjoint operator of $\tilde{A}$ we have to assert that $\tilde{A}$ is densely defined. For this we have to impose an additional assumption, which is *crucial* for the complete derivation of first order optimality conditions:

Assumption 5.2. The continuous embedding $U_\infty \hookrightarrow U$ is *dense*.

Under this assumption $\operatorname{dom} \tilde{A}$ is dense in $\tilde{Y}$, because $\operatorname{dom} A_\infty$ is dense in Y_∞, and the set of all y_U with $u_m(y_U) \in U_\infty$ is dense in Y_U by definition of $\|\cdot\|_{Y_U}$.

For a redefinition of B_∞ on U, we define $\operatorname{dom} \tilde{B} := U_\infty$, which is dense in U by Assumption 5.2, and

$$\begin{aligned} \tilde{B} : U \supset \operatorname{dom} \tilde{B} &\to R_\infty \\ u &\mapsto B_\infty u. \end{aligned}$$

Theorem 5.3. *Suppose that the Assumptions 2.1 and 2.2 hold. Additionally suppose that Assumption 5.2 holds. Let $x_* = (y_*, u_*)$ be the minimizer of (1). Then the following system of equations has a solution (m, p):*

$$\langle j_y(x_*), \delta y\rangle + \langle m, \delta y\rangle + \langle \tilde{A}^* p, \delta y\rangle = 0 \quad \forall\, \delta y \in \tilde{Y} \tag{23}$$

$$\langle j_u(x_*), \delta u\rangle - \langle \tilde{B}^* p, \delta u\rangle = 0 \quad \forall\, \delta u \in U \tag{24}$$

$$\langle m, \delta y\rangle \le 0 \quad \forall\, 0 \le \delta y \in \tilde{Y} \tag{25}$$

$$\langle m, y_* - \underline{y}\rangle = 0. \tag{26}$$

Here $m \in \tilde{Y}^*$ and $p \in \operatorname{dom} \tilde{B}^* \cap \operatorname{dom} \tilde{A}^*$. In particular, m has a representation as a positive measure on Q with additional regularity properties.

If in converse x_* is feasible and (23)-(26) *has a solution, then x_* is a minimizer of* (1).

Proof. By Theorem 5.1 there is $p \in \operatorname{dom} A_\infty^*$, such that (16) and (17) hold. Consider (17) first, which yields:

$$\langle j_u(x_*), \delta u\rangle = \langle B_\infty^* p, \delta u\rangle = \langle p, B_\infty \delta u\rangle = \langle p, \tilde{B}\delta u\rangle \quad \forall \delta u \in U_\infty \subset U.$$

Because $j_u(x_*) \in U^*$, also the linear functional $\langle p, \tilde{B}\cdot\rangle$ is continuous on $U_\infty = \operatorname{dom} \tilde{B} \subset U$, and hence $p \in \operatorname{dom} \tilde{B}^*$. Thus, $\tilde{B}^* p$ is well defined as an element of U^*.

Let now $\delta y = \delta y_C + \delta y_U \in \operatorname{dom} \tilde{A}$. Then $\delta y, \delta y_C, \delta y_U \in \operatorname{dom} A_\infty$, and we have

$$\langle p, \tilde{A}y\rangle = \langle p, A_\infty y_C\rangle + \langle p, A_\infty y_U\rangle$$

with

$$\langle p, A_\infty y_U\rangle = \langle p, B_\infty u_m(y_U)\rangle = \langle p, \tilde{B} u_m(y_U)\rangle = \langle \tilde{B}^* p, u_m(y_U)\rangle.$$

Because $A_\infty^* p \in Y_\infty^*$ by Theorem 5.1 and $\tilde{B}^* p \in U^*$ as shown above,

$$|\langle p, \tilde{A}y\rangle| \le \|A_\infty^* p\|_{Y_\infty^*} \|y_C\|_{Y_\infty} + \left\|\tilde{B}^* p\right\|_{U^*} \|y_U\|_{Y_U}.$$

Hence, the linear functional $\langle p, \tilde{A}\cdot\rangle$ is continuous on $\operatorname{dom} \tilde{A}$. Thus, $p \in \operatorname{dom} \tilde{A}^*$ and $\tilde{A}^* p \in \tilde{Y}^*$ is well defined.

Because also $j_y \in Y^*$, and $\tilde{Y} \hookrightarrow Y$ is continuous, $j_y \in \tilde{Y}^*$, and hence $m^* \in X^*$ has a unique continuous extension to $\tilde{Y} \times U$ that satisfies (23) and vanishes on U, which we call m. Now (25) and (26) follow from continuity of m and from (18) and (19), respectively.

To show sufficiency of our conditions we note that (16)-(19) follow from (23)-(26) and that X contains all feasible solutions of (1). Hence, a minimizer of F in X is also a minimizer of F in $U \times Y$.

Let now $x_* \in X$ be feasible and (16)-(19) have a solution (m^*, p). Clearly, $j' \in \partial j(x_*)$. Next, by (18) and (19) m^* satisfies (12) and (14) and hence $m^* \in \partial \iota_G(x_*)$.

Let $v^* := (-B_\infty^* p, A_\infty^* p)$ which means that

$$\langle v^*, x\rangle = \langle p, A_\infty y^V - B_\infty u^V\rangle + \langle p, A_\infty y^W\rangle \qquad \begin{array}{l} \forall (u^V, y^V) \in (U_\infty \times \operatorname{dom} A_\infty) \cap V, \\ \forall y^W \in \operatorname{dom} A_\infty, \end{array}$$

and $A_\infty y^V - B_\infty u^V = 0$. Because U_∞ is dense in U, $(U_\infty \times \operatorname{dom} A_\infty) \cap V$ is dense in V and we conclude $Ay^V - Bu^V = 0$ for all $(u^V, y^V) \in V$. Hence, $v^* = A_\infty^* p = T_\infty^* p$, and thus by Lemma 3.3 $v^* \in \partial \iota_V(x_*)$. It follows that (20) holds, and by the reverse direction of the sum-rule, $0 \in \partial F(x_*)$. This is equivalent to optimality of x_* in X and thus in $U \times Y$. ☐

6 State Constrained Elliptic Boundary Control

As an example and illustration of our abstract results we consider the following optimal control problem on a smoothly bounded domain $\Omega \subset \mathbb{R}^3$:

$$\min_{(u,y)\in U\times Y} j(y,u) = \|y-y_d\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|u\|^2_{L^2(\Gamma)} \tag{27}$$

subject to the state equation

$$\begin{aligned} -\Delta y + y &= 0 \qquad && \text{in } \Omega \\ \partial y &= u && \text{on } \Gamma \end{aligned} \tag{28}$$

and the pointwise state constraint

$$\underline{y} \le y \quad \text{ a.e. in } \Omega. \tag{29}$$

We assume that $\underline{y}$ is continuous and that there is $(\breve{u}, \breve{y})$ that satisfies the state equation, and $\breve{y} - \underline{y} \geq \varepsilon > 0$ a.e..

A class of control problems similar to our example has been analysed in [6, 2], however with the assumption that $u_* \in L_p(\Gamma)$ for some $p > 2$ and $y_* \in C(\overline{\Omega})$.

The state equation reads in the weak formulation:

$$\langle \nabla y, \nabla v\rangle_{L_2(\Omega)} + \langle y, v\rangle_{L_2(\Omega)} = \langle u, v\rangle_{L_2(\Gamma)} \quad \forall v \in H^1(\Omega).$$

Let us now define the spaces and operators needed to apply our general theory. In view of Assumption 2.1 we define

- $U := L_2(\Gamma)$, $Y := L_2(\Omega)$, $R := (H^1(\Omega))^*$.
- $A : L_2(\Omega) \supset H^1(\Omega) \to H^1(\Omega)^*$, $Ay := \langle \nabla y, \nabla \cdot\rangle_{L_2(\Omega)} + \langle y, \cdot\rangle_{L_2(\Omega)}$. The Lax-Milgram Lemma asserts closedness and bijectivity of A, and because $H^1(\Omega)$ is dense in $L_2(\Omega)$ it is also densely defined.
- $B : L_2(\Gamma) \to H^1(\Omega)^*$, $Bu := \langle u, \gamma(\cdot)\rangle_{L_2(\Gamma)}$. Here continuity of the trace operator γ asserts continuity of B.

Clearly, j satisfies Assumption 2.1(v) and is even strictly convex. All other parts of Assumption 2.1 have already been verified or imposed and we include the existence of a unique minimizer x_*.

Apart from this standard setting we have to define a suitable more regular setting that obeys Assumption 2.2, where the states are continuous. For this we need advanced regularity results for our equation. Fortunately, these results are available in the literature. By the Sobolev embedding theorem $W^{1,t} \hookrightarrow C(\overline{\Omega})$ for every $t > 3$ (cf. [1]). By [3, Theorem 2.3] our weak formulation defines an isomorphism $W^{1,t}(\Omega) \leftrightarrow (W^{1,s}(\Omega))^*$ for $1 = t^{-1} + s^{-1}$, thus $s < 3/2$. This result is available for a general class of elliptic PDEs, as long as the coefficients are smooth enough. With this information we define:

- $Y_\infty := C(\overline{\Omega})$, $R_\infty := (W^{1,s}(\Omega))^*$, $U_\infty := L_p(\Gamma)$ with $p > 2$ chosen such that the trace operator $\gamma : W^{1,s}(\Omega) \to L_q(\Gamma)$ is continuous for $q^{-1} = 1-p^{-1}$. With these definitions $R_\infty \hookrightarrow R$, $Y_\infty \hookrightarrow Y$ are continuous, and $U_\infty \hookrightarrow U$ is dense.

- $A_\infty : C(\overline{\Omega}) \supset W^{1,t}(\Omega) \to (W^{1,s}(\Omega))^*$, $Ay := \langle \nabla y, \nabla \cdot \rangle_{L_t(\Omega)\times L_s(\Omega)} + \langle y, \cdot \rangle_{L_2(\Omega)}$. Clearly, A_∞ is a restriction of A. The regularity result in [3, Theorem 2.3] yields closedness and bijectivity of A_∞. Density of $W^{1,t}(\Omega)$ in $C(\overline{\Omega})$ yields that A_∞ is densely defined.

- $B_\infty : L_p(\Gamma) \to (W^{1,s}(\Omega))^*$, $B_\infty u := \langle u, \gamma(\cdot)\rangle_{L_p(\Gamma)\times L_q(\Gamma)}$. By our choice of p the trace operator $\gamma : W^{1,s}(\Omega) \to L_q(\Gamma)$ is continuous and thus also B_∞.

The test space $\tilde{Y}$ is defined just as in (22), and contains all functions that are the sum of a continuous function on $\overline{\Omega}$ and solutions of (28) for any $u \in L_2(\Gamma)$. This implies that $\tilde{y} \in \tilde{Y}$ may have a discontinuous trace on Γ, and thus, for example, the Dirac measure $\delta_x \in M(\overline{\Omega})$ with $x \in \Gamma$ is not a representation of an element of $\tilde{Y}^*$.

Theorem 6.1. *Let (u_*, y_*) be the optimal solution of the problem (27)-(29). Then there exist $dm \in M(\overline{\Omega})$ and $p \in W^{1,s}(\Omega)$, which satisfy*

$$\int_\Omega \varphi_y \cdot (y_* - y_d)\, dt + \int_\Omega \langle \nabla \varphi_y, \nabla p\rangle + \varphi_y \cdot p\, dt + \int_{\overline{\Omega}} \varphi_y\, dm = 0 \quad \forall\, \varphi_y \in \tilde{Y} \tag{30}$$

$$\kappa u_* - \gamma(p) = 0 \quad a.e.\ in\ \Omega \tag{31}$$

$$\int_{\overline{\Omega}} \varphi_y\, dm \le 0 \quad \forall\, 0 \le \varphi_y \in \tilde{Y} \tag{32}$$

$$\int_{\overline{\Omega}} (y_* - \underline{y})\, dm = 0, \tag{33}$$

where all terms in (30)-(33) are well defined in the sense of unique continuous extension.

Here dm is the representation of $m \in \tilde{Y}^*$ as a measure with additional regularity properties. As regularity results we have $u_* = \kappa^{-1}\gamma(p) \in L_2(\Gamma) \cap W^{1-s,t}(\Gamma)$ and $p \in W^{1,s}(\Omega)$.

If in converse the system (30)-(33) is solvable for given feasible (u_*, y_*), then (u_*, y_*) is the unique optimal solution of the problem (27)-(29).

Proof. We have already asserted that the Assumptions 2.1, 2.2, and 5.2 are valid, and we can thus apply Theorem 5.3. This yields existence of p and m that satisfy the equations (23)-(26), the regularity results and sufficiency. The representations of the linear functional m and of the adjoint operators $\tilde{A}^*$ and $\tilde{B}^*$ are canonical.

In particular, by definition of $\tilde{Y}$ and the Riesz representation theorem, m has a representation as a measure, and hence $\int_{\overline{\Omega}} \varphi_y\, dm$ is well defined for all $\varphi_y \in C(\overline{\Omega})$. Theorem 5.3 asserts that this expression has a unique continuous extension to all $\varphi_y \in \tilde{Y}$, which yields (33) and (32).

As for $\tilde{A}^*$ we have

$$\langle \varphi_p, \tilde{A}\varphi_y\rangle = \int_\Omega \langle \nabla\varphi_y, \nabla\varphi_p\rangle + \varphi_y \cdot \varphi_p\, dt = \langle \tilde{A}^*\varphi_p, \varphi_y\rangle.$$

Strictly speaking, the middle expression is only defined for $\varphi_y, \varphi_p \in C^1(\overline{\Omega})$. It is well known that this expression can be extended canonically to $\varphi_y \in W^{1,t}(\Omega) = \operatorname{dom} A_\infty$ and $\varphi_p \in W^{1,s}(\Omega) = R^*_\infty$.

For the particular choice $\varphi_p = p$ Theorem 5.3 asserts that this expression even extends uniquely and continuously to all $\varphi_y \in \tilde{Y}$. This does, of course, not imply that $\nabla\varphi_y$ is well defined for all $\varphi_y \in \tilde{Y}$. Our assertion is, however, that the linear functional $\int_{\overline{\Omega}} \langle \nabla \cdot, \nabla p \rangle \, dt$, which is certainly defined on $W^{1,t}(\Omega) \hookrightarrow \tilde{Y}$ has a unique continuous extension onto $\tilde{Y}$. This yields (30).

The representation of $\tilde{B}^*$ is the following:

$$\langle \varphi_p, \tilde{B}\varphi_u \rangle = \int_\Gamma \varphi_u \cdot \gamma(\varphi_p)) \, ds = \langle \tilde{B}^* \varphi_p, \varphi_u \rangle.$$

It is defined for $\varphi_u \in L_p(\Gamma) = U_\infty$ with $p > 2$ and $\varphi_p \in W^{1,s}(\Omega) = R^*_\infty$. For the particular choice $\varphi_p = p$ Theorem 5.3 asserts that this expression extends to $\varphi_u \in L_2(\Gamma)$. Because in our case (24) is an equation in $L_2(\Gamma)^* = L_2(\Gamma)$ we can write it as a pointwise equation, which yields (31). □

7 Conclusion and Outlook

We have presented a new technique for the analysis of state constrained optimal control problems. It allows to derive first order optimality conditions for problems for which this was not yet possible. Our key was the construction of a suitable auxiliary space X, where the sum-rule of convex analysis could be applied. With this existence result it was possible to derive an abstract Lagrange multiplier rule in X^* that had to be carried over to more elementary dual spaces.

One particular result of our analysis is remarkable: not only that measure valued Lagrange multipliers exist, just as in the well known case of continuous states. The Lagrange multipliers are even more regular for less regular state equations. Because Lagrange multipliers reflect the sensitivity of the functional with respect to perturbations of the constraints, we can give a heuristic interpretation. The poor regularity properties of the state equation lead to some additional "flexibility" of the state. Hence, perturbations in the constraints can be compensated with relatively small effort for the control and thus for the functional.

While we have answered one question, many other theoretical and practical questions arise. First of all, the application of our results to various classes of optimal control problems may be explored systematically. In particular, the abstract assertion $dm \in \tilde{Y}^*$ can and should be used to deduce regularity results on the dual variables. Moreover, our results may be extended to more general inequality constraints. Finally, the extension to suitable classes of nonlinear problems remains to be done. This may either be completed by the use of non-convex variants of the sum-rule, or by a-priori linearization.

Equally important is the analysis of algorithms. It will be interesting to study the convergence behaviour of infeasible regularization methods (cf. e.g. [9, 10]) in the light of our new results, and it is very probable that barrier methods in function space (cf. e.g. [11]) can also be analysed in this setting.

Finally, the construction of discretization schemes and their analysis remains as a challenging topic, because L_∞-error estimates for the state will not be available in general.

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