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ZIB-Report 09-02 (January 2009)

Insensitivity bounds for the moments of the sojourn times in M/GI systems under state-dependent processor sharing

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Abstract

We consider a system with Poisson arrivals and i.i.d. service times and where the requests are served according to the state-dependent (Cohen's generalized) processor sharing discipline, where each request in the system receives a service capacity which depends on the actual number of requests in the system. For this system we derive asymptotically tight upper bounds for the moments of the conditional sojourn time of a request with given required service time. The bounds generalize corresponding results, recently given for the single-server processor sharing system by Cheung et al. and for the state-dependent processor sharing system with exponential service times by the authors. Analogous results hold for the waiting times.

Mathematics Subject Classification (MSC 2000): 60K25, 68M20, 90B22, 60E15, 60G10.

Keywords: Poisson arrivals; general service times; state-dependent processor sharing; Cohen's generalized processor sharing; conditional sojourn time; conditional waiting time; moments; bounds; insensitivity; many-server; $M/GI/m-PS$; Laplace-Stieltjes transform.

1 Introduction

Processor sharing (PS) systems have been widely used in the last decades for modeling and analyzing computer and communication systems, cf. e.g. [Kle], [CMT], [KY], [Ya3], [PG], [ZTK], [BBJ], [GRZ], [BB], and the references therein. In this paper we deal with insensitivity bounds for the moments of sojourn times of the following PS service system, denoted by $M/GI/SDPS$: At a node requests arrive according to a Poisson process of intensity λ with i.i.d. service times, which are independent of the arrival process and have the distribution function (df.) $B(x) := P(S \leq x)$, where S denotes a generic service time with finite mean $m_S := ES$. The requests are served according to the following state-dependent processor sharing (SDPS) discipline (Cohen's generalized processor sharing discipline[1]), cf. [Coh], [BBJ]: If there are $n \in \mathbb{N} := \{1, 2, \dots\}$ requests in the node then each of them receives a positive service capacity $\varphi(n)$, i.e., each of the n requests receives during an interval of length $\Delta\tau$ the amount $\varphi(n)\Delta\tau$ of service. In case of $\varphi_1(n) = 1/n$, $n \in \mathbb{N}$, we obtain the well known $M/GI/1-PS$ system (single-server system with egalitarian processor sharing discipline), cf. [CMT], [Ya2], [YY1]. In case of $\varphi_{1,k}(n) = 1/(n+k)$, $n \in \mathbb{N}$, we have a single-server PS system with $k \in \mathbb{N}$ permanent requests in the system, cf. [VB], in case of $\varphi_m(n) = \min(m/n, 1)$, $n \in \mathbb{N}$, an $M/GI/m-PS$ system, i.e. an m-server PS system, where all requests are served in a PS mode, but each request receives at most the capacity of one processor, cf. [Coh] p. 283, [Bra], [GRZ], in case of $\varphi_{m,k}(n) = \min(m/(n+k), 1)$, $n \in \mathbb{N}$, an m-server PS system with $k \in \mathbb{N}$ permanent requests. In case of $\varphi(n) = 1$, $n \in \mathbb{N}$, the system corresponds to an $M/GI/\infty$ system.

Networks with nodes working under the SDPS discipline are investigated in [Coh], [BP], [Zac], [BB]. In particular, for the $M/GI/SDPS$ system some basic results are known, cf. [Coh], which we will use and therefore shortly review in the following. Let $N(t)$ be the number of requests in the system at time t and $R^*(t) := (R_1^*(t), \dots, R_{N(t)}^*(t))$ be the vector of the residual service times of the $N(t)$ requests in the system at time t, ordered randomly. The vector process $(N(t); R^*(t))$, $t \in \mathbb{R}$, is a Markov process. The $M/GI/SDPS$ system is stable, i.e., there exists a unique stationary process $(N(t); R^*(t))$, $t \in \mathbb{R}$, if and only if

$$\sum_{n=0}^{\infty} \prod_{\ell=1}^{n} \frac{\varrho\chi(\ell)}{\ell} < \infty, \tag{1.1}$$

where $\chi(n) := 1/\varphi(n)$, $n \in \mathbb{N}$, and $\varrho := \lambda m_S$ denotes the offered load, cf. [Coh] (7.18). We assume in the following that the system is stable and in steady state, i.e., that (1.1) is fulfilled and $(N(t); R^*(t))$, $t \in \mathbb{R}$, is a

[1]The SDPS discipline seems to go back to Cohen, cf. [Coh], which he denoted as generalized processor sharing discipline. But nowadays this term is used for other classes of models such as Weighted Fair Queueing systems, being the reason that it is now called SDPS discipline, cf. e.g. [BBJ].

stationary Markov process. Then the stationary occupancy distribution $p(n) := P(N(t)=n)$, $n \in \mathbb{Z}_+$, and $P(N(t)=n;\, R_1^*(t) \le r_1, \ldots, R_n^*(t) \le r_n)$, $n \in \mathbb{Z}_+$, $r_1, \ldots, r_n \in \mathbb{R}_+$, i.e. the stationary distribution of $(N(t); R^*(t))$ on $\{N(t) = n\}$, are given by

$$p(n) = \Big(\sum_{m=0}^{\infty} \prod_{\ell=1}^{m} \frac{\varrho\chi(\ell)}{\ell} \Big)^{-1} \prod_{\ell=1}^{n} \frac{\varrho\chi(\ell)}{\ell}\,, \tag{1.2}$$

$$P(N(t)=n; R_1^*(t) \le r_1, \ldots, R_n^*(t) \le r_n) = p(n) \prod_{\ell=1}^{n} B_R(r_\ell), \tag{1.3}$$

where

$$B_R(x) := \frac{1}{m_S} \int_0^x (1-B(t))\,\mathrm{d}t, \quad x \in \mathbb{R}_+, \tag{1.4}$$

denotes the stationary residual service time distribution having the density $b_R(x) = (1 - B(x))/m_S$, $x \in \mathbb{R}_+$, cf. [Coh] (7.19) for the case of phase-type distributed service times, [Zac] for the general case. For the sojourn time V of an arbitrary arriving request with required service time S, from Little's law and (1.2) we find that

$$EV = \frac{1}{\lambda} \sum_{n=1}^{\infty} np(n) = m_S \sum_{n=0}^{\infty} \chi(n+1)p(n). \tag{1.5}$$

For the conditional sojourn time $V(\tau)$ of a request with required service time $\tau \in \mathbb{R}_+$ (abbr. τ-request) it is stated that

$$EV(\tau) = \frac{\tau}{m_S}\, EV, \tag{1.6}$$

cf. [Coh] (7.27). It seems that in case of the general $M/GI/SDPS$ system for V and $V(\tau)$ besides (1.5), (1.6) only asymptotic results are known for heavy tailed service times, cf. [GRZ]. However, for special cases several analytical results and numerical algorithms are known. We mention only a few references. For the $M/GI/1 - PS$ system and special cases, cf. e.g. [CMT], [KY], [Ya2], [Ya3], [YY1]. The $M/M/2 - PS$ system is treated in [Tol] and the $M/M/m - PS$ system in [Bra]. For the general $M/M/SDPS$ system cf. [BB].

The aim of this paper is to derive for the $M/GI/SDPS$ system insensitive asymptotically tight upper bounds for the moments of $V(\tau)$. These bounds generalize corresponding results, recently given for the $M/GI/1-PS$ system by [CBB] and for the $M/M/SDPS$ system by the authors [BB]. Corresponding results can be given for the waiting times, too.

The paper is organized as follows. In Section 2 we first derive a linear system of partial differential equations (PDEs) for the Laplace-Stieltjes transforms (LSTs) of the sojourn time of a request under the condition that the residual service times of the other requests in the system are given, too. Then we derive preliminary results (Lemma 2.1), needed later. Although an explicite solution of the PDEs can be given only for special cases, the LSTs for the sojourn time $V(n,\tau)$ of a τ-request, finding n requests at its arrival in the system, as well as for $V(\tau)$ and for V, can be given in terms of the solution of the PDEs (Theorem 2.1, Corollary 2.1). In Section 3 for the general $M/GI/SDPS$ system we derive the announced asymptotically tight upper bounds for the moments of $V(\tau)$ and an asymptotical result for the moments of $V(n,\tau)$ (Theorem 3.1) by using Hölder's inequality and the results of Section 2. Further, relations are given for the moments of sojourn times (Theorem 3.2, Theorem 3.3, Remark 3.2), which reduce the numerical complexity for computing sojourn time characteristics for special cases as e.g. given in [BB]. All results proved in the paper have a correspondence to results for waiting times. In Section 4 we summarize a few of them.

2 Sojourn times and preliminary results

As mentioned above, we assume in the following that the system is stable, i.e., that (1.1) is fulfilled, and in steady state. In particular m_S is finite. Moreover, for technical reasons – if not stated otherwise – we make in the following the assumption:

(A1) $B(x)$ has a continuous density and $B(x) < 1$ for $x \in \mathbb{R}_+$.

For notational convenience let $b(x) := \mathrm{d}B(x)/\mathrm{d}x$ be the density of $B(x)$, $\bar{B}(x) := 1 - B(x)$, $\bar{B}_R(x) := 1 - B_R(x)$, cf. (1.4), and $\beta(x) := b(x)/\bar{B}(x)$, $\beta_R(x) := b_R(x)/\bar{B}_R(x)$ be the complementary distributions and hazard rates of the service time df. and the stationary residual service time df., respectively. Further we will use several vector notations. If not stated otherwise, let $r := (r_1, \ldots, r_\ell) \in \mathbb{R}_+^\ell$ where $\ell = m$ or $\ell = n$, respectively, and

$$\Omega_\ell := \{r \in \mathbb{R}_+^\ell : 0 < r_1 < \ldots < r_\ell\}. \tag{2.1}$$

For $x, y \in \mathbb{R}^\ell$ let $x \le y$ if and only if $x_i \le y_i$ for $i = 1, \ldots, \ell$.

Besides the randomly ordered residual service times $R_1^*(t), \ldots, R_{N(t)}^*(t)$ we need them ordered increasingly: Let $0 \le R_1(t) \le \ldots \le R_{N(t)}(t)$ be the residual service times of the $N(t)$ requests at time t, ordered increasingly, and let $R(t) := (R_1(t), \ldots, R_{N(t)}(t))$ be the corresponding vector. In view of the SDPS discipline, this implies that the requests are ordered according to their departure times, too. For $n \in \mathbb{N}$, $r \in \Omega_n$, let

$$p(n;r) := \frac{\partial^n}{\partial r_1 \ldots \partial r_n} P(N(t) = n;\, R(t) \leq r)$$

be the density of $R(t)$ on $\{N(t) = n\}$, and on the boundary of Ω_n let $p(n;r)$ be defined by continuous continuation. The support of $p(n;r)$ is the closure $\bar{\Omega}_n$ of Ω_n. Denoting by $\mathcal{S}_n$ the set of all permutations of the set $\{1, \ldots, n\}$, from (1.3), (1.4) for $n \in \mathbb{N}$, $r \in \Omega_n$ for the densities we obtain that

$$\begin{aligned} p(n;r) &= \sum_{\pi \in \mathcal{S}_n} \frac{\partial^n}{\partial r_1 \ldots \partial r_n} P(N(t){=}n;\, R_1^*(t) {\leq} r_{\pi(1)}, \ldots, R_n^*(t) {\leq} r_{\pi(n)}) \\ &= n!\, p(n) \prod_{\ell=1}^{n} b_R(r_\ell). \end{aligned} \tag{2.2}$$

By continuous continuation it follows that (2.2) holds for $n \in \mathbb{N}$, $r \in \bar{\Omega}_n$, too.

2.1 Partial differential equations for LSTs of sojourn times

Let $V_\ell(t)$, $\ell = 1, \ldots, N(t)$, be the sojourn time of the request with residual service time $R_\ell(t)$ from time t on until its departure (finish of service), i.e., its prospective sojourn time from time t on. Since the $R_\ell(t)$ are ordered increasingly, the SDPS discipline implies that the $V_\ell(t)$ are ordered increasingly, too, i.e., $0 \leq V_1(t) \leq \ldots \leq V_{N(t)}(t)$. Further, $V_1(t) = 0$ if and only if $R_1(t) = 0$. In view of (A1) and the distributional and independence assumptions, for $0 < m \leq n$, $r \in \bar{\Omega}_n$ the LSTs

$$h_{n,m}(s;r) := E[e^{-sV_m(t)} \,|\, N(t) = n,\, R(t) = r] \tag{2.3}$$

of $V_m(t)$ conditioned that at time t there are n requests in the system with residual service times $R_\ell(t) = r_\ell$, $\ell = 1, \ldots, n$, are well defined for $s \in \mathbb{R}_+$. Note that for $0 < m \leq n$, $r \in \bar{\Omega}_n$ it holds $h_{n,m}(0;r) = 1$. In the following let $s \in \mathbb{R}_+$ be fixed. As the prospective sojourn time of a request with residual service time zero is zero, from (2.3) we conclude for $0 < n$, $r \in \bar{\Omega}_n$ where $r_1 = 0$ that

$$h_{n,1}(s; 0, r_2, \ldots, r_n) = 1. \tag{2.4}$$

Since a request with residual service time zero leaves the system immediately, for $1 < m \le n$, $r \in \bar{\Omega}_n$ where $r_1 = 0$ it follows that

$$h_{n,m}(s; 0, r_2, \ldots, r_n) = h_{n-1,m-1}(s; r_2, \ldots, r_n). \tag{2.5}$$

Now let $0 < m \le n$, $r \in \Omega_n$ and consider the time interval $[0, h]$. In view of $r_1 > 0$, for h sufficiently small there will be no departure of any of the n requests which are in the system at $t = 0$. If there is a request arrival during $[0, h]$, then there may occur two cases for the sampled service time $\tau > 0$, which is just its residual service time: $\tau \in (0, r_m)$ or $\tau \in [r_m, \infty)$. Taking into account the dynamics of the $M/GI/SDPS$ system during $[0, h]$, we obtain that

$$\begin{aligned}
h_{n,m}(s; r) &= (1-\lambda h) E[e^{-s(V_m(h)+h)} \,|\, N(h) = n,\, R(h) = r - \varphi(n) h \mathbf{1}_n] \\
&+ \lambda h \sum_{\ell=1}^{m} \int_{r_{\ell-1}}^{r_\ell} b(\tau) E[e^{-s(V_{m+1}(h)+h)} \,|\, N(h) = n+1, \\
&\qquad\qquad R(h) = r^{(\ell)}(r, \tau) - \varphi(n+1) h \mathbf{1}_{n+1}] \mathrm{d}\tau \\
&+ \lambda h \sum_{\ell=m+1}^{n+1} \int_{r_{\ell-1}}^{r_\ell} b(\tau) E[e^{-s(V_m(h)+h)} \,|\, N(h) = n+1, \\
&\qquad\qquad R(h) = r^{(\ell)}(r, \tau) - \varphi(n+1) h \mathbf{1}_{n+1}] \mathrm{d}\tau + o(h),
\end{aligned}$$

where $r_0 := 0$, $r_{n+1} := \infty$, $\mathbf{1}_n := (1, \ldots, 1) \in \mathbb{R}^n$ and

$$r^{(\ell)}(r, \tau) := (r_1, \ldots, r_{\ell-1}, \tau, r_\ell, \ldots, r_n), \quad \ell = 1, \ldots, n+1,$$

for $r \in \mathbb{R}^n$, $\tau \in \mathbb{R}$. Subtracting on both sides $h_{n,m}(s; r - \varphi(n) h \mathbf{1}_n)$, dividing by h and taking the limit $h \downarrow 0$ provides the following linear system of PDEs for $0 < m \le n$, $r \in \Omega_n$:

$$\begin{aligned}
&\varphi(n) \frac{\partial}{\partial \xi} h_{n,m}(s; r+\xi \mathbf{1}_n)\Big|_{\xi=0} \\
&\quad = -(\lambda+s)\, h_{n,m}(s; r) + \lambda \sum_{\ell=1}^{m} \int_{r_{\ell-1}}^{r_\ell} b(\tau)\, h_{n+1,m+1}(s; r^{(\ell)}(r, \tau)) \mathrm{d}\tau \\
&\qquad + \lambda \sum_{\ell=m+1}^{n+1} \int_{r_{\ell-1}}^{r_\ell} b(\tau)\, h_{n+1,m}(s; r^{(\ell)}(r, \tau)) \mathrm{d}\tau.
\end{aligned} \tag{2.6}$$

The following observation will be crucial and simplifies the analysis of the model considerably in the following: In view of the $SDPS$ discipline, for $0 < m \le n$, $r \in \bar{\Omega}_n$ the conditional sojourn time $V_m(t)$ given that $N(t) = n$,

$R(t) = r$ depends only on $r_1, \ldots, r_m$ and the total number n of requests in the system since the requests with residual service times $r_{m+1}, \ldots, r_n$ have residual service times of an amount greater or equal to r_m and are thus in the system at least as long as the request with service time r_m. Therefore

$$f_{n,m}(s; r_1, \ldots, r_m) := h_{n,m}(s; r_1, \ldots, r_n), \quad 0 < m \le n, \ r \in \bar{\Omega}_n, \quad (2.7)$$

is well defined. From (2.4)–(2.7) we obtain the following linear system of PDEs for $0 < m \le n$, $r \in \Omega_m$:

$$\begin{aligned} \varphi(n) \frac{\partial}{\partial \xi} f_{n,m}(s; r + \xi \mathbf{1}_m) \Big|_{\xi=0} \\ &= -(\lambda + s) f_{n,m}(s; r) + \lambda \sum_{\ell=1}^{m} \int_{r_{\ell-1}}^{r_\ell} b(\tau) f_{n+1,m+1}(s; r^{(\ell)}(r, \tau)) \mathrm{d}\tau \\ &\qquad + \lambda \bar{B}(r_m) f_{n+1,m}(s; r) \end{aligned} \quad (2.8)$$

with the initial conditions

$$f_{n,1}(s; 0) = 1, \quad (2.9)$$

$$f_{n,m}(s; 0, r_2, \ldots, r_m) = f_{n-1,m-1}(s; r_2, \ldots, r_m), \quad 1 < m \le n, \quad (2.10)$$

for $0 \le r_2 \le \ldots \le r_m$. In view of $h_{n,m}(0; r) = 1$ for $r \in \bar{\Omega}_n$, from (2.7) we find that $f_{n,m}(0; r) = 1$ for $0 < m \le n$, $r \in \bar{\Omega}_m$.

It seems that for general $\varphi(n)$, $n \in \mathbb{N}$, there is no explicite solution of (2.8)–(2.10). However, for $\varphi_{1,k}(n) = 1/(n + k)$, $n \in \mathbb{N}$, $k \in (-1, \infty)$, a solution can be given by adopting results of [KY], [Ya3], leading to the well known results for $M/GI/1 - PS$ systems. Note that for $k \in \mathbb{Z}_+$ we have a single-server PS system with k permanent requests.

Example 2.1 *Let $\varphi_{1,k}(n) = 1/(n + k)$, $n \in \mathbb{N}$, $k \in (-1, \infty)$, and $s \in \mathbb{R}_+$. We try, cf. [KY], [Ya1], the substitution*

$$f_{n,m}(s; r_1, \ldots, r_m) = \delta(s, r_m)^{n+k} \prod_{i=1}^{m-1} \frac{1}{\delta(s, r_m - r_i)}$$

for $0 < m \le n$, $0 \le r_1 \le \ldots \le r_m$, where $\delta(s, \tau)$ is a continuously differentiable function in $\tau \in \mathbb{R}_+$ with initial condition $\delta(s, 0) = 1$. The substitution satisfies (2.9) *and* (2.10)*. Inserting the substitution into* (2.8) *and using that $\varphi_{1,k}(n) = 1/(n + k)$, $n \in \mathbb{N}$, one finds after some algebra that the linear* *system of PDEs* (2.8) *is fulfilled if* $\delta(s,\tau)$ *satisfies the following differential equation*

$$0=\frac{\partial\delta(s,\tau)}{\partial\tau}+\Big(s+\lambda-\lambda\int_0^\tau\frac{\delta(s,\tau)}{\delta(s,\tau-y)}\,b(y)\mathrm{d}y-\lambda\delta(s,\tau)\bar{B}(\tau)\Big)\delta(s,\tau)$$

with initial condition $\delta(s,0)=1$*, which has a uniquely determined solution, cf. [Ya1]. The product form solution for the* $f_{n,m}(s;r_1,\ldots,r_m)$*, given above, has been proved for* $M/GI/1-PS$ *systems for the first time by a decomposition of the sojourn time in [KY], [Ya1] and for the single-server* PS *model with permanent requests in [YY2]. The differential equation for* $\delta(s,\tau)$*, given above, can be solved for particular cases explicitely. In the general case, the Laplace transform for* $\mathrm{e}^{-(s+\lambda)\tau}/\delta(s,\tau)$ *can be given explicitely, cf. [KY], [Ya1].*

For notational convenience, in the following we suppress the time parameter t for steady state r.v.s, i.e., we use V_m, N, $R=(R_1,\ldots,R_N)$ instead of $V_m(t)$, $N(t)$, $R(t)=(R_1(t),\ldots,R_{N(t)}(t))$.

In view of (2.2) and (2.3), for $0<m\le n$, $r\in\Omega_n$, the LSTs

$$v_{n,m}(s;r):=\frac{\partial^n}{\partial r_1\ldots\partial r_n}\,E[e^{-sV_m}\mathbb{I}\{N=n,\ R\le r\}] \tag{2.11}$$

of V_m on $\{N=n,R_1\in\mathrm{d}r_1,\ldots,R_n\in\mathrm{d}r_n\}$ are well defined for $s\in\mathbb{R}_+$, and in view of (2.2) and (2.7) given by

$$v_{n,m}(s;r)=u_{n,m}(s;r_1,\ldots,r_m)\prod_{\ell=m+1}^{n}b_R(r_\ell), \tag{2.12}$$

where

$$u_{n,m}(s;r_1,\ldots,r_m):=n!\,p(n)\Big(\prod_{\ell=1}^{m}b_R(r_\ell)\Big)f_{n,m}(s;r_1,\ldots,r_m). \tag{2.13}$$

For $0<m\le n$, let $u_{n,m}(s;r)$ and hence $v_{n,m}(s;r)$ be defined on the boundary of Ω_n by continuous continuation. Note that for $0<m\le n$, $r\in\Omega_n$ from (2.11)–(2.13) it follows that

$$\begin{aligned}&u_{n,m}(s;r_1,\ldots,r_m)\\&\quad=\frac{\partial^m}{\partial r_1\ldots\partial r_m}\,E[e^{-sV_m}\mathbb{I}\{N=n,\ R_1\le r_1,\ldots,R_m\le r_m\}\\&\qquad\qquad\qquad\mid R_{m+1}=r_{m+1},\ldots,R_n=r_n],\end{aligned} \tag{2.14}$$

and thus the conditional LST on the r.h.s. of (2.14) is independent of $r_{m+1},\ldots,r_n$. From $f_{n,m}(0;r)=1$ and (2.13) we obtain the boundary condition

$$u_{n,m}(0;r)=n!\,p(n)\prod_{\ell=1}^{m}b_R(r_\ell),\quad 0<m\le n,\ r\in\bar{\Omega}_m. \tag{2.15}$$

In the following let $s\in\mathbb{R}_+$ be fixed. Taking the derivative of $u_{n,m}(s;r+\xi\mathbf{1}_m)$ with respect to ξ by taking into account (2.13), (1.2) and (1.4), one finds after some algebra that (2.8) is equivalent to the following linear system of PDEs for $0<m\le n$, $r\in\Omega_m$:

$$\begin{aligned}\varphi(n)\frac{\partial}{\partial\xi}u_{n,m}(s;r+\xi\mathbf{1}_m)\Big|_{\xi=0}&=-(\lambda+s+\varphi(n)\beta(r))u_{n,m}(s;r)\\&\quad+\varphi(n+1)\sum_{\ell=1}^{m}\int_{r_{\ell-1}}^{r_\ell}\beta(\tau)\,u_{n+1,m+1}(s;r^{(\ell)}(r,\tau))\mathrm{d}\tau\\&\quad+\varphi(n+1)b_R(r_m)u_{n+1,m}(s;r),\end{aligned} \tag{2.16}$$

where $\beta(r):=\sum_{\ell=1}^{m}\beta(r_\ell)$. In view of (2.13), (1.2) and $b_R(0)=1/m_S$, the initial conditions (2.9) and (2.10) yield the initial conditions

$$u_{n,1}(s;0)=n!\,p(n)m_S^{-1}, \tag{2.17}$$

$$u_{n,m}(s;0,r_2,\ldots,r_m)=\frac{\lambda}{\varphi(n)}\,u_{n-1,m-1}(s;r_2,\ldots,r_m) \tag{2.18}$$

for $1<m\le n$, $0\le r_2\le\ldots\le r_m$.

2.2 Preliminary results

For deriving later expressions for the LSTs of $V(n,\tau)$ and $V(\tau)$, we consider the LSTs

$$g_{n,m}(s,x):=\frac{\partial}{\partial x}E[e^{-sV_m}\mathbb{I}\{N=n,\,R_m\le x\}] \tag{2.19}$$

for x, $s\in\mathbb{R}_+$, $0<m\le n$. From (2.12) and $b_R(x)=\bar{B}(x)/m_S$ by integrating $v_{n,m}(s;r)$ over $0\le r_1\le\ldots\le r_{m-1}\le x\le r_{m+1}\le\ldots\le r_n$ with respect to $\mathrm{d}r_1\ldots\mathrm{d}r_{m-1}\mathrm{d}r_{m+1}\ldots\mathrm{d}r_n$ we obtain that

$$g_{n,m}(s,x)=\bar{g}_{n,m}(s,x)\,d_{n-m}(x), \tag{2.20}$$

where

$$\bar{g}_{n,m}(s,x) := \int_{0\le r_1\le\ldots\le r_{m-1}\le x} u_{n,m}(s;r_1,\ldots,r_{m-1},x)\,\mathrm{d}r_1\ldots\mathrm{d}r_{m-1}, \tag{2.21}$$

$$d_\ell(x) := \frac{1}{\ell!}\Big(\int_x^\infty b_R(\eta)\mathrm{d}\eta\Big)^\ell = \frac{1}{\ell!}\bar{B}_R(x)^\ell. \tag{2.22}$$

Note that in case of $n = m = 1$ the r.h.s. of (2.19) is just $v_{1,1}(s;x)$ and in case of $1 = m \le n$ the r.h.s. of (2.21) is just $u_{n,1}(s;x)$. From (2.21), (2.15), (2.17) we find for $0 < m \le n$ that

$$\bar{g}_{n,m}(0,x) = n!\,p(n)b_R(x)\frac{1}{(m-1)!}\,B_R(x)^{m-1}, \quad x\in\mathbb{R}_+, \tag{2.23}$$

$$\bar{g}_{n,m}(s,0) = \mathbb{I}\{m=1\}\,n!\,p(n)\,m_S^{-1}, \quad s\in\mathbb{R}_+, \tag{2.24}$$

and thus from (2.20) and (2.22) for $0 < m \le n$ the boundary condition

$$g_{n,m}(s,0) = \mathbb{I}\{m=1\}\,n\,p(n)\,m_S^{-1}, \quad s\in\mathbb{R}_+. \tag{2.25}$$

Because of (2.20), (2.22) and (2.23), we obtain for $0 < m \le n$, $x\in\mathbb{R}_+$ that

$$g^{(0)}_{n,m}(x) := g_{n,m}(0,x) \tag{2.26}$$

$$= n\,p(n)\,b_R(x)\binom{n-1}{m-1}B_R(x)^{m-1}\bar{B}_R(x)^{n-m}, \tag{2.27}$$

and thus by taking into account (1.2), (1.4), it follows that

$$\sum_{m=1}^{n} g^{(0)}_{n,m}(x) = n\,p(n)\,b_R(x) = \frac{\lambda\bar{B}(x)}{\varphi(n)}\,p(n-1). \tag{2.28}$$

For deriving later expressions and estimates for the moments of $V(n,\tau)$, $V(\tau)$, we need some preliminary results for the $g_{n,m}(s,x)$ and its derivatives with respect to s. For $s\in(0,\infty)$, $0 < m \le n$, $x\in\mathbb{R}_+$, $k\in\mathbb{Z}_+$, let

$$g^{(k)}_{n,m}(s,x) := (-1)^k\frac{\partial^k}{\partial s^k}\,g_{n,m}(s,x) \tag{2.29}$$

$$= \frac{\partial}{\partial x}\,E[V_m^k\,e^{-sV_m}\mathbb{I}\{N=n,\,R_m\le x\}], \tag{2.30}$$

where the last equality follows in view of (2.19). Note that

$$g^{(k)}_{n,m}(s,x) = \lim_{h\downarrow 0} E[V_m^k\,e^{-sV_m}\mathbb{I}\{N\!=\!n,\,x\!<\!\mathbb{R}_m\!\le\! x\!+\!h\}]/h \ge 0.$$

From (2.20) and (2.24) we find that

$$g^{(k)}_{n,m}(s,0)=0,\quad k\in\mathbb{N}. \tag{2.31}$$

Taking into account $v^k e^{-sv}\le k!s^{-k}$ for $v\in\mathbb{R}_+$, from (2.30) and (2.26) we obtain for $s\in(0,\infty)$, $x\in\mathbb{R}_+$ that

$$g^{(k)}_{n,m}(s,x)\le k!s^{-k}g^{(0)}_{n,m}(x),\quad 0<m\le n,\ k\in\mathbb{Z}_+. \tag{2.32}$$

Lemma 2.1 *Let $EN<\infty$. For $s\in(0,\infty)$ and $k\in\mathbb{Z}_+$, it holds that*

$$\begin{aligned}\sum_{0<m\le n}\varphi(n)\frac{\partial}{\partial x}g^{(k)}_{n,m}(s,x)=-\sum_{0<m\le n}(\varphi(n)\beta(x)+s)g^{(k)}_{n,m}(s,x)\\ +k\sum_{0<m\le n}g^{(k-1)}_{n,m}(s,x).\end{aligned} \tag{2.33}$$

Proof First we will prove (2.33) for $k=0$. Let $s\in[0,\infty)$ be fixed. Replacing in (2.16) the variables r_i by $r_i+\eta$ for $i=1,\dots,m$ in the arguments, integrating over $[-r_1,0]$ with respect to η and applying (2.17), (2.18), one finds that the system of PDEs (2.16) with the initial conditions (2.17), (2.18) is equivalent to the following linear system of integral equations for $0<m\le n$, $r\in\bar{\Omega}_m$:

$$\begin{aligned}\varphi(n)u_{n,m}(s;r)&=\mathbb{I}\{m{=}1\}\varphi(n)\,n!\,p(n)\,m_S^{-1}\\ &+\mathbb{I}\{m{>}1\}\lambda u_{n-1,m-1}(s;r_2-r_1,\dots,r_m-r_1)\\ &-\int_{-r_1}^{0}(\lambda+s+\varphi(n)\,\beta(r+\eta\mathbf{1}_m))\,u_{n,m}(s;r+\eta\mathbf{1}_m)\mathrm{d}\eta\\ &+\varphi(n{+}1)\sum_{\ell=1}^{m}\int_{-r_1}^{0}\int_{r_{\ell-1}}^{r_\ell}\beta(\tau+\eta)u_{n+1,m+1}(s;r^{(\ell)}(r+\eta\mathbf{1}_m,\tau+\eta))\\ &\qquad\mathrm{d}\tau\mathrm{d}\eta\\ &+\varphi(n{+}1)\int_{-r_1}^{0}b_R(r_m+\eta)u_{n+1,m}(s;r+\eta\mathbf{1}_m)\mathrm{d}\eta.\end{aligned} \tag{2.34}$$

Integrating both sides of (2.34) over $0\le r_1\le\ldots\le r_{m-1}\le x$ with respect to $\mathrm{d}r_1\ldots\mathrm{d}r_{m-1}$ for fixed $x=r_m$, using Fubini's theorem and taking into account (2.21), we obtain after some algebra for $0<m\le n$, $x\in\mathbb{R}_+$ that

$$\begin{aligned}\varphi(n)\bar{g}_{n,m}(s,x)&=\mathbb{I}\{m{=}1\}\varphi(n)\,n!\,p(n)\,m_S^{-1}\\ &+\mathbb{I}\{m{>}1\}\lambda\int_0^x\bar{g}_{n-1,m-1}(s,\eta)\mathrm{d}\eta-(\lambda+s)\int_0^x\bar{g}_{n,m}(s,\eta)\mathrm{d}\eta\end{aligned}$$

$$- \varphi(n) \sum_{\ell=1}^{m-1} \int_0^x J_{n,m,\ell}(s,\eta) \mathrm{d}\eta - \varphi(n) \int_0^x \beta(\eta) \bar{g}_{n,m}(s,\eta) \mathrm{d}\eta$$

$$+ \varphi(n+1) \sum_{\ell=1}^{m} \int_0^x J_{n+1,m+1,\ell}(s,\eta) \mathrm{d}\eta$$

$$+ \varphi(n+1) \int_0^x b_R(\eta) \bar{g}_{n+1,m}(s,\eta) \, \mathrm{d}\eta, \tag{2.35}$$

where

$$J_{n,m,\ell}(s,x) := \int_{0 \le r_1 \le \ldots \le r_{m-1} \le x} \beta(r_\ell) u_{n,m}(s; r_1, \ldots, r_{m-1}, x) \, \mathrm{d}r_1 \ldots \mathrm{d}r_{m-1} \tag{2.36}$$

for $0 < \ell < m \le n$ and $x \in \mathbb{R}_+$. In view of $\mathrm{d}d_\ell(x)/\mathrm{d}x = -b_R(x) d_{\ell-1}(x)$ for $\ell \in \mathbb{N}$, cf. (2.22), (1.4), from (2.35) and (2.20) after some algebra we find the following system of differential equations for the $g_{n,m}(s,x)$ for $0 < m \le n$, $x \in \mathbb{R}_+$:

$$\varphi(n) \frac{\partial}{\partial x} g_{n,m}(s,x) = \mathbb{I}\{m > 1\} \lambda g_{n-1,m-1}(s,x) - (\lambda + s + \varphi(n)\beta(x)$$

$$+ \varphi(n)(n-m)\beta_R(x)) g_{n,m}(s,x) - \varphi(n) \sum_{\ell=1}^{m-1} J_{n,m,\ell}(s,x) \, d_{n-m}(x)$$

$$+ \varphi(n+1) \sum_{\ell=1}^{m} J_{n+1,m+1,\ell}(s,x) \, d_{n-m}(x)$$

$$+ \varphi(n+1)(n+1-m)\beta_R(x) \, g_{n+1,m}(s,x). \tag{2.37}$$

Now let $x^* \in \mathbb{R}_+$ be arbitrary but fixed and $\beta^* := \sup\{\beta(x) : 0 \le x \le x^*\}$, which is finite in view of (A1). Then from (2.36) and (2.20) we find that for $0 < \ell < m \le n$, $0 \le x \le x^*$ it holds

$$J_{n,m,\ell}(s,x) \, d_{n-m}(x) \le \beta^* g_{n,m}(s,x). \tag{2.38}$$

Because of (2.32) for $k = 0$, (2.28), (2.38) and $EN < \infty$, summing up (2.37) over $0 < m \le n \le n'$, taking the limit $n' \to \infty$ provides after some algebra that most of the summands cancel each other, and we obtain (2.33) for $x \in [0, x^*]$. Since x^* was chosen arbitrarily, (2.33) is valid for $x \in \mathbb{R}_+$, finishing the proof for $k = 0$.

For proving (2.33) for $k \in \mathbb{N}$, we start from (2.33) for $k = 0$. In view of $|g_{n,m}(s,x)| \le g_{n,m}(0,x)$ for $\Re s \ge 0$, we may take the kth derivative on both sides of (2.33) with respect to s for $\Re s > 0$ item by item due to Weierstrass's theorem, cf. [Ahl], which provides (2.33) for $s \in (0, \infty)$, $k \in \mathbb{N}$. $\square$

Remark 2.1 *Note that* (2.35)–(2.37) *and* (2.33) *for* $k = 0$ *are valid for* $s \in [0, \infty)$.

2.3 LSTs of sojourn times

Consider the $M/GI/SDPS$ system in steady state. For $\tau \in \mathbb{R}_+$, let $V(n, \tau)$ be the sojourn time of a tagged arriving request with required service time τ (τ-request) finding n requests at its arrival in the system. By $V(\tau)$ we denote the sojourn time of a tagged arriving τ-request.

Theorem 2.1 *For the $M/GI/SDPS$ system let the stability condition* (1.1) *and* (A1) *be satisfied. Then for $s \in \mathbb{R}_+$ and $\tau \in \mathbb{R}_+$ the LSTs of $V(n, \tau)$, $n \in \mathbb{Z}_+$, and $V(\tau)$ are given by*

$$E[e^{-sV(n,\tau)}] = \frac{\varphi(n+1)}{\lambda(\tau)p(n)} \sum_{m=1}^{n+1} g_{n+1,m}(s, \tau), \tag{2.39}$$

$$E[e^{-sV(\tau)}] = \frac{1}{\lambda(\tau)} \sum_{n=1}^{\infty} \varphi(n) \sum_{m=1}^{n} g_{n,m}(s, \tau), \tag{2.40}$$

respectively, where

$$\lambda(x) := \lambda \bar{B}(x), \quad x \in \mathbb{R}_+, \tag{2.41}$$

and the $g_{n,m}(s, x)$, $0 < m \leq n$, $x \in \mathbb{R}_+$, are given by (2.20)–(2.22).

Proof Consider a tagged arriving τ-request finding n requests at its arrival. From the PASTA property and conditioning with respect to the vector $R = r \in \bar{\Omega}_n$ of residual service times one obtains from (2.2), (2.3), (2.7) and (2.13) that

$$\begin{aligned} E[e^{-sV(n,\tau)}] &= \frac{1}{p(n)} \sum_{m=1}^{n+1} \int_{0 \leq r_1 \leq \ldots \leq r_{m-1} \leq \tau \leq r_m \leq \ldots \leq r_n} p(n; r) \\ &\qquad h_{n+1,m}(s; r^{(m)}(r, \tau)) \mathrm{d}r_1 \ldots \mathrm{d}r_n \\ &= \frac{1}{(n+1)p(n+1)b_R(\tau)} \sum_{m=1}^{n+1} \\ &\qquad \int_{0 \leq r_1 \leq \ldots \leq r_{m-1} \leq \tau} u_{n+1,m}(s; r_1, \ldots, r_{m-1}, \tau) \mathrm{d}r_1 \ldots \mathrm{d}r_{m-1} \\ &\qquad \int_{\tau \leq r_m \leq \ldots \leq r_n} \prod_{\ell=m}^{n} b_R(r_\ell) \mathrm{d}r_m \ldots \mathrm{d}r_n. \end{aligned} \tag{2.42}$$

From (2.42) the assertion (2.39) follows easily in view of $b_R(\tau) = \bar{B}(\tau)/m_S$, (1.2), (2.41), integration with respect to $\mathrm{d}r_m \dots \mathrm{d}r_n$ and (2.20)–(2.22). The assertion (2.40) follows directly from (2.39) and

$$E[e^{-sV(\tau)}] = \sum_{n=0}^{\infty} p(n) E[e^{-sV(n,\tau)}].$$

□

From (2.40), (2.41), $\beta(\tau) = b(\tau)/\bar{B}(\tau)$ and

$$E[e^{-sV}] = \int_{\mathbb{R}_+} E[e^{-sV(\tau)}]\, b(\tau)\mathrm{d}\tau \tag{2.43}$$

we obtain immediately the following representation for the LST of V.

Corollary 2.1 *Let the stability condition* (1.1) *for the $M/GI/SDPS$ system with* (A1) *be satisfied. Then for $s \in \mathbb{R}_+$ the LST of V is given by*

$$E[e^{-sV}] = \frac{1}{\lambda} \sum_{n=1}^{\infty} \varphi(n) \sum_{m=1}^{n} \int_{\mathbb{R}_+} \beta(\tau) g_{n,m}(s,\tau)\mathrm{d}\tau. \tag{2.44}$$

3 Bounds for the moments of $V(\tau)$

Theorem 3.1 *Let the stability condition* (1.1) *for the $M/GI/SDPS$ system with a general service time df. $B(x)$ be satisfied. Then the kth moment of $V(\tau)$, $\tau \in \mathbb{R}_+$, is finite if*

$$\sum_{n=0}^{\infty} \chi(n+1)^k p(n) < \infty. \tag{3.1}$$

For $k \in \mathbb{Z}_+$, it holds that

$$\tau^k \Big(\sum_{n=0}^{\infty} \chi(n+1) p(n) \Big)^k \le E[V^k(\tau)] \le \tau^k \sum_{n=0}^{\infty} \chi(n+1)^k p(n), \quad \tau \in \mathbb{R}_+, \tag{3.2}$$

$$\lim_{\tau \downarrow 0} \frac{E[V^k(\tau)]}{\tau^k} = \sum_{n=0}^{\infty} \chi(n+1)^k p(n). \tag{3.3}$$

If additionally (3.1) *is satisfied, then it holds that*

$$\lim_{\tau \downarrow 0} \frac{E[V^k(n,\tau)]}{\tau^k} = \chi(n+1)^k, \quad n \in \mathbb{Z}_+. \tag{3.4}$$

Proof The lower bound in (3.2) follows directly from (1.6), (1.5) and Hölder's inequality[2]. For proving the other assertions first we assume that $B(x)$ satisfies (A1), i.e. that $B(x)$ has a continuous density and $B(x) < 1$ for $x \in \mathbb{R}_+$. Then for $s \in (0, \infty)$, $k \in \mathbb{Z}_+$ from Theorem 2.1 and (2.29) we obtain that

$$\lambda(x) E[V^k(x) e^{-sV(x)}] = \sum_{0<m\le n} \varphi(n) g^{(k)}_{n,m}(s,x), \quad x \in \mathbb{R}_+. \tag{3.5}$$

In view of (2.30), for $k \in \mathbb{N}$, applying Hölder's inequality to the difference quotient $E[V_m^k e^{-sV_m} \mathbb{I}\{N = n, x < R_m \le x + h\}]/h$ and taking the limit $h \downarrow 0$ we find that

$$\begin{aligned} g^{(k)}_{n,m}(s,x) &\le \Big(\frac{\partial}{\partial x} E[e^{-sV_m} \mathbb{I}\{N = n, R_m \le x\}]\Big)^{1/(k+1)} \\ &\qquad\qquad \Big(\frac{\partial}{\partial x} E[V_m^{k+1} e^{-sV_m} \mathbb{I}\{N = n, R_m \le x\}]\Big)^{k/(k+1)} \\ &\le (g^{(0)}_{n,m}(x))^{1/(k+1)} (g^{(k+1)}_{n,m}(s,x))^{k/(k+1)}, \end{aligned}$$

where the last inequality follows from (2.30) and (2.32) for $k = 0$. Using the above inequality, applying Hölder's inequality to the series and taking into account (2.28), (3.5) provides

$$\begin{aligned} &\frac{1}{\lambda(x)} \sum_{0<m\le n} g^{(k)}_{n,m}(s,x) \\ &\quad\le \sum_{0<m\le n} \Big(\chi(n)^{k+1} \frac{\varphi(n)}{\lambda(x)} g^{(0)}_{n,m}(x)\Big)^{\frac{1}{k+1}} \Big(\frac{\varphi(n)}{\lambda(x)} g^{(k+1)}_{n,m}(s,x)\Big)^{\frac{k}{k+1}} \\ &\quad\le \Big(\sum_{0<m\le n} \chi(n)^{k+1} \frac{\varphi(n)}{\lambda(x)} g^{(0)}_{n,m}(x)\Big)^{\frac{1}{k+1}} \Big(\sum_{0<m\le n} \frac{\varphi(n)}{\lambda(x)} g^{(k+1)}_{n,m}(s,x)\Big)^{\frac{k}{k+1}} \\ &\quad= \Big(\sum_{n=0}^{\infty} \chi(n+1)^{k+1} p(n)\Big)^{\frac{1}{k+1}} \Big(E[V^{k+1}(x) e^{-sV(x)}]\Big)^{\frac{k}{k+1}}. \end{aligned} \tag{3.6}$$

Taking the derivative on both sides of (3.5) with respect to x, taking into account $\lambda(x) = \lambda \bar{B}(x)$, applying Lemma 2.1, $\beta(x) = b(x)/\bar{B}(x)$ and again (3.5), we obtain for $s \in (0, \infty)$, $k \in \mathbb{N}$ that

$$\lambda(x) \frac{\partial}{\partial x} E[V^k(x) e^{-sV(x)}] = \sum_{0<m\le n} (k g^{(k-1)}_{n,m}(s,x) - s g^{(k)}_{n,m}(s,x)), \quad x \in \mathbb{R}_+. \tag{3.7}$$

[2] In case of $k = 2$ the lower bound in (3.2) is equivalent to the fact that the variance of $V(\tau)$ is greater or equal to zero as we have equality in (3.2) for $k = 1$.

In view of $g^{(k)}_{n,m}(s,x) \geq 0$, in case of $k > 1$, (3.7) and (3.6) where k is replaced by $k-1$ imply

$$\frac{\partial}{\partial x}\, E[V^k(x)e^{-sV(x)}]$$
$$\leq k\Big(\sum_{n=0}^{\infty}\chi(n{+}1)^k p(n)\Big)^{1/k}(E[V^k(x)e^{-sV(x)}])^{(k-1)/k},$$

which is equivalent to

$$\frac{\partial}{\partial x}(E[V^k(x)e^{-sV(x)}])^{1/k} \leq \left(\sum_{n=0}^{\infty}\chi(n{+}1)^k p(n)\right)^{1/k}. \tag{3.8}$$

Note that (3.8) is valid for $k = 1$, too, in view of (3.7), (2.32), (2.28) and (2.41). For $k \in \mathbb{N}$, because of $E[V^k(0)e^{-sV(0)}] = 0$, integrating (3.8) over $[0, \tau]$ and taking the kth power yields that

$$E[V^k(\tau)e^{-sV(\tau)}] \leq \tau^k \sum_{n=0}^{\infty}\chi(n{+}1)^k p(n), \quad \tau \in \mathbb{R}_+. \tag{3.9}$$

The limit $s \downarrow 0$ provides the upper bound in (3.2). Thus (3.1) implies that $E[V^k(\tau)]$ is finite for $\tau \in \mathbb{R}_+$.

For proving (3.3) we derive a lower bound for $E[V^k(n, \tau)]$. Let

$$u^{(k)}_{n,m}(s;r) := (-1)^k \frac{\partial^k}{\partial s^k}\, u_{n,m}(s;r) \tag{3.10}$$

for $s \in (0,\infty)$, $0 < m \leq n$, $r \in \bar{\Omega}_m$, $k \in \mathbb{Z}_+$. Since $0 \leq v^k e^{-sv} \leq k!\, s^{-k}$ for $v \in \mathbb{R}_+$, from (2.14) and (2.15) we obtain that

$$0 \leq u^{(k)}_{n,m}(s;r) \leq k!\, s^{-k}\, u_{n,m}(0;r) = k!\, s^{-k}\, n!\, p(n) \prod_{\ell=1}^{m} b_R(r_\ell). \tag{3.11}$$

Further, from (2.17) and (2.18) we find that

$$u^{(k)}_{n,1}(s;0) = \mathbb{I}\{k{=}0\}\, n!\, p(n)\, m_S^{-1}, \tag{3.12}$$

$$u^{(k)}_{n,m}(s;0,r_2,\ldots,r_m) = \frac{\lambda}{\varphi(n)}\, u^{(k)}_{n-1,m-1}(s;r_2,\ldots,r_m) \tag{3.13}$$

for $1 < m \leq n$, $0 \leq r_2 \leq \ldots \leq r_m$. Taking the kth derivative on both sides of (2.16) with respect to s, multiplying the resulting equation by $(-1)^k$, replacing then the variables r_i by $r_i + \eta$, $i = 1, \dots, m$, multiplying both sides by

$$C_n(r;\eta) := \exp\Big(\int_0^\eta c_n(r+\xi\mathbf{1}_m)\mathrm{d}\xi\Big), \tag{3.14}$$

where

$$c_n(r) := (\lambda+s)/\varphi(n) + \beta(r), \quad r \in \bar{\Omega}_m, \tag{3.15}$$

integrating then over $[-r_1, 0]$ with respect to η, applying partial integration to the resulting l.h.s. and applying (3.13), one obtains the following linear system of integral equations for $0 < m \le n$, $r \in \bar{\Omega}_m$, $s \in (0,\infty)$, $k \in \mathbb{N}$:

$$\begin{aligned}\varphi(n)\, u^{(k)}_{n,m}(s;r) &= \mathbb{I}\{m>1\}\lambda\, u^{(k)}_{n-1,m-1}(s;r_2-r_1,\dots,r_m-r_1)C_n(r;-r_1)\\ &+\int_{-r_1}^0 \Big(\varphi(n+1)\Big(\sum_{\ell=1}^m \int_{r_{\ell-1}}^{r_\ell} \beta(\tau+\eta)\\ &\qquad\qquad u^{(k)}_{n+1,m+1}(s;r^{(\ell)}(r+\eta\mathbf{1}_m,\tau+\eta))\mathrm{d}\tau\\ &+ b_R(r_m+\eta)u^{(k)}_{n+1,m}(s;r+\eta\mathbf{1}_m)\Big)+ku^{(k-1)}_{n,m}(s;r+\eta\mathbf{1}_m)\Big)C_n(r;\eta)\mathrm{d}\eta.\end{aligned} \tag{3.16}$$

For $s \downarrow 0$ from (3.16) and (3.11) we find that for

$$u^{(k)}_{n,m}(r) := \lim_{s\downarrow 0} u^{(k)}_{n,m}(s;r), \quad 0 < m \le n,\ r \in \bar{\Omega}_m,\ k \in \mathbb{N}, \tag{3.17}$$

it holds

$$\varphi(n)\, u^{(k)}_{n,m}(r) \ge k \int_{-r_1}^0 u^{(k-1)}_{n,m}(r+\eta\mathbf{1}_m)C_n(r;\eta)\mathrm{d}\eta. \tag{3.18}$$

For $0 < m \le n$, $k \in \mathbb{N}$, let

$$\begin{aligned}\bar{g}^{(k)}_{n,m}(x) &:= \int_{0\le r_1\le\dots\le r_{m-1}\le x} u^{(k)}_{n,m}(r_1,\dots,r_{m-1},x)\mathrm{d}r_1\dots\mathrm{d}r_{m-1}\\ &= (-1)^k \lim_{s\downarrow 0} \frac{\partial^k}{\partial s^k}\, \bar{g}_{n,m}(s,x),\end{aligned} \tag{3.19}$$

cf. (2.21), (3.10) and (3.17). Now let $x^* \in \mathbb{R}_+$ be arbitrary but fixed, $\beta^* := \sup\{\beta(x) : 0 \le x \le x^*\}$ and

$$a_n(x) := \exp\Big(-\Big(\frac{\lambda}{\varphi(n)} + n\beta^*\Big)x\Big), \quad x \in \mathbb{R}_+. \tag{3.20}$$

Integrating (3.18) over $0 \le r_1 \le \ldots \le r_{m-1} \le x$ with respect to $\mathrm{d}r_1 \ldots \mathrm{d}r_{m-1}$, taking into account

$$C_n(r; -\eta) \ge a_n(x), \quad 0 \le \eta \le r_1 \le \ldots \le r_{m-1} \le r_m = x \le x^*,$$

and using Fubini's theorem, for $0 < m \le n$, $k \in \mathbb{N}$ we obtain that

$$\varphi(n)\, \bar{g}^{(k)}_{n,m}(x) \ge k\, a_n(x) \int_0^x \bar{g}^{(k-1)}_{n,m}(\eta)\mathrm{d}\eta, \quad 0 \le x \le x^*. \tag{3.21}$$

For $0 < m \le n$, $k \in \mathbb{Z}_+$, $x \in \mathbb{R}_+$, let

$$g^{(k)}_{n,m}(x) := \lim_{s \downarrow 0} g^{(k)}_{n,m}(s, x) = \frac{\partial}{\partial x} E[V^k_m\, \mathbb{I}\{N = n,\ R_m \le x\}], \tag{3.22}$$

cf. (2.26), (2.29), (2.30). Note that from (2.20), (2.29), (3.19), (3.22) it follows

$$g^{(k)}_{n,m}(x) = \bar{g}^{(k)}_{n,m}(x)\, d_{n-m}(x). \tag{3.23}$$

Now we will show by induction that for $k \in \mathbb{Z}_+$, $n \in \mathbb{N}$, $x \in [0, x^*]$ it holds

$$\varphi(n) \sum_{m=1}^{n} g^{(k)}_{n,m}(x) \ge \lambda(x)\, x^k\, p(n-1) \Big(\frac{a_n(x)\, \bar{B}_R(x)^{n-1}}{\varphi(n)} \Big)^k. \tag{3.24}$$

For $k = 0$ the assertion (3.24) follows directly from (2.28) and (2.41). Assume that (3.24) is true for $k \in \mathbb{Z}_+$. For $k+1$ from (3.23), (3.21), (2.22), the induction assumption for k and by taking into account that $\bar{B}_R(\eta)$, $a_n(\eta)$, $\lambda(\eta)$ are decreasing in η, $\bar{B}_R(\eta) \le 1$ and $\bar{B}_R(0) = 1$ we find that

$$\begin{aligned}
\varphi(n) \sum_{m=1}^{n} g^{(k+1)}_{n,m}(x) &\ge (k+1)\, a_n(x) \sum_{m=1}^{n} \int_0^x \bar{g}^{(k)}_{n,m}(\eta) d_{n-m}(x)\mathrm{d}\eta \\
&\ge (k+1)\, a_n(x) \int_0^x \sum_{m=1}^{n} \bar{g}^{(k)}_{n,m}(\eta) d_{n-m}(\eta) \bar{B}_R(x)^{n-m}\mathrm{d}\eta \\
&\ge (k+1)\, a_n(x)\, \bar{B}_R(x)^{n-1} \int_0^x \frac{1}{\varphi(n)}\, \lambda(\eta)\, \eta^k\, p(n-1) \\
&\qquad\qquad\qquad\qquad \Big(\frac{a_n(\eta)\, \bar{B}_R(\eta)^{n-1}}{\varphi(n)} \Big)^k \mathrm{d}\eta \\
&\ge \lambda(x)\, x^{k+1}\, p(n-1) \Big(\frac{a_n(x)\, \bar{B}_R(x)^{n-1}}{\varphi(n)} \Big)^{k+1},
\end{aligned}$$

finishing the induction step, i.e., (3.24) is proved. Now, from Theorem 2.1, (2.29), (3.22) and (3.24) we obtain for $k \in \mathbb{N}$ that

$$E[V^k(n,\tau)] \geq \tau^k \left(\frac{a_{n+1}(\tau)\,\bar{B}_R(\tau)^n}{\varphi(n+1)}\right)^k, \quad \tau \in \mathbb{R}_+,\ n \in \mathbb{Z}_+. \tag{3.25}$$

In view of $\lim_{\tau\downarrow 0} a_{n+1}(\tau) = \lim_{\tau\downarrow 0} \bar{B}_R(\tau) = 1$, cf. (3.20), thus we find that

$$\liminf_{\tau\downarrow 0} \frac{E[V^k(n,\tau)]}{\tau^k} \geq \chi(n+1)^k, \quad n \in \mathbb{Z}_+. \tag{3.26}$$

Multiplying both sides of (3.25) by $p(n)$, summing up over $n \in \mathbb{Z}_+$ and taking into account (3.2), we obtain (3.3). For fixed $n \in \mathbb{Z}_+$, from (3.26), (3.3) it follows that

$$\begin{aligned}
&\limsup_{\tau\downarrow 0} \frac{E[V^k(n,\tau)]}{\tau^k}\, p(n) + \sum_{j\in\mathbb{Z}_+\setminus\{n\}} \chi(j+1)^k p(j)\\
&\quad\leq \limsup_{\tau\downarrow 0} \frac{E[V^k(n,\tau)]}{\tau^k}\, p(n) + \sum_{j\in\mathbb{Z}_+\setminus\{n\}} \liminf_{\tau\downarrow 0} \frac{E[V^k(j,\tau)]}{\tau^k}\, p(j)\\
&\quad\leq \limsup_{\tau\downarrow 0} \sum_{j\in\mathbb{Z}_+} \frac{E[V^k(j,\tau)]}{\tau^k}\, p(j) = \limsup_{\tau\downarrow 0} \frac{E[V^k(\tau)]}{\tau^k}\\
&\quad= \sum_{j\in\mathbb{Z}_+} \chi(j+1)^k p(j),
\end{aligned}$$

which provides (3.4) in view of (3.26) if (3.1) is fulfilled.

The case of a general df. $B(x)$ with finite mean m_S is obtained by taking the limit in distribution of a sequence of service time distributions $B_\nu(x)$, $\nu = 1, 2, \ldots$, where the $B_\nu(x)$ have the given mean m_S, fulfill (A1) and converge weakly to $B(x)$. Since the assertions of the theorem hold for the $B_\nu(x)$, by arguments of continuity we obtain the assertions for $B(x)$ in view of (1.1) and (1.2). $\square$

Remark 3.1 *The results of Theorem 3.1 are insensitivity results with respect to the service time distribution for given m_S. Note that in case of $k = 1$ from Cohen's result* (1.6) *indeed we know that even equality holds on the r.h.s. of* (3.2)*, cf. also the discussion after the proof of Theorem 3.2 below. In case of $k > 1$ the r.h.s. of* (3.2) *provides an upper bound for $E[V^k(\tau)]$ and for small positive values of τ a good approximation for $E[V^k(\tau)]$ because of* (3.3).

For the $M/M/SDPS$ system and waiting times Theorem 3.1 has been proved recently by the authors in [BB] Theorem 2.1. For the $M/GI/1-PS$ system, i.e. for $\varphi_1(n) = 1/n$, $n \in \mathbb{N}$, Theorem 3.1 has been proved in [CBB] Theorem 5.11 and Theorem 4.1 by using stochastic ordering theory and particular results known for $M/GI/1-PS$ systems. Analogously to [CBB] for the $M/GI/1-PS$ system, the r.h.s. of (3.2) can be interpreted as follows: consider a tagged τ-request finding at its arrival n requests in the system. If during the service of the τ-request no arrival and no departure occurs, then $\hat{V}(\tau) = \tau\,\chi(n+1)$ is the sojourn time of the τ-request, and its kth moment is $\hat{V}^k(\tau) = \tau^k\,\chi(n+1)^k$. Thus, if τ is small, then $\hat{V}(\tau) = \tau\,\chi(N+1)$ is approximately the sojourn time of a τ-request, and therefore $\hat{V}(\tau)$ is called in [CBB] the instantaneous sojourn time as $\tau \downarrow 0$. Note that the r.h.s. of (3.2) is just the kth moment of the instantaneous sojourn time $\hat{V}(\tau)$, i.e., $E[\hat{V}^k(\tau)]$ is for all $\tau \in \mathbb{R}_+$ an upper bound for $E[V^k(\tau)]$, and thus Theorem 3.1 generalizes the result of [CBB] to the general $M/GI/SDPS$ system. In case of a stable $M/GI/1-PS$ system, the assumption (3.1) is satisfied for all $k \in \mathbb{N}$, in view of $p(n) = (1-\varrho)\varrho^n$, $n \in \mathbb{Z}_+$, and $\varrho < 1$, and hence all moments of $V(\tau)$ are finite. However, for a stable $M/GI/SDPS$ system (3.1) is not fulfilled in general, e.g. in case of $\varphi(n) := (n+k+1)/n^2$, $n \in \mathbb{N}$ and $\varrho := 1$ for $k \in \mathbb{N}$.

Theorem 3.2 *For the $M/GI/SDPS$ system let the stability condition* (1.1) *and* (A1) *be satisfied, and let $k \in \mathbb{N}$ such that* (3.1) *is fulfilled. Then the kth moment of $V(\tau)$ is finite, and it holds that*

$$E[V^k(\tau)] = k \sum_{n=1}^{\infty} \sum_{m=1}^{n} \int_0^{\tau} \frac{g_{n,m}^{(k-1)}(x)}{\lambda(x)}\,\mathrm{d}x, \quad \tau \in \mathbb{R}_+, \tag{3.27}$$

where the $g_{n,m}^{(k-1)}(x)$ are given by (2.27), (2.29) *and* (3.22).

Proof In case of $k > 1$, (3.6) where k is replaced by $k-1$, (3.9) and taking the limit $s \downarrow 0$ provide

$$\frac{1}{\lambda(x)} \sum_{0<m\le n} g_{n,m}^{(k-1)}(x) \le x^{k-1} \sum_{n=0}^{\infty} \chi(n+1)^k p(n), \quad x > 0. \tag{3.28}$$

Note that (3.28) holds for $k = 1$, too, in view of (2.28). Taking into account $xe^{-x} \le 1$ for $x \in \mathbb{R}_+$, from (2.30) we find that

$$\begin{aligned} s g_{n,m}^{(k)}(s,x) &= \lim_{h\downarrow 0} E[V_m^{k-1}(sV_m e^{-sV_m})\mathbb{I}\{N=n,\, x<R_m \le x+h\}]/h \\ &\le g_{n,m}^{(k-1)}(x). \end{aligned}$$

Moreover, from (2.40) and (3.2) we obtain that

$$\varphi(n) g_{n,m}^{(k)}(s,x) \leq \lambda(x) E[V^k(x)] \leq \lambda(x) x^k \sum_{n=0}^{\infty} \chi(n+1)^k p(n),$$

which implies $\lim_{s \downarrow 0} s g_{n,m}^{(k)}(s,x) = 0$. Due to (3.28) and Lebesgue's theorem, thus it follows that

$$\lim_{s \downarrow 0} \frac{1}{\lambda(x)} \sum_{0<m \leq n} s g_{n,m}^{(k)}(s,x) = 0, \quad x > 0. \tag{3.29}$$

Because of (3.28) and (3.29), taking the limit $s \downarrow 0$ in (3.7) provides

$$\frac{\mathrm{d}}{\mathrm{d}x} E[V^k(x)] = \frac{k}{\lambda(x)} \sum_{n=1}^{\infty} \sum_{m=1}^{n} g_{n,m}^{(k-1)}(x), \quad x > 0. \tag{3.30}$$

In view of $E[V^k(0)] = 0$, integrating on $[0, \tau]$ yields (3.27). □

Note that, because of (2.28), for $k = 1$ from (3.27) it follows

$$E[V(\tau)] = \tau \sum_{n=0}^{\infty} \chi(n+1)\, p(n), \qquad \tau \in \mathbb{R}_+,$$

corresponding to Cohen's general result (1.6), cf. (1.5). From (3.27) and

$$E[V^k] = \int_0^{\infty} E[V^k(\tau)]\, b(\tau)\, \mathrm{d}\tau$$

via Fubini's theorem and Theorem 3.1 we obtain the following results for the unconditional sojourn time V.

Theorem 3.3 *For the $M/GI/SDPS$ system let the stability condition* (1.1) *and* (A1) *be satisfied, and let $k \in \mathbb{N}$ such that* (3.1) *is fulfilled and that $E[S^k] < \infty$. Then the kth moment of V is finite, and it holds that*

$$E[V^k] = \frac{k}{\lambda} \sum_{n=1}^{\infty} \sum_{m=1}^{n} \int_0^{\infty} g_{n,m}^{(k-1)}(x)\, \mathrm{d}x, \tag{3.31}$$

where the $g_{n,m}^{(k-1)}(x)$ are given by (2.27), (2.29) *and* (3.22).

Remark 3.2 *The equations* (3.27) *and* (3.31) *are useful for a numerical computation of sojourn time characteristics for particular $M/GI/SDPS$ systems, because they allow a reduction of the numerical complexity (from k to $k-1$), cf. [BB] for the $M/M/SDPS$ system.*

4 Waiting times

The waiting time in an $M/GI/SDPS$ system is defined as the difference of the sojourn and required service time of a request, cf. [Ya2] p. 107. Let $W := V - S$ and $W(\tau) := V(\tau) - \tau$ be the waiting time of an arbitrary arriving request and an arriving τ-request, respectively. In order to ensure that the waiting times are non-negative we assume in the following that $0 < \varphi(n) \le 1$, $n \in \mathbb{N}$, and that there exists an $n \in \mathbb{N}$ such that $\varphi(n) < 1$[1]. We have immediately that $EW = EV - ES$, $var(W(\tau)) = var(V(\tau))$, and from (1.6) that $EW(\tau) = (\tau/ES)EW$. For the variances $var(V)$, $var(W)$ or equivalently for the squared coefficients of variation[2] $c_V^2 := var(V)/(EV)^2$, $c_W^2 := var(W)/(EW)^2$ it holds that

$$(EV)^2(c_V^2 - c_S^2) = (EW)^2(c_W^2 - c_S^2) = \int_{\mathbb{R}_+} var(W(\tau))\, \mathrm{d}B(\tau) \ge 0, \quad (4.1)$$

cf. [BB] (4.4), which implies

$$var(V) = var(W) + var(S) + 2c_S^2 EWES. \quad (4.2)$$

Note that due to the SDPS discipline waiting and service times are not independent in contrast to the FCFS discipline, cf. (4.2). Moreover, from (4.1) it follows that $c_S^2 \le c_V^2 \le c_W^2$. Note that in (4.1) the sojourn and waiting times occur symmetrically, reflecting the fact that results for sojourn times have a correspondence to results for waiting times and vice versa. Below we will shortly summarize some corresponding results for waiting times for the $M/GI/SDPS$ system. The proofs are analogous to the arguments given in Sections 2 and 3, where one has only to take into account that the waiting time increases by $(1 - \varphi(n))h$ during an interval $[0, h]$ if n requests are in the system and if there is no arrival and no departure. Thus in the corresponding equation leading to (2.6) the term $e^{-s(V_m(h)+h)}$ has to be replaced by $e^{-s(W_m(h)+(1-\varphi(n))h)}$ and in (2.6) the term $\lambda + s$ has to be replaced by $\lambda + s(1 - \varphi(n))$. Paralleling the proof of Theorem 3.1 one finds

Theorem 4.1 *Let the stability condition* (1.1) *for the $M/GI/SDPS$ system with a general service time df. $B(x)$ be satisfied. Then the kth moment of $W(\tau)$, $\tau \in \mathbb{R}_+$, is finite if*

$$\sum_{n=0}^{\infty} (\chi(n+1) - 1)^k p(n) < \infty. \quad (4.3)$$

[1] In case of $\varphi(n) = 1$, $n \in \mathbb{N}$, the system corresponds to an $M/GI/\infty$ system, where no waiting occurs.

[2] For a r.v. X the squared coefficient of variation is denoted by $c_X^2 := var(X)/(EX)^2$.

For $k \in \mathbb{Z}_+$, it holds that

$$\tau^k \Big(\sum_{n=0}^{\infty} (\chi(n+1)-1)p(n) \Big)^k \leq E[W^k(\tau)] \leq \tau^k \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n), \qquad \tau \in \mathbb{R}_+,$$

$$\lim_{\tau \downarrow 0} \frac{E[W^k(\tau)]}{\tau^k} = \sum_{n=0}^{\infty} (\chi(n+1)-1)^k p(n).$$

If additionally (4.3) *is satisfied, then it holds that*

$$\lim_{\tau \downarrow 0} \frac{E[W^k(n,\tau)]}{\tau^k} = (\chi(n+1)-1)^k, \quad n \in \mathbb{Z}_+.$$

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