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Hyperdeterminants as integrable discrete systems

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Hyperdeterminants as integrable discrete systems

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Abstract

We give the basic definitions and some theoretical results about hyperdeterminants, introduced by A. Cayley in 1845. We prove integrability (understood as $4d$-consistency) of a nonlinear difference equation defined by the $2\times2\times2$ - hyperdeterminant. This result gives rise to the following hypothesis: the difference equations defined by hyperdeterminants of any size are integrable.

We show that this hypothesis already fails in the case of the $2\times2\times2\times2$ - hyperdeterminant.

1 Introduction

Discrete integrable equations have become a very vivid topic in the last decade. A number of important results on the classification of different classes of such equations, based on the notion of consistency [3], were obtained in [1, 2, 17] (cf. also references to earlier publications given there). As a rule, discrete equations describe relations on the scalar field variables $f_{i_1\ldots i_n} \in \mathbb{C}$ associated with the points of a lattice $\mathbb{Z}^n$ with vertices at integer points in the n-dimensional space $\mathbb{R}^n = \{(x_1,\ldots,x_n)|x_s \in \mathbb{R}\}$. If we take the elementary cubic cell $K_n = \{(i_1,\ldots,i_n) \mid i_s \in \{0,1\}\}$ of this lattice and the field variables $f_{i_1\ldots i_n}$ associated to its 2^n vertices, an n-dimensional discrete system of the type considered here is given by an equation of the form

$$Q_n(\mathbf{f}) = 0. \tag{1}$$

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Hereafter we use the short notation $\mathbf{f}$ for the set $(f_{00\ldots 0}, \ldots, f_{11\ldots 1})$ of all these 2^n variables. For the other elementary cubic cells of $\mathbb{Z}^n$ the equation is the same, after shifting the indices of $\mathbf{f}$ suitably.

The equations mostly investigated so far [1, 2, 17] were supposed to have the following properties:

1) Quasilinearity. Equation (1) is affine linear w.r.t. every $f_{i_1 i_2 \ldots i_n}$, i.e. Q has degree 1 in any of its four variables.

2) Symmetry. Equation (1) should be invariant w.r.t. the symmetry group of elementary cubic cell K_n or its suitably chosen subgroup.

On the other hand a number of interesting discrete equations which do not enjoy one or both of these properties has been found. In this publication we investigate an important class of symmetric discrete equations which do not have the quasilinearity property and are given by the equations $H_n(f_{00\ldots 0}, \ldots, f_{11\ldots 1}) = 0$, were H_n denotes the n-dimensional hyperdeterminant of the corresponding n-index array $(f_{00\ldots 0}, \ldots, f_{11\ldots 1})$. We give the precise definition of hyperdeterminants in Section 2. In the simplest two-dimensional case of the 2×2 matrix the hyperdeterminant is just the usual determinant:

$$H_2(\mathbf{f}) = f_{00}f_{11} - f_{01}f_{10}. \tag{2}$$

The next nontrivial case is the 3-dimensional $2 \times 2 \times 2$ - hyperdeterminant:

$$\begin{aligned} H_3(\mathbf{f}) = \; & f_{111}^2 f_{000}^2 + f_{100}^2 f_{011}^2 + f_{101}^2 f_{010}^2 + f_{110}^2 f_{001}^2 \\ & -2f_{111}f_{110}f_{001}f_{000} - 2f_{111}f_{101}f_{010}f_{000} \\ & -2f_{111}f_{100}f_{011}f_{000} - 2f_{110}f_{101}f_{010}f_{001} \\ & -2f_{110}f_{100}f_{011}f_{001} - 2f_{101}f_{100}f_{011}f_{010} \\ & +4f_{111}f_{100}f_{010}f_{001} + 4f_{110}f_{101}f_{011}f_{000}. \end{aligned} \tag{3}$$

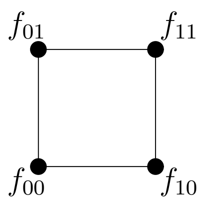

Figure 1: Square K_2.

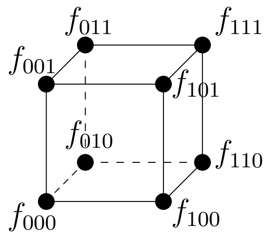

Figure 2: Cube K_3.

The corresponding elementary cells K_2, K_3 and the field variables associated with the vertices are shown on Figures 1, 2.

The general definition of hyperdeterminants was given by A. Cayley [7], who also gave the explicit form (3) of the first nontrivial $2\times 2\times 2$ - hyperdeterminant. In the last decades, following the modern and much more general approach of $\mathcal{A}$-discriminants [9], the theory of hyperdeterminants found important applications in quantum informatics [4], biomathematics [5], numerical analysis and data analysis [6] as well as other fields.

As one can easily see, the expressions (3.7) in [15] and (6.11) in [16], describing some discrete integrable equations, are nothing but the classical Cayley's $2 \times 2 \times 2$ -

hyperdeterminant (3). We prove below in Section 3 that (3) is also integrable in the sense of $(n+1)$-*dimensional consistency* [3]:

An n-dimensional discrete equation (1) is called consistent, if it may be imposed in a consistent way on all n-dimensional faces of a $(n+1)$-dimensional cube.

We give the accurate formulation of this general consistency principle for the case of non-quasilinear expressions similar to (3) in Section 3. For the two-dimensional determinant (2) (which is quasilinear) consistency can be established by a trivial computation; the equation $H_2(f_{00}, f_{11}, f_{01}, f_{10}) = 0$ is obviously linearized by the substitution $f_{ij} = \exp \tilde{f}_{ij}$. Using a result on Principal Minor Assignment Problem proved in [14] we establish $4d$-consistency of the $2 \times 2 \times 2$ - hyperdeterminant (3) in Section 3, cf. Theorem 2 below for the precise formulation.

This result gives rise to the following *Conjecture*: the difference equations defined by hyperdeterminants of any size are integrable in the sense of $(n+1)$-dimensional consistency. Nevertheless as we show in Section 4, this Conjecture fails already in the case of the $2\times2\times2\times2$ - hyperdeterminant. The computation of this $4d$ - hyperdeterminant turns out to be highly nontrivial (compared to the relatively simple expressions (2), (3)) and was completed only recently [13]. We report in Section 4 a more straightforward and simpler computation of the same hyperdeterminant with the free symbolic computation program FORM [18]. The size of this hyperdeterminant (2 894 276 terms, total degree 24, degree 9 w.r.t. each of the field variables) implies that checking its $5d$-consistency can be done only numerically, using high precision computation of roots of respective polynomial equations on the $4d$-faces of the 5-dimensional cube K_5. This was done using two different computer algebra systems REDUCE [19] and SINGULAR [20]. As our computations have shown (cf. their description in Section 4), the $4d$ - hyperdeterminantal equation $H_4(\mathbf{f}) = 0$ is not $5d$-consistent. This non-integrability result should be investigated further since recent examples [11] show that consistency is not the only possible definition for discrete integrability.

2 The definition of hyperdeterminants and its variations

The remarkable definition of hyperdeterminants given by A. Cayley in 1845 [7] and still used today [9] describes the condition of singularity of an appropriate hypersurface. Let $A = (a_{i_1 i_2 \cdots i_r})$ be a hypermatrix (an array with r indices) with $i_s = 0, \ldots, n_s$. The polylinear form

$$U = \sum_{i_1 \cdots i_r} a_{i_1 \cdots i_r} x^{(1)}_{i_1} \cdots x^{(r)}_{i_r}$$

defines a hypersurface $U = 0$ in $\mathbb{C}P^{n_1} \times \ldots \times \mathbb{C}P^{n_r}$. Here $x^{(k)}_{i_k}$ denote the homogeneous coordinates in the respective complex projective space $\mathbb{C}P^{n_k}$. This hypersurface is singular, i.e. has at least one point where the condition of smoothness is not satisfied iff the following set of $(n_1+1)\cdot\ldots\cdot(n_r+1)$ equations

$$\left\{\forall s=1,\ldots,r,\quad \forall k=1,\ldots,n_s,\quad \frac{\partial U}{\partial x^{(k)}_{i_s}}=0\right\} \tag{4}$$

has a nontrivial solution $x^{(k)}_{i_s}\in\mathbb{C}P^{n_k}$. As one can show (cf. [9]), if a certain condition (5) on the dimensions n_k of the array A is satisfied, elimination of the variables $x^{(k)}_{i_s}$ from (4) results in a single polynomial equation in the array elements $a_{i_1 i_2\cdots i_r}$: $H_r(A)=0$. This polynomial is irreducible and enjoys practically the same symmetry properties as the usual determinant of a square matrix. Following Cayley this polynomial $H_r(A)$ (defined uniquely up to a constant factor) is called the *hyperdeterminant of the array A*. The necessary and sufficient condition of existence of a *single* polynomial condition $H_r(A)=0$ for the hypersurface $U=0$ to be singular, i.e. the condition for the corresponding hyperdeterminant of A to be correctly defined, is as follows:

$$\forall k,\qquad n_k\le\sum_{s\neq k}n_s. \tag{5}$$

In particular, if $r=2$, so for usual $(n_1{+}1)\times(n_2{+}1)$-matrices, this condition implies $n_1=n_2$, and in this case the hyperdeterminant H_2 coincides with the classical determinant of the matrix $A_{i_1 i_2}$. Note that for a given set $\{n_1,\ldots,n_r\}$ of array dimensions one says that we have the corresponding $(n_1{+}1)\times\ldots\times(n_r{+}1)$ - hyperdeterminant since the array indices range from 0 to n_k. The hyperdeterminant is $SL(\mathbb{C},n_1{+}1)\times\cdots\times SL(\mathbb{C},n_r{+}1)$-invariant, which means that if one adds to one *slice* $A_{k,p}=\{(a_{i_1 i_2\cdots i_r})\ \text{with fixed}\ \ i_k=p\}$ another parallel slice $A_{k,q}$, $q\neq p$, multiplied by some constant λ, the value of H_r is unchanged; swapping the slices $A_{k,p}$, $A_{k,q}$ either leave H_r again invariant or changes its sign depending on the parity of the dimensions n_i; finally, multiplication of a slice $A_{k,p}$ with a constant λ results in multiplication of the hyperdeterminant by an appropriate power of λ. H_r is also invariant w.r.t. the transposition of any two indices i_l, i_m of the hypermatrix $A=(a_{i_1 i_2\cdots i_r})$.

As we have stated in the introduction, the first nontrivial $2\times2\times2$ - hyperdeterminant (3) was computed by A. Cayley himself [7]. Amazingly enough, already the next step, computation of the $2\times2\times2\times2$ - hyperdeterminant is very difficult. The problem of computation of an *explicit* polynomial expression for this case was proposed by I. M. Gel'fand in his Fall 2005 research seminar at Rutgers University. The monomial expansion of the $2\times2\times2\times2$ - hyperdeterminant is related to some combinatorial problems, and was done (using an inductive algorithm of L. Schläfli [8]) for the first time in [13], using a dedicated C code; this computation required a serious programming effort since the standard computer algebra systems like Maple can not cope with the intermediate large expressions. The resulting polynomial expression for the $2\times2\times2\times2$ - hyperdeterminant has 2 894 276 terms, total degree 24, and has degree 9 w.r.t. each of the array entries $a_{i_1 i_2 i_3 i_4}$. The size of this expression in usual text format is around 200 megabytes.

In October 2007 we re-checked this computation of the $2\times2\times2\times2$ - hyperdeterminant using a free symbolic computation program Form [18] and the same inductive algorithm of L. Schläfli [8]. The computation required 8 hours on a 3GHz processor of one of the SHARCNET nodes (`www.sharcnet.ca`) and some 800 Mb of temporary disk storage. Due to the efficient design of FORM one had no need to write any special low-level code, the standard FORM routines are completely sufficient. The obtained expression can be saved either in text format (around 200 Mb) or in the internal binary FORM format; both can be used as input to other FORM runs and require only around 1 minute to be read into a session. Moreover, FORM could cope with a straightforward check of the invariance of the obtained hyperdeterminant w.r.t. addition of one slice $A_{k,p}$ multiplied by a symbolic constant λ to another parallel slice $A_{k,q}$, $q \neq p$.

This more challenging computation still used the usual FORM routines and required some 10 hours of CPU time and around 200 Gb of temporary disk storage (the size of the intermediate expression reached $\sim 800\,000\,000$ terms!). This may suggest that some other hyperdeterminants of higher size may also be computed using available software and hardware.

Amazingly enough, if one reads the original Cayley papers [7], then a *relatively small* expression with some 340 terms which Cayley describes as *the* $2 \times 2 \times 2 \times 2$ - hyperdeterminant can be found! A straightforward check shows that this expression enjoys the invariance properties for hyperdeterminants stated above. On the other hand, Proposition 1.6 in [9, p. 447] states that if a polynomial in the entries $a_{i_1 i_2 \cdots i_r}$ of a hypermatrix A has these invariance properties *and meets some extra weak condition on the stars of the monomial powers* then it should be divisible by the respective hyperdeterminant of A. The Cayley's expression does satisfy the necessary invariance conditions; only a few terms do not meet the required extra condition on the stars of the monomial powers, so this statement of Cayley on the explicit form of the $2\times2\times2\times2$ - hyperdeterminant is wrong. This was already remarked by L. Schläfli [8] who gave an inductive algorithm for the computation of hyperdeterminants. Unfortunately, as shown in [10], the Schläfli algorithm works only for very special hyperdeterminants, in particular the only hyperdeterminants with $n_1 = \ldots = n_r = 1$ which it can compute are precisely the $2 \times 2 \times 2$- and $2 \times 2 \times 2 \times 2$ - hyperdeterminants. Already for the $2 \times 2 \times 2 \times 2 \times 2$ - hyperdeterminant there seems to be no better way to compute the explicit expression other than the elimination procedure given in the definition of hyperdeterminants.

Based on the above symmetry discussion one could adopt other definitions for hyperdeterminants. Historically this resulted in a few other definitions of hyperdeterminants as invariant expressions. Many of them have a much simpler form than the definition we apply. A review of various definitions can be found in [12].

3 The Cayley $2 \times 2 \times 2$ - hyperdeterminant as a discrete integrable system and the Principal Minor Assignment Problem

The integrability definition applied in this publication is based on the requirement of consistency which is described in this section in detail for the case of the $3d$ - hyperdeterminant (3). Suppose we have a $4d$-cube K_4 shown on figure 4 with field values $f_{ijkl},\ i,j,k,l \in \{0,1\}$. One should impose the formula (3) on every $3d$-face of K_4, by fixing one of the indices i,j,k,l, and making it 0 for the faces which we will call below "initial faces", or respectively 1 for the faces which we will call "final faces". Further one needs to fix some mapping from the initial "standard" $3d$-cube shown on figure 3 (with the vertices labelled f_{ijk}) onto every one of the eight $3d$-faces (for example $\{f_{i1kl}\}$ on $\{x_2 = 1\}$). Due to the symmetry properties of the $3d$ - hyperdeterminant (3) this can be done, for example, using the trivial lexicographic correspondence of the type $f_{ijk} \mapsto f_{i1jk}$. The initial data are some arbitrary complex values of the field variables f_{ijkl} assigned to the vertices shown on figure 4 by black circles. Then, using the equation (3) imposed on the initial faces we can find the values of the field variables for the last (8^{th}) vertex of the respective initial face, such vertices are shown on figures 3, 4 as white circles. Obviously since the equations (3) for these values are quadratic we obtain two possible values. On the next step we impose (3) to hold on the 4 final faces using the initial data and the values found on the previous step. From each of the 4 final $3d$-faces we again obtain possible values for the final vertex f_{1111} shown on figure 4 by a small white box. For generic initial data we have on *each* final face $2^3 = 8$ different choices for the intermediate values (white circles) so we find in principle $8 \cdot 2 = 16$ different possible values for f_{1111} from each final face. How many of them coincide among the 4 final faces?

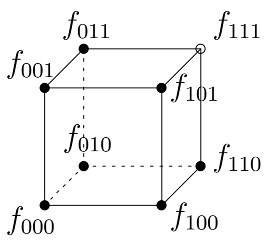

Figure 3: Cube K_3.

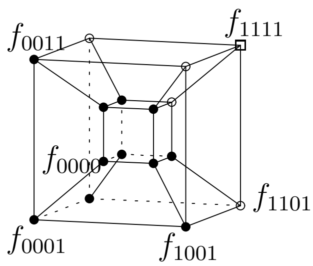

Figure 4: Cube K_4.

As we prove below, 8 of every 16 values for f_{1111} found for each of the final $3d$-faces are common. All other $8 \cdot 4 = 32$ are in general not shared between the final faces as it has been confirmed by numerical examples.

This result should be considered as the proof of consistency for face formula (3); our considerations below are based on a remarkable result proved in [14].

Let us first formulate the necessary definitions and results of [14]. Suppose we have a real symmetric $n\times n$-matrix $M = (m_{ij})$. Its principal minors form a vector of length 2^n with entries indexed by subsets I of the set $\{1, 2, \ldots, n\}$. Namely, M_I denotes the minor of M whose rows and columns are indexed by I. This includes the 0×0-minor $M_\emptyset = 1$. The famous Principal Minor Assignment Problem considers the description of a suitable complete set of algebraic relations among the minors of a generic symmetric $n\times n$-matrix $M = (m_{ij})$. The first observation (formula (2)) of [14] consists in the fact that for principal minors of a 3×3 symmetric matrix one has the following relation:

$$\begin{array}{l} M_\emptyset^2 M_{123}^2 + M_1^2 M_{23}^2 + M_2^2 M_{13}^2 + M_3^2 M_{12}^2 + 4M_\emptyset M_{12} M_{13} M_{23} + 4M_1 M_2 M_3 M_{123} \\ -2M_\emptyset M_1 M_{23} M_{123} - 2M_\emptyset M_2 M_{13} M_{123} - 2M_\emptyset M_3 M_{12} M_{123} - 2M_1 M_2 M_{13} M_{23} \\ -2M_1 M_3 M_{12} M_{23} - 2M_2 M_3 M_{12} M_{13} \quad = \quad 0. \end{array}$$

This obviously gives us the Cayley's hyperdeterminant (3) if we interpret every minor M_I as the field variable $f_{i_1 i_2 i_3}$ with $i_s = 1$ if $s \in I$ and $i_s = 0$ otherwise, for example $M_{13} = f_{101}$. For the initial vertex we have $f_{000} = M_\emptyset = 1$. For symmetric matrices M of larger size the $2 \times 2 \times 2$ - hyperdeterminantal relations are also fulfilled for "shifted" principal minors, in our terminology this means that the hyperdeterminant (3) vanishes on every $3d$-face of the n-dimensional hypercube with field variables $f_{i_1 \ldots i_n}$ equal to the principal minors M_I such that $i_s = 1$ if $s \in I$ and $i_s = 0$ otherwise. As a remarkable fact (not necessary to us) we mention that for $d > 2$ on any d-dimensional face of this n-dimensional hypercube the corresponding d-dimensional hyperdeterminant also vanishes:

Theorem 1 ([14]) *Let $M = (m_{ij})$ be a symmetric $n\times n$ matrix. Then the vector M_* of all principal minors of M is a common zero of all the hyperdeterminants of formats from $2\times 2\times 2$ up to $\underbrace{2\times 2\times \cdots \times 2}_{n\ terms}$.*

For $n \geq 2$, the entries of the symmetric matrix $M = (m_{ij})$ are determined up to sign by their principal minors of size 1×1 and 2×2 since $m_{ii} = M_i$, $m_{ij}^2 = M_i M_j - M_{ij} M_\emptyset$. As we see, the value $f_{111} = M_{123}$ for the final vertex is not defined uniquely from the relation (3); the choice of one of the two possible values corresponds to a suitable choice of the signs for the 3 off-diagonal elements m_{12}, m_{13}, m_{23}. These $2^3 = 8$ sign combinations give only two different values for M_{123}, because changing simultaneously the signs of a certain row and the column symmetric to it we do not change the principal minors of the matrix, so we can fix the signs of m_{12} and m_{13}.

If we take a 4×4 symmetric matrix, all its elements will be defined up to the sign of the off-diagonal elements by the principal minors of size 1×1 and 2×2; this corresponds precisely to the choice of the initial data (black circles) on figure 4. So provided we choose the initial data f_{ijkl} with $f_{0000} = 1$, we can find a corresponding 4×4 symmetric matrix with fixed 1×1 and 2×2 minors; the off-diagonal elements of the matrix are fixed up to the signs. So, fixing 3 signs of the 6 off-diagonal elements (changing simultaneously the signs of a certain row and the column symmetric to it) we have only 3 essential sign choices; as the numerical examples show, they really give

$2^3 = 8$ different minors $M_{1234} = \det M$. Precisely these 8 values should give the 8 values for f_{1111} that should coincide after computation of 16 possible candidates for f_{1111} from each of the 4 final faces from the initial data.

If we choose initial data with $f_{0000} \neq 1$, $f_{0000} \neq 0$, homogeneity of the face equations (3) allows us to reduce this situation to the case $f_{0000} = 1$ considered above. Thus the following statement has been proved:

Theorem 2 *Let some generic initial data* $\{f_{0000}, f_{1000}, f_{0100}, f_{0010}, f_{0001}, f_{1100}, f_{1010}, f_{1001}, f_{0110}, f_{0101}, f_{0011}\}$ *on the cube* K_4 *be given. After computation of the two possible values for each of the intermediate vertices* $\{f_{1110}, f_{1101}, f_{1011}, f_{0111}\}$ *from the face relations (3) on the respective* 3*-dimensional initial faces, among the sets of 16 possible values of* f_{1111} *for each if the 4 final faces, subsets of 8 values coincide for all of them. They are equal to the 8 possible values of* $\det M$ *for the symmetric matrices having given* $1{\times}1$ *and* $2{\times}2$ *principal minors corresponding to the initial data for* f_{ijkl}.

4 The next step: the $2\times2\times2\times2$ - hyperdeterminant and its $5d$-inconsistency

The definition of hypothetical $5d$-consistency for the $2 \times 2 \times 2 \times 2$ - hyperdeterminant which has degree 9 w.r.t. each of its 16 variables is easily formulated along the lines given in the previous Section. So in this case not only the size of the face equations, but also the number of possible choices of the intermediate values f_{ijklm} with $i+j+k+l+m = 4$ for the computation of the final f_{11111} from each of the 5 final faces of the 5-dimensional hypercube K_5 is dramatically increased.

The strategy of numerical checking the hypothesis of $5d$-consistency adopted by us involved the following steps.

1) We assign some random integer values for the initial data f_{ijklm} with $i + j + k + l + m \leq 3$ and use Form to substitute them into the expressions of the $2 \times 2 \times 2 \times 2$ - hyperdeterminant on the 5 initial $4d$-faces obtaining univariate polynomials for each of the intermediate $f_{01111}, \ldots, f_{11110}$ and output the resulting expressions into a text file for further processing by Reduce and independently by Singular. For initial data being random integers in the range $[1, 100]$, the obtained equations have integer coefficients with approximately 40 decimal digits.

2) The same substitution of the initial data into the final faces is performed with Form, resulting in much larger multivariate polynomials for the intermediate values $f_{01111}, \ldots, f_{11110}$ and the final f_{11111}. These polynomials (each has the size of $\sim$360 Kb) are output into a text file for further processing by Reduce and independently by Singular.

3) The 5 univariate polynomial equations for the intermediate $f_{01111}, \ldots, f_{11110}$ are solved with a guaranteed precision of 150 digits.

4) For each one of the 9^5 combinations of the 9 complex roots the following computation is performed.

a) The set of complex roots for $f_{01111}, \ldots, f_{11110}$ is replaced in the 5 final face relations obtained on step 2, which makes them univariate polynomials for f_{11111} with complex rounded coefficients.

Starting with a guaranteed precision of 20 digits:

b) One of the obtained polynomials is solved for f_{11111} and

c) successively the other 4 polynomials are solved for f_{11111} as long as there is a non-empty *approximate intersection* (with a definite relative tolerance, see below) of the sets of roots for the f_{11111} for all the polynomials solved so far.

d) If all five face relations have at least one common approximate solution then execution continues with step b) with twice as many guaranteed precise digits, up to a maximum of 80 digits. This was never necessary.

To increase safety, two complex values u, v were only considered NOT to be approximately equal if for p precise digits the difference $u - v$ differed significantly from zero, more precisely if $|u - v|/|u| > 10^{-p/2}$.

The computation performed on the nodes of SHARCNET has shown that *no equal values for* f_{11111} are obtained from the 5 final faces. The details of the computation and the code used can be obtained from the authors or downloaded from `http://lie.math.brocku.ca/twolf/papers/TsWo2008/`.

This results in the conclusion that:
The $2 \times 2 \times 2 \times 2$ *- hyperdeterminant is not* $5d$*-consistent.*

The safety of our numerical inconsistency result is increased by the fact that it is obtained by two completely different computer algebra systems. The SINGULAR package is written in C using an arbitrary precision C library for their numerical computations whereas REDUCE uses a long number arithmetic implemented in LISP.

We have also checked that the relatively small expression given by Cayley [7] for the $2 \times 2 \times 2 \times 2$ - hyperdeterminant is also $5d$-inconsistent.

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