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ZIB-Report 09-25 (September 2009)

Approximations for the second moments of sojourn times in M/GI systems under state-dependent processor sharing

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Abstract

We consider a system with Poisson arrivals and general service times, where the requests are served according to the State-Dependent Processor Sharing (SDPS) discipline (Cohen's generalized processor sharing discipline), where each request receives a service capacity which depends on the actual number of requests in the system. For this system, denoted by $M/GI/SDPS$, we derive approximations for the squared coefficients of variation of the conditional sojourn time of a request given its service time and of the unconditional sojourn time by means of two-moment fittings of the service times. The approximations are given in terms of the squared coefficients of variation of the conditional and unconditional sojourn time in related $M/D/SDPS$ and $M/M/SDPS$ systems, respectively. The numerical results presented for $M/GI/m-PS$ systems illustrate that the proposed approximations work well.

Mathematics Subject Classification (MSC 2000): 60K25, 68M20, 90B22,

Keywords: Poisson arrivals; general service times; state-dependent processor sharing; Cohen's generalized processor sharing; many-server; permanent customers; $M/GI/m-PS$; sojourn time; moments; approximations; two-moment fitting.

1 Introduction

Processor Sharing (PS) systems have been widely used in the last decades for modeling and analyzing computer and communication systems, cf. e.g.

[BBJ], [BB2], [GRZ], [Ott], [PG], [Yas], and the references therein. In this paper we deal with approximations for the second moment of the conditional sojourn time $V(\tau)$ of a request with required service time τ (τ-request) and of the unconditional sojourn time V in the following system, denoted by $M/GI/SDPS$: At a node requests arrive according to a Poisson process of intensity λ with i.i.d. service times, which are independent of the arrival process and have the df. $B(x) := P(S \le x)$ with finite mean $m_S := ES$, where S denotes a generic service time. The requests are served according to the following State-Dependent Processor Sharing (SDPS) discipline, cf. [Coh], [BBJ]: If there are $n \in \mathbb{N} := \{1, 2, \ldots\}$ requests in the node then each of them receives a positive service capacity $\varphi(n)$, i.e., each of the n requests receives during an interval of length $\Delta\tau$ the amount $\varphi(n)\Delta\tau$ of service. In case of $\varphi_1(n) = 1/n$, $n \in \mathbb{N}$, we obtain the well known $M/GI/1-PS$ system, cf. e.g. [Yas]. In case of $\varphi_{1,k}(n) = 1/(n+k)$, $n \in \mathbb{N}$, we have a single-server PS system with $k \in \mathbb{N}$ permanent requests in the system, in case of $\varphi_m(n) = \min(m/n, 1)$, $n \in \mathbb{N}$, an $M/GI/m-PS$ system, i.e. an m-server PS system, where all requests are served in a PS mode, but each request receives at most the capacity of one processor, cf. [Coh] p. 283, [Bra], [GRZ]. Finally, in case of $\varphi(n) = 1$, $n \in \mathbb{N}$, we have an $M/GI/\infty$ system.

The $M/GI/SDPS$ system is stable if and only if

$$\sum_{n=0}^{\infty} \prod_{j=1}^{n} \frac{\varrho}{j\varphi(j)} < \infty, \tag{1.1}$$

where $\varrho := \lambda m_S$ denotes the offered load, cf. [Coh] (7.18). We assume in the following that the system is stable and in steady state. Then the distribution $p(n) := P(N = n)$, $n \in \mathbb{Z}_+$, of the stationary number N of requests in the system is given by, cf. [Coh] (7.19),

$$p(n) = \left(\sum_{m=0}^{\infty} \prod_{j=1}^{m} \frac{\varrho}{j\varphi(j)} \right)^{-1} \prod_{j=1}^{n} \frac{\varrho}{j\varphi(j)}, \quad n \in \mathbb{Z}_+. \tag{1.2}$$

For the unconditional sojourn time V of an arriving request from Little's law and (1.2) we find that

$$EV = \frac{1}{\lambda} \sum_{n=0}^{\infty} np(n) = m_S \sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)}\, p(n), \tag{1.3}$$

and for the conditional sojourn time $V(\tau)$ of a τ-request it holds

$$EV(\tau) = \frac{\tau}{m_S}\, EV = \tau \sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)}\, p(n), \quad \tau \in \mathbb{R}_+, \tag{1.4}$$

cf. [Coh] (7.27). More generally, for $\tau \in \mathbb{R}_+$, $k \in \mathbb{N}$ we have the estimate

$$\tau^k \Big(\sum_{n=0}^{\infty} \frac{1}{\varphi(n+1)} \, p(n) \Big)^k \leq E[V^k(\tau)] \leq \tau^k \sum_{n=0}^{\infty} \Big(\frac{1}{\varphi(n+1)} \Big)^k p(n), \quad (1.5)$$

and it holds

$$\lim_{\tau \downarrow 0} \frac{E[V^k(\tau)]}{\tau^k} = \sum_{n=0}^{\infty} \Big(\frac{1}{\varphi(n+1)} \Big)^k p(n), \quad (1.6)$$

cf. [BB3] Theorem 3.1. It seems that for the general $M/GI/SDPS$ system for V and $V(\tau)$ besides (1.3)–(1.6) there are known only asymptotic results for heavy tailed service times, cf. [GRZ]. However, for special cases several results and numerical algorithms are known. For the $M/GI/1-PS$ system and special cases, cf. e.g. [Yas], [SGB], for the $M/M/m-PS$ system cf. [Bra], for the general $M/M/SDPS$ system cf. [BB2]. For the $M/GI/SDPS$ system with service times exponentially distributed in a neighborhood of zero as well as for the $M/D/SDPS$ system and in particular for the $M/D/2-PS$ system cf. [BB4]. In [vBe] there are given simple approximations for the second moments of $V(\tau)$ and V in the $M/GI/1-PS$ system. For an approximation of V in the $GI/GI/1-PS$ system see [Sen], for an approximation and an upper bound of EV in the $G/GI/1-PS$ system see [BB1].

The aim of this paper is to derive for the $M/GI/SDPS$ system in Section 2 approximations for the second moment of $V(\tau)$ and in Section 3 approximations for the second moment of V. They are based on two two-moment fittings of the service times, and are given in terms of the first two moments of $V(\tau)$ and V in related $M/D/SDPS$ and $M/M/SDPS$ systems, respectively. The numerical results presented for $M/GI/m-PS$ systems in Section 4 illustrate that the proposed approximations work well and can be computed efficiently.

2 Approximations for the second moment of $V(\tau)$

We assume that the $M/GI/SDPS$ system is stable and in steady state. Further, we assume that $\sum_{n=0}^{\infty} \varphi(n+1)^{-2} \, p(n)$ is finite, ensuring that $E[V^2(\tau)]$ is finite, too, cf. (1.5).

Instead of dealing with the second moment of a r.v. X, we consider its squared coefficient of variation (scv) $c^2(X) := var(X)/(EX)^2$ in the following. Note that from (1.4)–(1.6) it follows that

$$c^2(V(\tau)) \leq c^2(1/\varphi(N+1)), \quad \tau \in (0, \infty), \quad (2.1)$$

with equality for $c^2(V(0)) := \lim_{\tau\downarrow 0} c^2(V(\tau))$. For obtaining approximations for $c^2(V(\tau))$ based on a two-moment characterization of $B(x)$ given by m_S and $c^2(S)$, we model the service time by a mixture of a zero and a positive service time d_0:

$$B_{D,p}(x) := (1-p)\,\mathbb{I}\{x \geq 0\} + p\,\mathbb{I}\{x \geq d_0\}, \quad x \in \mathbb{R}_+, \tag{2.2}$$

where $p \in (0,1]$, $d_0 \in (0,\infty)$. The mean and scv of (2.2) are given by $m_S = p\,d_0$, $c^2(S) = (1-p)/p$. The df. (2.2) can be used for modeling arbitrary service times with given mean $m_S \in (0,\infty)$ and scv $c^2(S) \in [0,\infty)$ since the parameters

$$p := 1/(c^2(S)+1), \qquad d_0 := m_S/p \tag{2.3}$$

provide the desired mean and scv, i.e., (2.2), (2.3) provide a two-moment fitting for arbitrary service times. Assume now that S has the df. $B_{D,p}(x)$. Note that under the $SDPS$ discipline the sojourn times of the zero service time requests are zero and that they do not have any impact on the system dynamics. Thus the dynamics of the $M/GI/SDPS$ system correspond to these of an $M/D/SDPS$ system with arrival intensity $\lambda_0 := p\lambda$ and deterministic service times $d_0 = m_S/p$. Therefore the sojourn time $V^{D,p}(\tau)$ of a τ-request in the $M/GI/SDPS$ system with service time df. (2.2), (2.3) equals the sojourn time of a τ-request in the $M/D/SDPS$ system with arrival intensity λ_0 and deterministic service times d_0 in distribution. Time scaling provides that $V^{D,p}(\tau)$ equals $p^{-1}V^D(p\tau)$ in distribution, where $V^D(\tau)$ denotes the sojourn time of a τ-request in the $M/D/SDPS$ system with arrival intensity λ and deterministic service times $d := m_S$. In particular, we find that

$$c^2(V^{D,p}(\tau)) = c^2(V^D(p\tau)) = c^2(V^D(\tau/(c^2(S)+1))), \tag{2.4}$$

cf. (2.3). Note that (2.4) implies that (2.1) is tight for any $\tau \in (0,\infty)$ with respect to $B(x)$ given m_S by choosing p sufficiently small, i.e., within a one-moment characterization of the service times (2.1) cannot be improved. In case of an arbitrary $M/GI/SDPS$ system with service time df. $B(x)$, characterized by m_S and $c^2(S)$, the r.h.s. of (2.4) provides the following first approximation for $c^2(V(\tau))$:

$$c^2(V(\tau)) \approx c^2_{app1}(V(\tau)) := c^2(V^D(\tau/(c^2(S)+1))). \tag{2.5}$$

The approximation (2.5) bases on approximating $B(x)$ via the two-moment matching (2.2), (2.3). Note that for any approximation of $c^2(V(\tau))$ in $M/GI/SDPS$ given by a two-moment fitting of arbitrary service times,

$c^2(V^D(\tau))$ is needed in case of $c^2(S) = 0$. In [BB4] Theorem 3.2 there is given an algorithm for computing $E[(V^D(\tau))^2]$, $\tau \in [0, d)$, based on solving numerically an infinite linear system of ordinary differential equations (ODEs) with constant coefficients. Moreover, there is derived a representation for $var(V(\tau))$, $\tau \in [0, d]$, in $M/D/2 - PS$, cf. [BB4] Corollary 4.1, which provides that

$$c^2(V^D(\tau)) = \frac{\varrho}{2} - \Big(1+\frac{\varrho}{2}\Big)\Big(1-\frac{\varrho}{2}\Big)\frac{e^\xi - 1 - \xi}{\xi^2} + \frac{1}{18}\Big(1+\frac{\varrho}{2}\Big)\Big(1-\frac{\varrho}{2}\Big)^3 \frac{(12\xi-10)e^\xi + 9 + e^{-2\xi}}{\xi^2}\Big|_{\xi=\lambda\tau/2}, \quad \tau \in (0, d], \tag{2.6}$$

for the $M/D/2 - PS$ system. Note that

$$c^2(V^D(\tau)) = 1 - 2(1-\varrho)\frac{e^\xi - 1 - \xi}{\xi^2}\Big|_{\xi=\lambda\tau}, \quad \tau \in (0, d], \tag{2.7}$$

for the $M/D/1-PS$ system, cf. e.g. [Ott] Section 5, [vBe] (2.1), (2.12). The numerical results given in Section 4 illustrate that in case of $M/GI/m-PS$ systems the approximation (2.5) works well. However, the numerical complexity for solving the ODEs may become rather high. Moreover, only in case of $\tau \in (0, (c^2(S)+1)m_S)$ we can apply the algorithm given in [BB4] Theorem 3.2 for computing $c^2_{app1}(V(\tau))$, cf. (2.5). Therefore we are interested in further approximations for $c^2(V(\tau))$.

Applying the approximation (2.5) to exponential service times, we find for $c^2(V^D(\tau))$ the approximation

$$c^2(V^D(\tau)) \approx c^2(V^M(2\tau)), \tag{2.8}$$

where $V^M(\tau)$ denotes the sojourn time of a τ-request in the corresponding $M/M/SDPS$ system with arrival intensity λ and exponential service times with mean $d = m_S$. Inserting now approximation (2.8) into (2.5), we obtain the following second approximation for $c^2(V(\tau))$ for a general $M/GI/SDPS$ model:

$$c^2(V(\tau)) \approx c^2_{app2}(V(\tau)) := c^2(V^M(2\tau/(c^2(S)+1))). \tag{2.9}$$

The advantage of approximation (2.9) is that $E[(V^M(\tau))^2]$ can be computed for all $\tau \in (0, \infty)$ by means of the faster algorithm given in [BB2] (2.24), (2.26)–(2.28) for $M/M/SDPS$ systems, based again on solving numerically an infinite linear system of ODEs with constant coefficients. Remember that

$$c^2(V^M(\tau)) = 2\varrho \, \frac{e^{-\xi}-1+\xi}{\xi^2} \Big|_{\xi=\lambda\tau(1-\varrho)/\varrho}, \quad \tau \in (0,\infty), \tag{2.10}$$

for the $M/M/1-PS$ system, cf. e.g. [Yas] (2.6), (2.7). Alternatively to the modeling of $B(x)$ by (2.2), (2.3), in case of $c^2(S) \geq 1$ one can also model $B(x)$ by a mixture of a zero and an exponential service time with mean d_0:

$$B_{M,p}(x) := (1-p)\, \mathbb{I}\{x \geq 0\} + p\,(1-\exp(-x/d_0)), \quad x \in \mathbb{R}_+, \tag{2.11}$$

where $p \in (0,1]$, $d_0 \in (0,\infty)$. Since the mean and scv of (2.11) are given by $m_S = p\, d_0$ and $c^2(S) = (2-p)/p$, respectively, the df. (2.11) can be used for modeling arbitrary service times with given mean $m_S \in (0,\infty)$ and scv $c^2(S) \in [1,\infty)$, since the parameters

$$p := 2/(c^2(S)+1), \qquad d_0 := m_S/p \tag{2.12}$$

provide the desired mean and scv, i.e., (2.11), (2.12) provide a two-moment fitting of the service times in case of $c^2(S) \in [1,\infty)$. Analogously to the derivations of (2.4), by applying the arguments given for the df. (2.2), (2.3) to the df. (2.11), (2.12), one finds for the sojourn time $V^{M,p}(\tau)$ of a τ-request in the $M/GI/SDPS$ system with service time df. (2.11), (2.12) that

$$c^2(V^{M,p}(\tau)) = c^2(V^M(p\tau)) = c^2(V^M(2\tau/(c^2(S)+1))), \tag{2.13}$$

cf. (2.12). Thus (2.9) is exact in case of the service time df. (2.11), (2.12). Therefore we have two heuristics for the approximation (2.9) in case of $c^2(S) \geq 1$, which thus should be a good approximation at least for $c^2(S) \geq 1$ from a theoretical point of view. However, for $c^2(S)$ small (2.5) will be in general a better approximation than (2.9), since approximation (2.5) is exact for $c^2(S) = 0$. Note that the approximations (2.5) and (2.9) satisfy (2.1) with equality in the limiting case of $\tau \downarrow 0$, and that they thus are exact in the limiting case of $\tau \downarrow 0$. Therefore the approximations (2.5) and (2.9) should work very well for τ small. The corresponding approximations for $c^2(V((c^2(S)+1)\tau))$ are independent of $B(x)$ given m_S, and they take values in $[0, c^2(1/\varphi(N+1))]$ because of (2.1), cf. Table 4.1 below.

Remark 2.1 *The quality of the approximations* (2.5) *for* $c^2(S) \geq 0$ *and* (2.9) *for* $c^2(S) \geq 1$ *is determined by the sensitivity of* $c^2(V(\tau))$ *with respect to* $B(x)$ *given* m_S *and* $c^2(S)$*. The approximations* (2.5) *for* $c^2(S) \geq 0$ *and* (2.9) *for* $c^2(S) \geq 1$ *cannot be improved for any* $B(x)$ *without using further characteristics of* $B(x)$*.*

3 Approximations for the second moment of V

We assume again that the $M/GI/SDPS$ system is stable and in steady state. Further, we assume that $\sum_{n=0}^{\infty}\varphi(n+1)^{-2}\,p(n)$ and $E[S^2]$ are finite, ensuring that $E[V^2]$ is finite, too, cf. (1.5).

Because of (1.4), it holds

$$\frac{E[V^2]}{(EV)^2}-\frac{E[S^2]}{(ES)^2}=\frac{1}{(EV)^2}\int_{\mathbb{R}_+}E[V^2(\tau)]\mathrm{d}B(\tau)-\frac{1}{(ES)^2}\int_{\mathbb{R}_+}\tau^2\mathrm{d}B(\tau)$$

$$=\frac{1}{(ES)^2}\int_{\mathbb{R}_+}\Big(\frac{E[V^2(\tau)]}{(EV(\tau))^2}-1\Big)\tau^2\mathrm{d}B(\tau),$$

which provides

$$c^2(V)-c^2(S)=m_S^{-2}\int_{\mathbb{R}_+}c^2(V(\tau))\tau^2\mathrm{d}B(\tau), \tag{3.1}$$

cf. [BB2] (4.4). Note that (3.1) and (2.1) imply the estimate

$$c^2(S)\le c^2(V)\le (c^2(S)+1)c^2(1/\varphi(N+1))+c^2(S). \tag{3.2}$$

In view of (3.1), any approximation for $c^2(V(\tau))$, $\tau\in(0,\infty)$, provides an approximation for $c^2(V)$. For obtaining a first approximation for $c^2(V)$ we use again the two-moment matching of $B(x)$ by (2.2), (2.3). For the unconditional sojourn time $V^{D,p}$ in the $M/GI/SDPS$ system with service time df. (2.2), (2.3) from (3.1), (2.4) we obtain that

$$c^2(V^{D,p})-c^2(S)=m_S^{-2}\int_{\mathbb{R}_+}c^2(V^{D,p}(\tau))\tau^2\mathrm{d}B_{D,p}(\tau)$$

$$=m_S^{-2}\int_{\mathbb{R}_+}c^2(V^D(p\tau))\tau^2\mathrm{d}B_{D,p}(\tau)=m_S^{-2}p\,c^2(V^D(pd_0))d_0^2$$

$$=p^{-1}c^2(V^D(m_S))=(c^2(S)+1)c^2(V^D),$$

where V^D denotes the unconditional sojourn time in the $M/D/SDPS$ system with arrival intensity λ and deterministic service times $d:=m_S$. Thus it holds

$$c^2(V^{D,p})=(c^2(S)+1)c^2(V^D)+c^2(S). \tag{3.3}$$

In case of an arbitrary $M/GI/SDPS$ system with service time df. $B(x)$, characterized by m_S and $c^2(S)$, the r.h.s. of (3.3) provides the following first approximation for $c^2(V)$:

$$c^2(V)\approx c^2_{app1}(V):=(c^2(S)+1)c^2(V^D)+c^2(S). \tag{3.4}$$

The approximation (3.4) bases on approximating $B(x)$ via the two-moment matching (2.2), (2.3). In [BB4] Theorem 3.3 there is given an algorithm for computing $E[(V^D)^2]$ based on solving numerically an infinite linear system of ODEs with constant coefficients. Moreover, (2.6) and (2.7) yield explicite expressions for $c^2(V^D) = c^2(V^D(d))$ in $M/D/2-PS$ and $M/D/1-PS$ systems, respectively. The numerical results presented in Section 4 illustrate that in case of $M/GI/m-PS$ systems the approximation (3.4) works well. However, the numerical complexity for solving the ODEs may become rather high. Thus we are interested in approximations of lower complexity for $c^2(V)$.

Applying the approximation (3.4) to exponential service times, we find for $c^2(V^D)$ the approximation

$$c^2(V^D) \approx (c^2(V^M)-1)/2, \tag{3.5}$$

where V^M denotes the unconditional sojourn time in the corresponding $M/M/SDPS$ system with arrival intensity λ and exponential service times with mean $d = m_S$. Inserting now approximation (3.5) into (3.4), we obtain the following second approximation for $c^2(V)$ for a general $M/GI/SDPS$ model:

$$c^2(V) \approx c^2_{app2}(V) := ((c^2(S)+1)c^2(V^M) + (c^2(S)-1))/2. \tag{3.6}$$

The advantage of approximation (3.6) is that $E[(V^M)^2]$ can be computed by means of the very fast recursive algorithm given in [BB2] Algorithm 3.1 for $M/M/SDPS$ systems. Note that from (3.1), (2.10) it follows that

$$c^2(V^M) = (2+\varrho)/(2-\varrho) \tag{3.7}$$

for the $M/M/1-PS$ system, cf. e.g. [vBe] (2.14), (2.17). In case of $c^2(S) \geq 1$, for the unconditional sojourn time $V^{M,p}$ in the $M/GI/SDPS$ system with service time df. (2.11), (2.12) from (3.1), (2.13) we obtain that

$$\begin{aligned}
c^2(V^{M,p}) - c^2(S) &= m_S^{-2} \int_{\mathbb{R}_+} c^2(V^{M,p}(\tau))\tau^2 \mathrm{d}B_{M,p}(\tau) \\
&= m_S^{-2} p \int_{\mathbb{R}_+} c^2(V^M(p\tau))(\exp(-\tau/d_0)/d_0)\tau^2 \,\mathrm{d}\tau \\
&= p^{-1} m_S^{-2} \int_{\mathbb{R}_+} c^2(V^M(x))(\exp(-x/m_S)/m_S)x^2 \,\mathrm{d}x \\
&= ((c^2(S)+1)/2)(c^2(V^M)-1),
\end{aligned}$$

which implies

$$c^2(V^{M,p}) = ((c^2(S)+1)c^2(V^M) + (c^2(S)-1))/2.$$

Thus (3.6) is exact in case of the service time df. (2.11), (2.12). Therefore we have two heuristics for the approximation (3.6) in case of $c^2(S) \geq 1$, which thus should be a good approximation at least for $c^2(S) \geq 1$. However, for $c^2(S)$ small (3.4) will be in general a better approximation than (3.6), since approximation (3.4) is exact for $c^2(S) = 0$. Note that the approximations (3.4) and (3.6) satisfy (3.2) since (3.2) holds for V^D and V^M. The corresponding approximations for $(c^2(V) - c^2(S))/(c^2(S)+1)$ are independent of $B(x)$ given m_S, and they take values in $[0, c^2(1/\varphi(N+1))]$ because of (3.2), cf. Table 4.2 below.

Remark 3.1 *The quality of the approximations* (3.4) *for* $c^2(S) \geq 0$ *and* (3.6) *for* $c^2(S) \geq 1$ *is determined by the sensitivity of* $c^2(V)$ *with respect to* $B(x)$ *given* m_S *and* $c^2(S)$. *The approximations* (3.4) *for* $c^2(S) \geq 0$ *and* (3.6) *for* $c^2(S) \geq 1$ *cannot be improved for any* $B(x)$ *without using further characteristics of* $B(x)$.

4 Numerical results

In this section we study the quality of the approximations (2.5), (2.9) and (3.4), (3.6) for $M/GI/m-PS$ systems, i.e. for $\varphi_m(n) = \min(m/n, 1)$.

In Table 4.1 there is given $c^2(V((c^2(S)+1)\tau))$ in $M/D/m-PS$ and in $M/M/m-PS$ for $m = 1, 2$. Remember that $c^2(1/\varphi_m(N+1))$ is an upper bound for $c^2(V(\tau))$, $\tau \in (0, \infty)$, with equality in the limiting case of $\tau \downarrow 0$. For the examples given in Table 4.1 the approximations $c^2_{app1}(V(\tau))$ and $c^2_{app2}(V(\tau))$ work well. Remember that $c^2(1/\varphi_m(N+1))$ can be considered as an approximation for $c^2(V(\tau))$, $\tau \in (0, \infty)$, based on a sequence of one-moment fittings of the service times given by (2.2), (2.3) for $p \downarrow 0$, cf. (2.4). Table 4.1 shows that this simple approximation works well for τ small and in case of higher load.

Table 4.1: $c^2(V((c^2(S)+1)\tau))$ *in* $M/D/m-PS$ *and* $M/M/m-PS$ *for* $m=1,\,2$.

ϱ/m	τ/m_S	$c^2(V((c^2(S){+}1)\tau))$			
		$M/D/1{-}PS$	$M/M/1{-}PS$	$M/D/2{-}PS$	$M/M/2{-}PS$
		$c^2(V^D(\tau))$	$c^2_{app2}(V^D(\tau))$	$c^2(V^D(\tau))$	$c^2_{app2}(V^D(\tau))$
		$c^2_{app1}(V^M(2\tau))$	$c^2(V^M(2\tau))$	$c^2_{app1}(V^M(2\tau))$	$c^2(V^M(2\tau))$
0.50	0.00	0.500000	0.500000	0.218750	0.218750
0.50	0.50	0.455593	0.426123	0.194754	0.179580
0.50	1.00	0.405115	0.367879	0.169378	0.151071
0.80	0.00	0.800000	0.800000	0.627200	0.627200
0.80	0.50	0.770438	0.749230	0.601854	0.583827
0.80	1.00	0.734037	0.703200	0.571023	0.545056
0.95	0.00	0.950000	0.950000	0.901372	0.901372
0.95	0.50	0.941046	0.934363	0.892669	0.886176
0.95	1.00	0.929561	0.919109	0.881514	0.871367

Table 4.2: $(c^2(V)-c^2(S))/(c^2(S)+1)$ *in* $M/D/m-PS$, $M/M/m-PS$, *and the upper bound* $c^2(1/\varphi_m(N+1))$ *for* $m=1,\,2,\,4,\,8$.

ϱ/m	m	$(c^2(V){-}c^2(S))/(c^2(S){+}1)$		upper bound
		$M/D/m{-}PS$	$M/M/m{-}PS$	$c^2(1/\varphi_m(N{+}1))$
		$c^2(V^D)$	$c^2_{app2}(V^D)$	
		$(c^2_{app1}(V^M){-}1)/2$	$(c^2(V^M){-}1)/2$	
0.50	1	0.405115	0.333333	0.500000
0.80	1	0.734037	0.666667	0.800000
0.95	1	0.929561	0.904762	0.950000
0.50	2	0.169378	0.135787	0.218750
0.80	2	0.571023	0.514929	0.627200
0.95	2	0.881514	0.857461	0.901372
0.50	4	0.034771	0.026712	0.048800
0.80	4	0.329685	0.293047	0.368122
0.95	4	0.773738	0.751506	0.792178
0.50	8	0.003180	0.002293	0.005164
0.80	8	0.125968	0.109163	0.145096
0.95	8	0.587403	0.568731	0.603059

Table 4.2 provides $(c^2(V) - c^2(S))/(c^2(S)+1)$ in $M/D/m-PS$ and in $M/M/m-PS$ for $m = 1, 2, 4, 8$ as well as its upper bound $c^2(1/\varphi_m(N+1))$. Remember that $(c^2(V) - c^2(S))/(c^2(S) + 1) \geq 0$, cf. (3.2). For the examples given in Table 4.2 the approximations $c^2_{app1}(V)$ and $c^2_{app2}(V)$ work well, in particular in case of higher load. Moreover, Table 4.2 shows that $c^2(1/\varphi_m(N+1))$ is not only an upper bound but also a good approximation for $(c^2(V) - c^2(S))/(c^2(S)+1)$ in case of higher load. Note that $c^2_{app2}(V^D)$ is a worse approximation for $c^2(V^D) = c^2(V^D(m_S))$ than $c^2_{app2}(V^D(m_S))$, cf. also Table 4.1, but the complexity for computing $c^2_{app2}(V^D)$ is lower than for $c^2_{app2}(V^D(m_S))$.

References

[BBJ] Borst, S., Boxma, O., Jelenkovic, P., Reduced-load equivalence and induced burstiness in GPS queues with long-tailed traffic flows. Queueing Syst. 43 (2003) 273–306.

[Bra] Braband, J., Waiting time distributions for $M/M/N$ processor sharing queues. Commun. Stat., Stochastic Models 10 (1994) 533–548.

[BB1] Brandt, A., Brandt, M., A sample path relation for the sojourn times in $G/G/1-PS$ systems and its applications. Queueing Syst. 52 (2006) 281–286.

[BB2] Brandt, A., Brandt, M., Waiting times for M/M systems under state-dependent processor sharing. Queueing Syst. 59 (2008) 297–319.

[BB3] Brandt, A., Brandt, M., Insensitivity bounds for the moments of the sojourn times in M/GI systems under state-dependent processor sharing. Konrad-Zuse-Zentrum für Informationstechnik Berlin, ZIB-Report 09-02 (2009) 1–25.
Available at `http://opus.kobv.de/zib/volltexte/2009/1150/`.

[BB4] Brandt, M., Brandt, A., On sojourn times in M/GI systems under state-dependent processor sharing. Konrad-Zuse-Zentrum für Informationstechnik Berlin, ZIB-Report 09-22 (2009) 1–39.
Available at `http://opus.kobv.de/zib/volltexte/2009/1185/`.

[Coh] Cohen, J.W., The multiple phase service network with generalized processor sharing. Acta Inf. 12 (1979) 245–284.

[GRZ] Guillemin, F., Robert, P., Zwart, B., Tail asymptotics for processor-sharing queues. Adv. Appl. Probab. 36 (2004) 525–543.

[Ott] Ott, T.J., The sojourn-time distribution in the $M/G/1$ queue with processor sharing. J. Appl. Probab. 21 (1984) 360–378.

[PG] Parekh, A.K., Gallager, R.G., A generalized processor sharing approach to flow control in integrated services networks: the single-node case. IEEE/ACM Transaction on Networking 1, No. 3 (1993) 344–357.

[Sen] Sengupta, B., An approximation for the sojourn-time distribution for the $GI/G/1$ processor-sharing queue. Commun. Statist., Stochastic Models 8 (1992) 35–57.

[SGB] Sericola, B., Guillemin, F., Boyer, J., Sojourn times in the $M/PH/1$ processor sharing queue. Queueing Syst. 50 (2005) 109–130.

[vBe] van den Berg, J.L., Simple approximations for second moment characteristics of the sojourn time in the $M/G/1$ processor sharing queue, Informatik-Fachberichte 218 (1989) 105–120.

[Yas] Yashkov, S.F., Mathematical problems in the theory of shared-processor systems. Itogi Nauki Tekh., Ser. Teor. Veroyatn., Mat. Sta., Teor. Kibern. 29 (1990) 3–82.