Symmetric Newton Polytopes for Solving Sparse Polynomial Systems

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Abstract

The aim of this paper is to compute all isolated solutions to symmetric polynomial systems. Recently, it has been proved that modelling the sparse structure of the system by its Newton polytopes leads to a computational breakthrough in solving the system. In this paper, it will be shown how the Lifting Algorithm, proposed by Huber and Sturmfels, can be applied to symmetric Newton polytopes. This symmetric version of the Lifting Algorithm enables the efficient construction of the symmetric subdivision, giving rise to a symmetric homotopy, so that only the generating solutions have to be computed. Efficiency is obtained by combination with the product homotopy. Applications illustrate the practical significance of the presented approach.

Keywords. polynomial systems, symmetry, Newton polytopes, homotopy continuation, mixed volume

1991 AMS subject classification. 65H10, 52A39, 52B15, 14Q99

1 Introduction

Most polynomial systems coming from practical applications give rise to systems which often have a symmetric structure. Computing all isolated solutions of polynomial systems can be done efficiently by homotopy continuation methods, see [18] for an introduction. This paper deals with methods for constructing symmetric homotopies.

In contrast to the problem class in [11], [12] the parameter is introduced by the homotopy. Secondly one is interested in **all** complex solutions which can not be assured by the approach in [12].

Recently, by a paper of Canny and Rojas [6], attention has been drawn on a root count for the number of solutions in $\mathbb{C}_0^n$, $\mathbb{C}_0 = \mathbb{C}\backslash\{0\}$, for systems of polynomials, where also negative exponents are allowed, the so-called Laurent polynomial systems. This root count has been developed by Bernshteĭn [3], Kushnirenko [16] and Khovanskiĭ [15], therefore it is also named the BKK bound.
First, the following definitions are needed:

Definition 1.1 Given a Laurent polynomial, denoted by $f = \sum_{q \in \mathbb{Z}^n} c_q \mathbf{x}^q$, where $c_q \in \mathbb{C}$ and $\mathbf{x}^q = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$. Its *support* is the set $\mathcal{A} = \{ \mathbf{q} \mid c_q \neq 0 \}$. The *Newton polytope* is defined as the convex hull of its support.

Then the BKK bound can be defined by the following combinatorial formula:

Definition 1.2 The *BKK bound* of a Laurent polynomial system F is defined as the mixed volume $V_n(\mathcal{P})$ of an n-tuple of Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)$:

$$V_n(\mathcal{P}) \;=\; \sum_{I \subset \{1,2,\ldots,n\}} (-1)^{n-\#I} \;\mathrm{vol}(\sum_{i \in I} P_i), \tag{1}$$

where $\mathrm{vol}(P)$ stands for the volume of a polytope P in $\mathbb{R}^n$.

When all polytopes are the same: $V_n(P, P, \ldots, P) = n!\mathrm{vol}(P)$. The mixed volume has some interesting properties: it is multilinear and invariant under a shift of the polytopes. The main theorem proved in [3] can be stated as follows:

Theorem 1.3 *Let F be a system of Laurent polynomials, with Newton polytopes $\mathcal{P} = (P_1, P_2, \ldots, P_n)$. Then the number of isolated solutions in $\mathbb{C}_0^n$ is bounded by the mixed volume $V_n(\mathcal{P})$. For almost all choices of the coefficients of F, the number of isolated solutions equals $V_n(\mathcal{P})$.*

Verschelde, Verlinden and Cools presented in [23] an implementation of Bernshteĭn's proof. Independently of their work, Huber and Sturmfels developed in [14] another constructive proof of Theorem 1.3, based on the mixed subdivisions induced by lifting of the Newton polytopes. This approach, called the Lifting Algorithm in [9], has been applied for the calculation of mixed resultants by Canny and Emiris in [7, 9]. See [13] for more on the relation between resultants and Newton polytopes. In applications it turns out to be useful to combine this algorithm with the homotopy approach in [22] and [21].

There are at least two reasons to consider the symmetry of the given system of equations. On the one hand the approach in [14] works generically for almost all choices of coefficients, but symmetry may force the exceptional situation, see Section 3. On the other hand the use of symmetry makes the algorithm much more efficient, see Example 4.14. So the aim of this paper is to investigate the influence of symmetry.

This paper is structured as follows. We start by recalling the idea behind the Lifting Algorithm. Terminology and notation concerning symmetry groups applied to Laurent polynomial systems is the topic of the third section. In the fourth section, symmetric Newton polytopes and subdivisions are discussed. The symmetric lifting function, which leads to the construction of a symmetric mixed subdivision and to a symmetric homotopy, is explained in the fifth section. Applications such as the cyclic n-roots problem [5], the four-bar problem [19], and a problem from neurofysiology show the practical significance of our method.

2 Homotopy defined by coherent subdivision

In this section we recall the notations from the theory of polytopes used in the approach by Huber and Sturmfels [14]. Summarizing their algorithm we give a sketch of their proof of Thm. 1.3. A lifting is used which defines a homotopy and generically gives a fine mixed subdivision. For more on polytopes see [24].

We are interested in systems of equations $F = (F^{(1)}, F^{(2)}, \ldots, F^{(r)})$, where r equals the number of different supports. When $r = 1$, the system is called to be *unmixed*, i.e. there is only one polytope. For $r = n$ it is called *fully mixed* and for $1 < r < n$ *semi-mixed*, see [14]. The system can then be denoted as

$$\mathbf{0} = F^{(ij)}(\mathbf{x}) = \sum_{q \in \mathcal{A}^{(i)}} c_q^{(ij)} \mathbf{x}^q, \qquad \begin{array}{c} i = 1, \ldots, r \\ j = 1, \ldots, k_i \end{array}, \quad \sum_{i=1}^{r} k_i = n, \tag{2}$$

where $c_q^{(ij)} \in \mathbb{C}$, $\mathcal{A}^{(i)} \subset \mathbb{Z}^n$, which is the support of the polynomials $F^{(i1)}, \ldots, F^{(ik_i)}$.

Definition 2.1 *(i) A* cell *C of $\mathcal{A} = (\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(r)})$ is a tuple $C = (C^{(1)}, \ldots, C^{(r)})$ of subsets $S^{(i)} \subset \mathcal{A}^{(i)}, i = 1, \ldots, r$.*

(ii) A face *$\mathcal{F}$ of $\mathcal{A}$ is a cell such that some linear functional $\alpha(\mathbf{x}) =< \gamma, \mathbf{x} >\in I\!R_0^n$ attains its minimum over $\mathcal{A}^{(i)}$ at $\mathcal{F}^{(i)}, i = 1, \ldots, r$. The vector γ is called the* inner normal *of $\mathcal{F}$.*

The following conventions are used. The convex hull of a cell C is denoted by conv(C). Its volume is written as vol$(C) = $ vol(conv(C)).

$$\mathit{type}(C) = (\dim(\mathrm{conv}(C^{(1)})), \ldots, \dim(\mathrm{conv}(C^{(r)}))) \in I\!N^r$$

$$\mathrm{conv}(C) = \mathrm{conv}(C^{(1)} + \cdots + C^{(r)}) \subset I\!R^n$$

The proof of Theorem 1.3 in [14] is based on cells of $\mathcal{A}$ with special properties.

Definition 2.2 ([14])

(i) A subdivision *of $\mathcal{A}$ is a collection $S = \{C_1, \ldots, C_m\}$ of m cells $C_j = (C_j^{(1)}, \ldots, C_j^{(r)})$ such that*

(a) $\dim(\mathrm{conv}(C_j)) = n$ *for* $j = 1, \ldots, m$,

(b) $C_j \cap C_k$ *is a common face of* C_j and *of* C_k *for all pairs* $C_j, C_k \in S$,

(c) $\cup_{j=1}^{m} \mathrm{conv}(C_j) = \mathrm{conv}(\mathcal{A})$.

(ii) The subdivision is called mixed *if the additional property*

(d) $\sum_{i=1}^{r} \dim(\mathrm{conv}(C_j^{(i)})) = n$ *for all cells* $C_j \in S$ *holds.*

(iii) *The subdivision is called* fine mixed *if*

(e) $\sum_{i=1}^{r}(\#(C_j^{(i)}) - 1) = n$ *for all cells* $C_j \in S$.

This definition is consistent with definitions by Lee [17]. In the case $r = 1$ the fine mixed subdivisions are usually called *triangulations*. To compute the mixed volume of a tuple of polytopes, Betke [4] proposed to embed the polytopes in $(n+1)$-dimensional space. In [14], this is called the lifting of the polytopes.

Definition 2.3 *An* r*-tuple of functions* $\omega = (\omega^{(1)}, \ldots, \omega^{(r)})$, $\omega^{(i)} : \mathcal{A}^{(i)} \to I\!R, i = 1, \ldots, r$ *is called a* lifting function *on* $\mathcal{A}$.

The embedding of the tuple $\mathcal{A}$ goes in the following way: $\mathcal{A}^{(i)}$ is lifted to $\hat{\mathcal{A}}^{(i)} = \{(\mathbf{q}, \omega^{(i)}(\mathbf{q})) : \mathbf{q} \in \mathcal{A}^{(i)}\} \subset I\!R^{n+1}$ and $\mathcal{A}$ is lifted to $\hat{\mathcal{A}} = (\hat{\mathcal{A}}^{(1)}, \ldots, \hat{\mathcal{A}}^{(r)})$. Then one denotes $\hat{Q}^{(i)} = conv(\hat{\mathcal{A}}^{(i)})$ and $\hat{Q} = \sum_{i=1}^{r} \hat{Q}^{(i)}$. Then the lower hull of $\hat{Q}$ gives a subdivision of $\mathcal{A}$.

Definition 2.4 *Let* S_ω *be the set of cells* C *of* $\mathcal{A}$ *which satisfy*

(a) $\dim(\mathrm{conv}(\hat{C})) = n$,

(b) $\hat{C}$ *is a face of* $\hat{\mathcal{A}}$ *whose inner normal* $\gamma \in I\!R_0^{n+1}$ *has positive last coordinate,* $\gamma_{n+1} > 0$.

Lemma 2.5 ([14]) S_ω *is a subdivision of* $\mathcal{A}$.

Definition 2.6 S_ω *is called the subdivision* induced *by* ω *and is said to be* coherent.

Huber and Sturmfels [14] discuss that for a generic choice of ω the induced subdivision S_ω is fine mixed. They give a combinatorial algorithm for the computation of the cells of type $(k_1, \ldots, k_r)$ of the subdivision, which are needed in Algorithm 2.7. But there is a more efficient way, see Canny and Emiris [7].

Application to solving systems of Laurent polynomials

Considering the system (2) a solution process is described based on the following homotopy

$$\mathcal{H}^{(ij)}(\mathbf{x}, t) = \sum_{q \in \mathcal{A}^{(i)}} c_q^{(ij)} \mathbf{x}^q t^{\omega^{(i)}(q)}, \qquad \begin{matrix} i = 1, \ldots, r \\ j = 1, \ldots, k_i \end{matrix}, \qquad \sum_{i=1}^{r} k_i = n. \tag{3}$$

The homotopy $\mathcal{H}$ is defined by the r-tuple of lifting functions ω on the supports $\mathcal{A}$ which yields $\mathcal{H}(\mathbf{x}, 1) = F(\mathbf{x})$. For polynomial continuation, $w^{(i)}$ must be an *integer valued* lifting function, which will be assumed for the following.

A generic choice of the lifting function implies that the induced subdivision is fine mixed. For the proof we only need to assume that the subdivision is mixed. The advantages of fine mixed are discussed after the proof.

Proof of Thm. 1.3:
The proof of Thm. 1.3 starts in [14] with the observation that for generic choices of coefficients the solutions of $\mathcal{H}(\mathbf{x}, t) = 0$ corresponds to the solutions at t very close to zero. The solutions for $t \approx 0$ are approximated by Puiseux series

$$\mathbf{x}(t) = (x_{10}t^{\gamma_1}, \ldots, x_{n0}t^{\gamma_n}) + \text{higher order terms}, \tag{4}$$

where the γ_i are rational numbers. For an introduction to Puiseux series see [20, Chapter 2]. Substituting (4) into (3) gives

$$\sum_{q \in \mathcal{A}^{(i)}} c_q^{(ij)} \mathbf{x}_0^q t^{<\gamma, q> + \omega^{(i)}(q)} + \text{higher order terms} = 0, \qquad \begin{array}{c} i = 1, \ldots, r, \\ j = 1, \ldots, k_i. \end{array} \tag{5}$$

The lower order terms in (5) give approximations of solutions. If we assume that $(\gamma, 1)$ is the inner normal of a cell C_γ of type $(k_1, \ldots, k_r)$ of the coherent subdivision S_ω this system is

$$F_\gamma^{(ij)} = init_{\gamma + \omega^{(i)}}(F^{(ij)}) = \sum_{q \in C_\gamma^{(i)}} c_q^{(ij)} \mathbf{x}^q = 0, \qquad \begin{array}{c} i = 1, \ldots, r, \\ j = 1, \ldots, k_i. \end{array} \tag{6}$$

We call it the *initial form system*. For almost all choices of coefficients $c_q^{(ij)}$ the homotopy (3) has only solutions of the form (5) for a mixed subdivision S_ω, where $(\gamma, 1)$ denotes the inner normal of cells C_γ of type $(k_1, \ldots, k_r)$.

The system $F_\gamma(\mathbf{x}) = 0$ has generically $k_1! \cdots k_r! \cdot \mathrm{vol}(C_\gamma)$ solutions. Since the mixed volume of the Newton polytopes (Thm. 2.4 in [14]) satisfies

$$V_n(\mathcal{P}) = \sum_{\substack{C \in S_\omega \\ type(C) = (k_1, \ldots, k_r)}} k_1! \cdots k_r! \cdot \mathrm{vol}(C). \tag{7}$$

Theorem 1.3 is proved. □

For generic choices ω the subdivision is even fine mixed. Then the initial form system $F_\gamma(\mathbf{x}) = 0$ in (6) can easily be solved in the following way.

(a) For each $i = 1, \ldots, r$ the subsystem $F_\gamma^{(i)}(\mathbf{x}) = 0$ consists of k_i equations and $k_i + 1$ terms in each equation. Thus by multiplication with monomials and Gaussian elimination one generically obtains for each $i = 1, \ldots, r$ a system

$$\tilde{c}^{(ij)} \mathbf{x}^{q_i} = 1, \quad j = 1, \ldots, k_i. \tag{8}$$

(b) Then the Smith normal form decomposition $UAV = \mathrm{diag}(m_1, \ldots, m_n), m_i \in \mathbb{Z}$ can be used, where the vectors $\mathbf{q}_i$ form the columns of the matrix $A = (q_{ij})$, see [14]. For the computation of a Smith normal form see e.g. [1].

Altogether the following algorithm was derived:

Algorithm 2.7 [14] The *Lifting Algorithm* solves a system (2) of type $(k_1, \ldots, k_r)$:

1.) Choose a generic lifting function $\omega = (\omega^{(1)}, \ldots, \omega^{(r)})$ and
compute the cells C_γ of type $(k_1, \ldots, k_r)$ of the fine mixed subdivision S_ω.

2.) For each computed cell C_γ solve $F_\gamma(\mathbf{x}_0) = 0$ by
   2a.) for $i = 1, \ldots, r$ do Gaussian elimination for $F_\gamma^{(i)}(\mathbf{x}_0)$
   2b.) compute the Smith normal form
   2c.) compute the solutions $\mathbf{x}_\gamma^\nu, \nu = 1, \ldots, k_1! \cdots k_r! \cdot \mathrm{vol}(C_\gamma)$ of $F_\gamma(\mathbf{x}_0) = 0$.

3.) For each cell C_γ do
   for each ν do
      path tracking of $\mathcal{H}(\mathbf{x}, t)$ starting at $(\mathbf{x}_\gamma^\nu, \epsilon)$ giving a solution for $t = 1$,
      after $x_i \to \tilde{x}_i t^{\gamma_i}$, $i = 1, 2, \ldots, n$ has been transformed in $\mathcal{H}(\mathbf{x}, t)$

The computation of the subdivision in Step 1) includes the determination of the BKK bound with (7). Huber and Sturmfels describe how to compute the volume of a fine mixed cell C of type $(k_1, \ldots, k_r)$. Then $C = (C^{(1)}, \ldots, C^{(r)})$ with $C^{(i)} = \{\mathbf{q}_0^{(i)}, \ldots, \mathbf{q}_{k_i}^{(i)}\}$. The determinant of the matrix with columns $\mathbf{q}_0^{(i)} - \mathbf{q}_\nu^{(i)}, i = 1, \ldots, r, \nu = 1, \ldots, k_i$ is equal to $k_1! \cdots k_r! \mathrm{vol}(C)$.

Remark 2.8 The numerical continuation in Step 3) is done by standard procedures, see e.g. [2], [8]. In view of easy evaluation of $\mathcal{H}(\mathbf{x}, t)$ and good numeric properties in the beginning of the path tracking in Step 3), the lifting function ω should be chosen as simple as possible.

Useful ideas and combinations with other methods

(i) By Theorem 1.3, a generic choice of the coefficients will lead to a system with as many regular solutions as its BKK bound. Consider a system $F'(\mathbf{x})$ with the same support as $F(\mathbf{x})$ but with randomly chosen complex coefficients $d_q^{(ij)} \in \mathbb{C}_0$. The homotopy

$$\tilde{\mathcal{H}}(\mathbf{x}, t) = (1-t)F'(\mathbf{x}) + tF(\mathbf{x}), \quad t \in [0,1] \tag{9}$$

defines continuation paths between the solutions of F' at $t = 0$ to the solutions of F at $t = 1$. The generic choice of the coefficients implies that the systems ${F'}_\gamma(\mathbf{x}) = 0$ have exactly $k_1! \cdots k_r! \cdot \mathrm{vol}(C_\gamma)$ regular solutions.
In order to avoid this intermediate stage of solving F', Huber and Sturmfels propose to use

$$\mathcal{H}^{(ij)}(\mathbf{x}, t) = \sum_{q \in \mathcal{A}^{(i)}} (d_q^{(ij)} + (c_q^{(ij)} - d_q^{(ij)})\, t\,) \mathbf{x}^q t^{\omega^{(i)}(q)}, \tag{10}$$

which defines another homotopy with $\mathcal{H}(\mathbf{x}, 1) = F(\mathbf{x})$ such that the Puiseux series leads to initial form systems with the coefficients $d_q^{(ij)}$ of F'.
However, it can often be desirable to know all solutions of F', as it can then be used as start system for a whole family of problems. Secondly, the numerical experience shows that it is better to proceed in two steps. Solving F' can be done with any standard path tracker, as the generic choice of the coefficients implies the homotopy (9) to be well conditioned, whereas several numerical difficulties can occur when it comes to solving F. Such as paths leading to singularities or diverging to infinity, which force the use of special path trackers. Additionally, often F' may possess more symmetry properties than F which can be exploited.

(ii) Observe that the mixed volume $V_n(\mathcal{P})$ does not depend on points in the middle of the Newton polytopes. Even more one often notices that subdivisions do not contain points in the middle. This can be forced by giving these points $\mathbf{q}$ a sufficiently high lifting value $\omega^{(i)}(\mathbf{q})$. We call this *lifting out* a point.

(iii) In contrast to (ii) it may be helpful to have an additional middle point which does not change the volume. This may help to find a subdivision or may help to exploit symmetry. This is applied in combination with (i).

(iv) Verschelde and Cools [22] introduced a product homotopy. If the system contains a lot of terms, a combination of this method with the Lifting Algorithm is promising. Apply the product homotopy first on the system as a whole, and then apply the Lifting algorithm to the subsystems of the constructed start systems, which are in general much sparser than the original problem. While in [22] the subsystems are assumed to be linear, the combination with the Lifting Algorithm extends the product approach to nonlinear subsystems. This advantage makes the product homotopy even more efficient, i.e. with fewer paths to follow.

Recursion makes any subdivision worthy

For randomly chosen lifting functions, the subdivision is fine mixed. However, this random choice is often prohibitive for the exploitation of the symmetry as will become clear in the following sections. Therefore, Algorithm 2.7 will be applied recursively. To explain this, we proceed by reading the proof of Theorem 1.3 backwards.
Formula (7) can be written as

$$V_n(\mathcal{P}) = \sum_{j=1}^{m} k_1! \cdots k_r! \cdot \mathrm{vol}(C_{\gamma^{(j)}}), \tag{11}$$

as each cell $C_{\gamma^{(j)}}$ is characterized by its inner normal $\gamma^{(j)}$, $j = 1, 2, \ldots, m$. Only when the subdivision is mixed, one can compute the mixed volume by (11), whereas the following

$$V_n(\mathcal{P}) = \sum_{j=1}^{m} V_n(C_{\gamma^{(j)}}) \tag{12}$$

holds for any subdivision $S_\omega = \{C_{\gamma^{(1)}}, C_{\gamma^{(2)}}, \ldots, C_{\gamma^{(m)}}\}$. Construct for each cell $C_{\gamma^{(j)}}$ a subdivision $S_\omega^{(j)}$, which is now assumed to be fine mixed (otherwise, apply the recursion again). Then

$$V_n(\mathcal{P}) = \sum_{j=1}^{m} \sum_{\substack{C \in S_\omega^{(j)} \\ type(C) = (k_1, \ldots, k_r)}} k_1! \cdots k_r! \cdot \mathrm{vol}(C) \tag{13}$$

can be used to compute the mixed volume. By Steps 2) and 3) in Algorithm 2.7, the initial form systems $F_{\gamma^{(j)}}$ induced by the cells $C_{\gamma^{(j)}}$ in the subdivision S_ω can be solved, by the use of the fine mixed subdivision $S_\omega^{(j)}$. The solutions of $F_{\gamma^{(j)}}(\mathbf{x}) = \mathbf{0}$ will serve as start solutions in Step 3) of Algorithm 2.7, but now for the homotopy defined by the subdivision S_ω.
Note that (12) generalizes both the recursion formula used by Bernshteĭn [3] and the non-recursive approach of Huber and Sturmfels [14].

Non-generic systems

If the Gaussian elimination in Step 2a.) of Algorithm 2.7 fails then this is a signal that the system has less solutions than indicated by the BKK bound. In this situation the system may be non-generic. But this is only a hint and the system may have as well as many solutions as stated by the BKK bound. The following theorem states explicitly when the BKK bound is sharp, see Canny, Rojas [6] and Bernshteĭn [3].

Theorem 2.9 *If for all faces C of $\mathcal{P}$ with inner normal $\gamma \in I\!R^n$ the corresponding initial form systems $F_\gamma^{(ij)}$ have no solutions in $\mathbb{C}_0^n$ then the system $F(\mathbf{x}) = 0$ has exactly $V_n(\mathcal{P})$ solutions, counting multiplicities.*

Due to a symmetric choice of the coefficients, the actual number of solutions can be lower than the BKK bound, as illustrated below in Example 3.6.

3 Laurent systems with symmetry

This section demonstrates which problems arise when the concept of Section 2 is applied to symmetric systems. We start with summarizing the well-known solution structure of systems with symmetry and continue with an example.

Let $F = (f_1, \ldots, f_n)$ be a systems of Laurent polynomials as in (2) and let $\mathcal{A} = (\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(r)})$ denote its tuple of supports. Let G denote a finite group and assume $D^1, D^2 : G \to Gl(\mathbb{C}^n)$ are two matrix representations, i.e.

$$D^i(g_1)D^i(g_2) = D^i(g_1 g_2), \quad \forall\, g_1, g_2 \in G, i = 1, 2. \tag{14}$$

Definition 3.1 *If*

$$D^2(g)F(\mathbf{x}) = F(D^1(g)\mathbf{x}), \quad \forall\, g \in G, \forall\, \mathbf{x} \in \mathbb{C}^n, \tag{15}$$

then the system F is said to be (G, D^1, D^2)-symmetric.

Of course we assume that $D^1(G)$ and G are isomorphic as groups. Otherwise one would consider a subgroup of G.

Example 3.2 We give a simple example where $G = Z_2(s_1) \times Z_2(s_2) = \{id, s_1, s_2, s_1 s_2\}$ and D^1 is a representation with

$$D^1(id) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D^1(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D^1(s_2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{16}$$

$D^1(s_1 s_2) = D^1(s_1)D^1(s_2)$ and $D^2(s_1) = D^1(s_1), D^2(s_2) = D^1(id)$. The system

$$F(\mathbf{x}) = \begin{pmatrix} x_1^2 x_2^2 + 3x_1^2 + x_2^2 + x_1 x_2 + 5 \\ x_1^2 x_2^2 + x_1^2 + 3x_2^2 + x_1 x_2 + 5 \end{pmatrix} = 0 \tag{17}$$

is $(Z_2(s_1) \times Z_2(s_2), D^1, D^2)$-symmetric. Since $Z_2(s_2)$ does not transport to the Newton polytopes (drawn in Figure 1) we consider F also as $(Z_2(s_1), D^1, D^1)$-symmetric. ■

Remark 3.3 Symmetric systems with $D^2 = D^1$ are usually called *equivariant*.

Example 3.4 Let $G = Z_2(s_1) \times Z_2(s_2)$ and D^1 as in Example 3.2. Let D^2 be given by $D^2(s_1) = D^2(s_2) = D^1(id)$. Then

$$F(\mathbf{x}) = \begin{pmatrix} x_1^2 x_2^2 + x_1^2 + x_2^2 + x_1 x_2 + 3 \\ x_1^2 x_2^2 + 3x_1^2 + 3x_2^2 + x_1 x_2 + 5 \end{pmatrix} = 0 \tag{18}$$

is $(Z_2(s_1) \times Z_2(s_2), D^1, D^2)$-symmetric and consists of two invariant polynomials. ■

Solutions of symmetric systems have a special structure: If $\mathbf{x}$ is a solution then it *generates* also *conjugate* solutions $D^1(g)\mathbf{x}$ which form the *orbit* $\mathcal{O}_{\mathbf{x}}$. It suffices to know one *generator* of an orbit. The cardinality of $\mathcal{O}_{\mathbf{x}}$ depends of course on the *isotropy* $G_{\mathbf{x}} = \{g \in G | D^1(g)\mathbf{x} = \mathbf{x}\}$ of $\mathbf{x}$. So one distinguishes different types of orbits. Usually one determines solutions with special isotropy H by restricting to the fixed point space of H

$$\mathrm{Fix}(H, D^1) = \{\mathbf{x} \in \mathbb{C}^n |\, D^1(g)\mathbf{x} = \mathbf{x}\} \tag{19}$$

and considering the *reduced system* $F^{red}(\mathbf{x}) = 0$,

$$F^{red} : \mathrm{Fix}(H, D^1) \to \mathrm{Fix}(H, D^2), \quad \mathbf{x} \to F(\mathbf{x}). \tag{20}$$

Generically, the reduced system is expected to have solutions, if the dimensions satisfy $\dim(\mathrm{Fix}(H, D^1)) \geq \dim(\mathrm{Fix}(H, D^2))$.

The system (17) may have solutions with trivial isotropy $(x_1, x_2), x_1 \neq x_2, x_1 \neq 0, x_2 \neq 0$ with conjugates $D^1(s_1)\mathbf{x} = (x_2, x_1), D^1(s_2) = (-x_1, -x_2), D^1(s_1 s_2) = (-x_2, -x_1)$ and solutions with isotropy $Z_2(s_1)$ of type $(x_1, x_1), x_1 \neq 0$ with conjugate $D^1(s_2)\mathbf{x} = (-x_1, -x_1)$ and solutions with isotropy $Z_2(s_1 s_2)$ of type $(x_1, -x_1)$ with conjugate $D^1(s_2)\mathbf{x} = (-x_1, x_1)$. But this structure is not automatically exploited by Algorithm 2.7 which is demonstrated in the following.

Example 3.5 (Example 3.2 continued.) We discuss three different lifting functions for the system (17) with corresponding induced subdivisions and systems for initial solutions to start the homotopy.
Since the two polynomials in (17) have the same support we have $r = 1, k_1 = 2, \omega = (\omega^{(1)})$ and $\mathcal{A} = (\mathcal{A}^{(1)})$ with

$$\mathcal{A}^{(1)} = \left\{ \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \tag{21}$$

Choosing $\omega^{(1)}(\mathcal{A}^{(1)})$ as $\omega_1 = [1, 7, 7, 8, 1]$, $\omega_2 = [7, 1, 1, 8, 7]$ or $\omega_3 = [7, 7, 7, 1, 7]$ three different triangulations $S_{\omega_i} = \triangle_i$ are induced which are shown in Figure 1. They are

$$\begin{aligned}
\triangle_1 &= \left\{ \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 2 \\ 2 \end{pmatrix}; \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}; \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 2 \\ 2 \end{pmatrix}; \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\} \right\}; \\
\triangle_2 &= \left\{ \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 2 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}; \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 2 \end{pmatrix}; \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \right\}; \\
\triangle_3 &= \left\{ \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 2 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}; \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 2 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}; \\
&\quad \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}; \begin{pmatrix} 2 \\ 2 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}; \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}; \begin{pmatrix} 2 \\ 2 \end{pmatrix}; \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \right\}:
\end{aligned} \tag{22}$$

In $\triangle_1$ and $\triangle_2$ the point $(1, 1)^t$ does not appear because the lifting value is greater than the others and it is in the middle. We say that this point was *lifted out*.

The two cells of subdivision $\triangle_2$ are both of type $(k_1, \ldots, k_r) = (2)$ and thus contribute solutions. The induced initial form systems are

$$F_{\gamma_1}(\mathbf{x}) = \begin{pmatrix} 3x_1^2 + x_2^2 + 5 \\ x_1^2 + 3x_2^2 + 5 \end{pmatrix} = 0 \qquad F_{\gamma_2}(\mathbf{x}) = \begin{pmatrix} x_1^2 x_2^2 + 3x_1^2 + x_2^2 \\ x_1^2 x_2^2 + x_1^2 + 3x_2^2 \end{pmatrix} = 0. \tag{23}$$

Each equation of the second system F_{γ_2} will be divided by x_2^2. After Gaussian elimination the following systems are obtained

$$F'_{\gamma_1}(\mathbf{x}) = \begin{pmatrix} 8x_1^2 + 10 \\ 8x_2^2 + 10 \end{pmatrix} = 0 \qquad F'_{\gamma_2}(\mathbf{x}) = \begin{pmatrix} 2x_1^2 x_2^{-2} - 2 \\ 2x_1^2 + 8 \end{pmatrix} = 0. \tag{24}$$

They have 4 solutions each. These solutions are used as start solutions of a homotopy leading to 8 solutions of (17). For practical computations one would use random complex coefficients of a start system F' as described in Section 2.

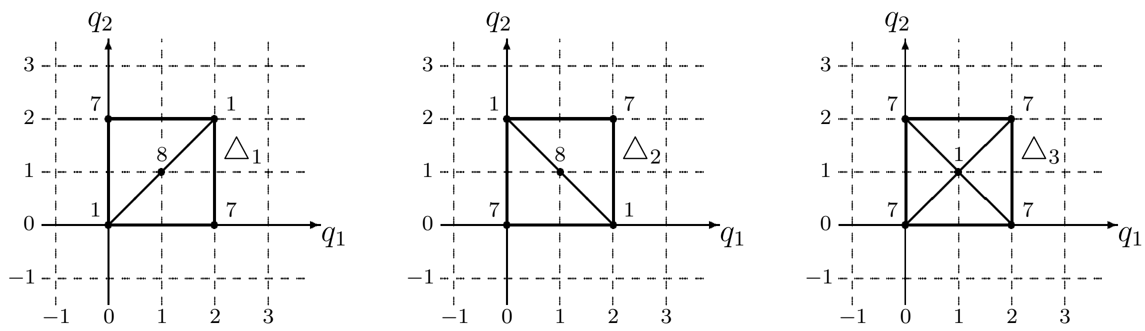

Figure 1: The Newton polytope P_1 of system (17) with three possible subdivisions (triangulations) $\triangle_1$, $\triangle_2$ and $\triangle_3$ which are induced by different lifting functions $\omega_1, \omega_2, \omega_3$.

But the two other subdivisions contain cells leading to initial form systems without solutions because Step 2a.) in Algorithm 1 fails. For example the first cell of $\triangle_3$ leads to a system without solutions:

$$\begin{pmatrix} x_1x_2 + 3x_1^2 + 5 \\ x_1x_2 + x_1^2 + 5 \end{pmatrix} = 0. \tag{25}$$

This is no contradiction to Thm. 1.3 and to Algorithm 2.7 because they work for almost all choices of coefficients. But system (17) is $(Z_2(s_1), D^1, D^1)$-symmetric putting restrictions on the coefficients and forcing them to be among the exceptions. This is also shown by observation of system (18) which has the same support as (17). Here the first cell of $\triangle_3$ gives the initial form system

$$\begin{pmatrix} x_1x_2 + x_1^2 + 3 \\ x_1x_2 + 3x_1^2 + 5 \end{pmatrix} = 0 \tag{26}$$

which has two solutions. ■

Although (18) is $(Z_2(s_1) \times Z_2(s_2), D^1, D^2)$-symmetric this sort of symmetry does not affect Algorithm 2.7. It is the existence of solutions (x_1, x_1) with isotropy $Z_2(s_1)$ in system (17) which needs special attention and a modification of Thm. 1.3 and a modification of Algorithm 2.7.

Next we give an example which demonstrates that symmetry may cause the BKK bound to be **not** sharp.

Example 3.6 The construction of a quadrature formula (numerical integration) leads to a system of algebraic equations. In order that the formula $w_1 f(y_1) + w_1 f(y_2)$ approximates the integral $\int_{-1}^{1} f(y)dy$ precisely for all polynomials f up to degree 3, the unknowns $\mathbf{x} = (w_1, w_2, y_1, y_2)$ have to fulfill

$$F(w_1, w_2, y_1, y_2) = \begin{pmatrix} w_1 + w_2 - 2 \\ w_1y_1 + w_2y_2 \\ w_1y_1^2 + w_2y_2^2 - \frac{2}{3} \\ w_1y_1^3 + w_2y_2^3 \end{pmatrix} = 0. \tag{27}$$

System (27) is $(Z_2(s_1) \times Z_2(s_2), D^1, D^2)$-symmetric, where $D^1(s_1)\mathbf{x} = (w_2, w_1, y_2, y_1)$, $D^1(s_2)\mathbf{x} = (w_1, w_2, -y_1, -y_2)$, and $D^2(s_1) = D^1(id)$, $D^2(s_2)(f_1, f_2, f_3, f_4) = (f_1, -f_2, f_3, -f_4)$.

According to Theorem 2.9 we choose $\gamma = (-1, -1, -1, -1)^t$. The corresponding face yields the system

$$F_\gamma(\mathbf{x}) = \begin{pmatrix} w_1 + w_2 \\ w_1 y_1 + w_2 y_2 \\ w_1 y_1^2 + w_2 y_2^2 \\ w_1 y_1^3 + w_2 y_2^3 \end{pmatrix} = 0 \tag{28}$$

which has solutions $(w_1, -w_1, y_1, y_1), w_1, y_1 \in \mathbb{C}$, because of the symmetry. By Theorem 2.9 we expect that the number of solutions of (27) is less than the BKK bound which is 4 in this example. Indeed it is well-known that the system (27) has two solutions with isotropy $Z_2(s_1 s_2)$ which are conjugate to each other by $D^1(s_1)$. ∎

4 Symmetric Newton Polytopes

In this section the consequences of the symmetry of systems to the Newton polytopes and the exploitation of symmetry with symmetric lifting functions and symmetric and conjugate cells are shown. The efficiency of using symmetry is demonstrated with an example.

Definition 3.1 describes with matrix representations how a group G operates on a system of equations. Obviously G operates as well on the Newton polytopes $(P_1, \ldots, P_n)$. It is important to distinguish this induced symmetry from an additional symmetry which is present in the support $\mathcal{A}$. The exploitation of this second symmetry makes the construction of a coherent subdivision more efficient even when the system is not symmetric, see Example 4.14.

We restrict to cases where the group operation leaves the support $\mathcal{A} = (\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(r)})$ invariant and can be expressed by two permutation representations which are not necessarily the same.

Definition 4.1 *Let D^3, D^4 be two matrix representations of G with $D^i(g) \in Z_2^{r\times r}$, for $i = 3, 4$, $\forall g \in G$, where $Z_2 = \{0, 1\}$. A support $\mathcal{A}$ is called (G, D^3, D^4)-symmetric, if*

$$D^4(g)\mathcal{A} = \mathcal{A} \circ D^3(g), \quad \forall\, g \in G. \tag{29}$$

Here the operation on the left means permutation of supports $\mathcal{A}^{(i)}$ while the operation on the right means a permutation of vector components of all vectors $\mathbf{q} \in \mathcal{A}^{(i)}, i = 1, \ldots, r$.

Example 4.2 (Example 3.2 continued.) Recall $G = Z_2(s_1) \times Z_2(s_2), D^1$ and D^2 from Example 3.2. The support is $\mathcal{A} = \{\mathcal{A}^{(1)}, \mathcal{A}^{(2)}\}$, where $\mathcal{A}^{(1)} = \mathcal{A}^{(2)}$. In order to use the symmetry we do not use the equality. Then $Z_2(s_1)$ is acting on the support with $D^3(s_1) = D^1(s_1), D^3(s_2) = id, D^4(s_1) = D^2(s_1), D^4(s_2) = id.$ The group action s_1 transports $(2, 0)^t \in \mathcal{A}^{(1)}$ to $(0, 2)^t \in \mathcal{A}^{(2)}$. The element s_2 does not have any effect. Please note that G and $D^3(G)$ are not isomorphic as groups. ∎

Assume that $\mathcal{A}$ is (G, D^3, D^4)-symmetric for two permutation representations D^3, D^4. For convenience we introduce

$$\hat{D}^i(g) = \begin{pmatrix} D^i(g) & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 3, 4, g \in G. \tag{30}$$

Definition 4.3 *An r-tuple of lifting functions ω is called (G, D^3, D^4)-symmetric if*

$$\hat{D}^4(g)\hat{\mathcal{A}} = \hat{\mathcal{A}} \circ \hat{D}^3(g), \quad \forall g \in G, \tag{31}$$

where $\mathcal{A}$ is lifted to $\hat{\mathcal{A}}$ with respect to ω.

Lemma 4.4 *Assume F is (G, D^1, D^2)-symmetric with two matrix representations $D^i(g) \in Z_2^{r\times r}$ and its support $\mathcal{A}$ is (G, D^3, D^4)-symmetric, induced by D^1, D^2. If the lifting function ω is (G, D^3, D^4)-symmetric, then the homotopy defined by (3) is (G, D^1, D^2)-symmetric, i.e.*

$$\mathcal{H}(D^1(g)\mathbf{x}, t) = D^2(g)\mathcal{H}(\mathbf{x}, t), \quad \forall g \in G, \forall \mathbf{x} \in \mathbb{C}^n, t \in \mathbb{C}. \tag{32}$$

Then for almost all choices of coefficients in F and almost all $t \in [0, 1]$ the system $\mathcal{H}(\mathbf{x}, t) = 0$ has at $t = t_0$ the same solution structure with respect to G than $F(\mathbf{x}) = 0$.

A proof can be given similar to techniques in [21].

This lemma has important practical relevance. If a solution $\mathbf{x}_0$ of an initial form system $F_\gamma(\mathbf{x}) = 0$ has isotropy $H = G_{x_0}$ than one may restrict to the fixed point space and do path following of $\mathcal{H}^{red}(\mathbf{x}, t)$. Secondly one may restrict to generators of an orbit $\mathcal{O}_x$. So the question to be addressed is whether the solution structure of the initial systems corresponding to a subdivision induced by a (G, D^3, D^4)-symmetric lifting equals the solution structure of $\mathcal{H}(\mathbf{x}, 1) = 0$. The subdivision is invariant and the structure of conjugates and isotropies transports to cells.

Proposition 4.5 *Let $\mathcal{A}$ be (G, D^3, D^4)-symmetric and let ω be a (G, D^3, D^4)-symmetric lifting function. Then the cells C in the subdivision S_ω induced by ω satisfy:*

$$D^4(g^{-1})C \circ D^3(g) \in S_\omega, \quad \forall g \in G, \forall C \in S_\omega. \tag{33}$$

and $(D^3(g)\gamma, 1)$ is its inner normal, where $(\gamma, 1)$ is the inner normal of $\hat{C}$.

Proof: Let $C = (C^{(1)}, \ldots, C^{(r)})$ be a cell with inner normal $(\gamma, 1)$. Then $< \gamma, \mathbf{q} > + \, \omega^{(i)}(\mathbf{q})$ attains its minimum over $\hat{\mathcal{A}}^{(i)}$ at $\hat{C}^{(i)}$. Because $\mathcal{A}$ is (G, D^3, D^4)-symmetric, $\mathcal{A}^{(i)}$ is permuted to some $\mathcal{A}^{(j)}$ for some $j \in \{1, \ldots, r\}$ and $C^{(i)}$ is permuted to a subset $\tilde{C}^{(i)}$ of $\mathcal{A}^{(j)}$. Since ω is a symmetric lifting function $< \gamma, D^3(g^{-1})\tilde{\mathbf{q}} > + \, \omega^{(j)}(\tilde{\mathbf{q}})$, where $\tilde{\mathbf{q}} = D^3(g)\mathbf{q}$, $\mathbf{q} \in C^{(i)}$, attains its minimum over $\hat{\mathcal{A}}^{(j)}$ at $\hat{\tilde{C}}^{(i)}$. Condition (b) of Definition 2.4 holds because the argumentation works for each i. The permutation does not change the dimension and thus condition (a) of Definition 2.4 holds.
This completes the proof. □

Definition 4.6 *Let $\mathcal{A}$ and ω be (G, D^3, D^4)-symmetric as in Proposition 4.5.*

$$G_C := \{g \in G | \, D^4(g^{-1})C \circ D^3(g) = C\} \tag{34}$$

is the isotropy *group of a cell $C \in S_\omega$. If $D^4(g^{-1})C \circ D^3(g) \neq C$ then it is called the* conjugate *to C. The conjugates form the* orbit *$\mathcal{O}_C$ and C is called a generator.*

Observe that this structure of the cells is very similar to the solution structure.

Proposition 4.7 *Assume F is (G, D^1, D^2)-symmetric with $D^2 = D^1$ and assume its support $\mathcal{A}$ is (G, D^3, D^4)-symmetric, where D^3, D^4 are induced by D^1. Let S_ω be a fine mixed subdivision induced by a (G, D^3, D^4)-symmetric lifting ω. Then the following statements hold.*

(a) *The initial form system $F_\gamma(x) = 0$ of a cell with isotropy G is (G, D^1, D^1)-symmetric.*

(b) *If C is a cell whose isotropy is not G then for almost all choices of coefficients in F the corresponding initial form system $F_\gamma(\mathbf{x}) = 0$ is not (G, D^1, D^1)-symmetric and thus in general has no solutions with isotropy G.*

Proof: (a) The support of the initial form system F_γ is given by C_γ. Since C_γ is assumed to be invariant with respect to D^3 and F is (G, D^1, D^1)-symmetric, F_γ is also (G, D^1, D^1)-symmetric.
(b) If the isotropy of a cell C is not G, then there is a $g \in G$ with

$$D^4(g^{-1})C \circ D^3(g) \neq C. \tag{35}$$

Since C is the support of the corresponding initial form system, (35) implies that $D^1(g^{-1})F_\gamma(D^1(g)\mathbf{x})$ has a different support unequal C. Thus F_γ is not (G, D^1, D^1)-symmetric. In general systems without symmetry properties are not expected to have solutions with isotropy G or to have conjugate solutions. □

A consequence of Prop. 4.7 is that the solutions in $\mathrm{Fix}(G, D^1)$ can only be obtained from cells with isotropy G. Secondly, for the (G, D^1, D^1)-symmetric initial form systems the restriction to the fixed point space can help to solve them.

Conclusion 4.8 Constructing a (G, D^3, D^4)-symmetric lifting and an invariant subdivision one should derive as much G-symmetric cells as possible. The same holds for all other isotropy subgroups of G.

Example 4.9 (Examples 3.2, 3.5, 4.2 continued.) Since $D^3(s_2) = D^4(s_2) = id$ all lifting functions are $(Z_2(s_2), D^3, D^4)$-symmetric and all cells of the induced subdivision have isotropy $Z_2(s_2)$. In order to get cells with isotropy $Z_2(s_1)$ we consider the following lifting:

$$\begin{aligned}
\hat{\mathcal{A}}^{(1)} &= \left\{ \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} \right\}, \\
\hat{\mathcal{A}}^{(2)} &= \left\{ \begin{pmatrix} 2 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} \right\}.
\end{aligned} \tag{36}$$

The cells of type $(1, 1)$ of the induced subdivision have isotropy $Z_2(s_1) \times Z_2(s_2)$:

$$\mathrm{C}_1 = \left\{ \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\} \right\}, \quad \mathrm{C}_2 = \left\{ \left\{ \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\}, \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\} \right\}. \tag{37}$$

The initial form systems are $(Z_2(s_1) \times Z_2(s_2), D^1, D^2)$-symmetric:

$$F_{\gamma_1} = \begin{pmatrix} x_2^2 + 5 \\ x_1^2 + 5 \end{pmatrix} = 0, \quad F_{\gamma_2} = \begin{pmatrix} x_2^2 + x_1^2 x_2^2 \\ x_1^2 + x_1^2 x_2^2 \end{pmatrix} = 0. \tag{38}$$

Both systems have the same solution structure: two orbits of different types. There are two solutions with isotropy $Z_2(s_1)$ which are conjugate to each other by $D^1(s_2)$ and two solutions with isotropy $Z_2(s_1 s_2)$ which are conjugate to each other by $D^1(s_1)$. See Figure 2 for the symmetric continuation diagram. ■

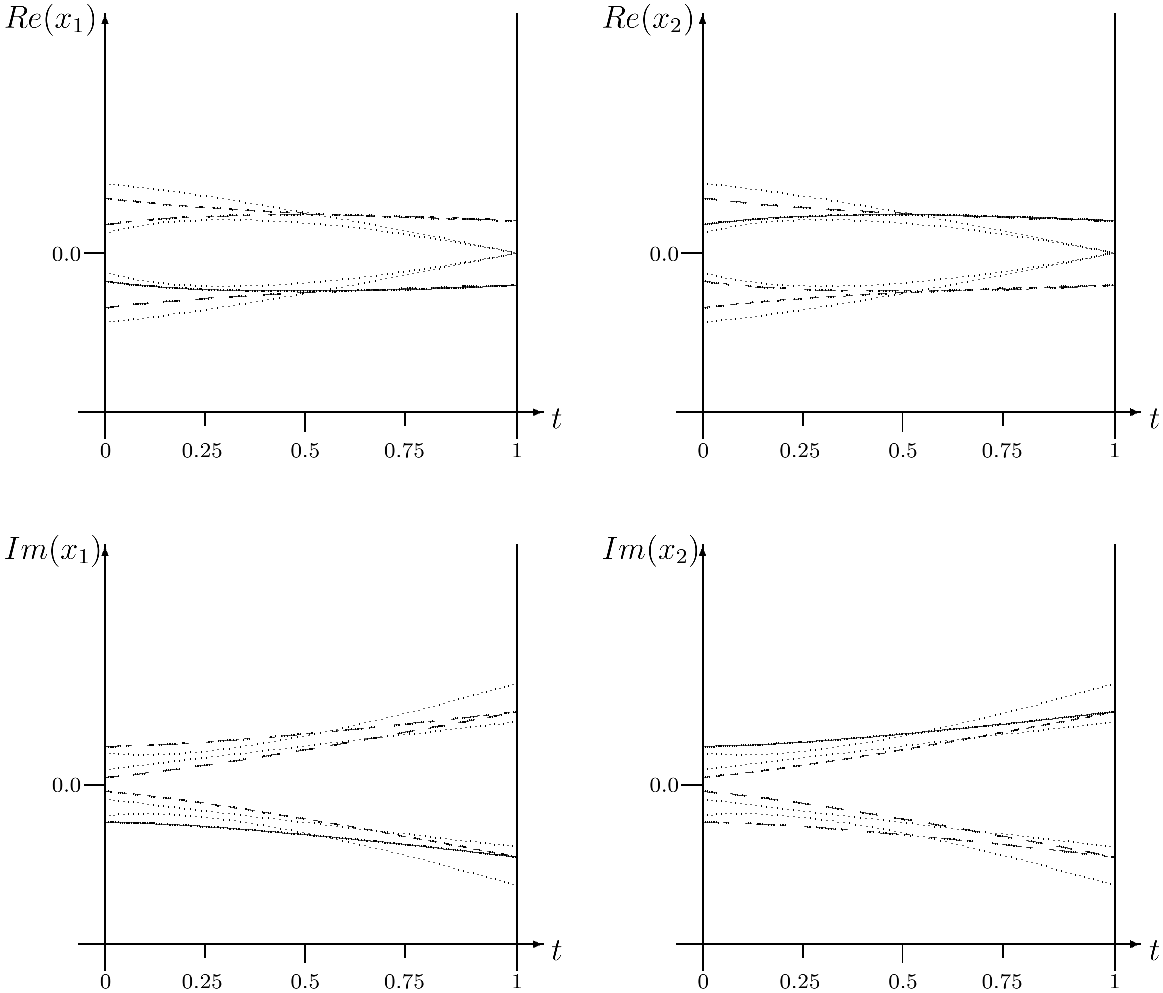

Figure 2: A symmetric continuation diagram for system (17). The homotopy is defined by $\tilde{\mathcal{H}}(\mathbf{x}, t) = (1-t)F'(\mathbf{x}) + tF(\mathbf{x})$. Paths with isotropy $Z_2(s_1)$ are drawn with dotted lines. The paths with isotropy $Z_2(s_1 s_2)$ are drawn as a line or dashed.

Remark 4.10 One may ask the question why not first reduce to fixed point spaces and then construct the homotopy. In order to find solutions with trivial isotropy this restriction is no effort. Eventually, this restriction can be done during continuation, for tracing the paths starting at solutions which belong to a particular fixed point space.

Genericity

It is important to note that symmetric lifting functions are often no longer generic. This means that the induced subdivision may fail to be mixed or fine mixed. Even for non-linear liftings this is a problem.

Example 4.11 (Examples 3.2, 3.5, 4.2, 4.9 continued.) $\mathcal{A}$ is $(Z_2(s_1) \times Z_2(s_2), D^3, D^4)$-symmetric with D^3 as defined in the beginning of Example 4.2. In view of Lemma 4.4 we choose a $(Z_2(s_1) \times Z_2(s_2), D^3, D^4)$-symmetric lifting function $\omega = (\omega^{(1)}, \omega^{(2)})$ with

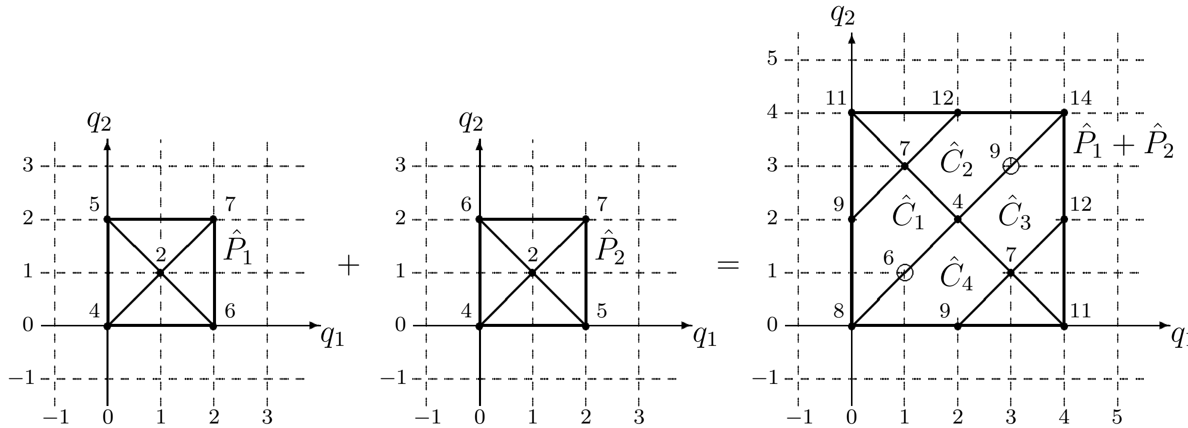

Figure 3: Minkowski sum of the lifted Newton polytopes $\hat{P}_1, \hat{P}_2$. The vertices with non-unique representation are marked with a circle. This non-uniqueness implies that the cells in the lower hull $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_4$ define a subdivision which is not mixed.

$\omega^{(1)}(\mathcal{A}^{(1)}) = [7, 6, 5, 2, 4]$ and $\omega^{(2)}(\mathcal{A}^{(2)}) = [7, 5, 6, 2, 4]$. But the induced subdivision has no cells of type $(1, 1)$ and is not mixed. The problem arises because points with a low lifting value have a non-unique representation as sum of points in $\mathcal{A}^{(i)}$, see Figure 3. The non-uniqueness results from the $Z_2(s_1)$-symmetry of the lifting. The lower hull of $P_1 + P_2$ contains 8 cells, 4 of them give cells of the coherent subdivision. For the other 4, condition (a) in Definition 2.4 is violated. ∎

Conclusion 4.12 In order to avoid non-mixed coherent subdivisions induced by a (G, D^3, D^4)-symmetric lifting functions, the points with non-unique representation should get a sufficiently high lifting value such that they are not members of the lower hull.

Efficiency

Proposition 4.5 together with the structure described in Definition 4.6 are very helpful for the computation of the subdivision and the computation of the mixed volume even when the support is symmetric but the system F is **not**.

Proposition 4.13 *Assume $\mathcal{A} = (\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(r)})$ where the supports have multiplicity $(k_1, \ldots, k_r)$, $\sum_{j=1}^{r} k_j = n$. Let D^3 and D^4 be two permutation representations of G such that $\mathcal{A}$ is (G, D^3, D^4)-symmetric. Let ω be a (G, D^3, D^4)-symmetric lifting with mixed subdivision S_ω. Then*

$$V_n(\mathcal{P}) = \sum_{\mathcal{O}_C} k_1! \cdots k_r! \cdot \mathrm{vol}(C) \cdot \#\mathcal{O}_C, \tag{39}$$

where $C \in S_\omega$ and C is of type $(k_1, \ldots, k_r)$ and generates the orbit $\mathcal{O}_C$.

Example 4.14 This example describes the treatment of a family of systems. The general formula for generating the system $F^{(n)}$ is defined as

$$f^{(i)}(\mathbf{x}) = \prod_{j=1}^{n} x_j^2 + 2x_i^2 + \sum_{j=1}^{n} x_j^2 + \prod_{j=1}^{n} x_j + 5 = 0, \; i = 1, 2, \ldots, n. \tag{40}$$

n	d	B	$V_n(\mathcal{P})$	$=$	$\#$ C	$\times$	vol(C)
2	16	8	8	$=$	4	$\times$	2
3	216	48	24	$=$	6	$\times$	4
4	4096	384	64	$=$	8	$\times$	8
5	10000	3840	160	$=$	10	$\times$	16
6	2985984	46080	384	$=$	12	$\times$	32
7	105413504	645120	896	$=$	14	$\times$	64
n	$(2n)^n$	$n! \times 2^n$	$n \times 2^n$	$=$	$2n$	$\times$	2^{n-1}

Table 1: The total degree bound d, the n-homogeneous bound B and the BKK bound for the number of solutions of system (40) depending on n.

The system (17) in Example 3.2 is included as case $n = 2$. For each n the system (40) is (S_n, D^1, D^1)-symmetric, where D^1 gives the permutation of the variables. In addition the polytopes have Z_2-symmetry by interchanging $(2, 2, \ldots, 2) \leftrightarrow (0, 0, \ldots, 0)$. While the polytope P_1 is a square for $n = 2$ (see Figure 1) and has $Z_2 \times Z_2$-symmetry the symmetry for arbitrary n is $Z_2 \times S_n$. Thus one can exploit the symmetry of $Z_2 \times S_n$ for the computation of the mixed volume $V_n(\mathcal{P})$. Generalizing the triangulation $\triangle_3$ in Example 3.5 we choose 1 as lifting value for the vertex associated with $x_1 x_2 \cdots x_n$ and 7 for the other vertices. Then the triangulation consists of one orbit. The cell $\{(0, \ldots, 0)^t, (1, \ldots, 1)^t, (2, 0, \ldots, 0)^t, \ldots, (0, \ldots, 0, 2, 0)^t\}$ has volume $\frac{2^{n-1}}{n!}$. Having isotropy S_{n-1} it generates $2n - 1$ other cells. By formula (39) $V_n(\mathcal{P}) = n! \cdot \frac{2^{n-1}}{n!} \cdot (2n)$.
Table 1 demonstrates that this bound is much better than the other bounds. As n increases, solving the problem based on the total degree bound d or the n-homogeneous bound B becomes hopeless, where the mixed volume remains exact. ■

5 Symmetric and Dynamic Lifting

This section gives a characterization of symmetric lifting functions and an algorithm based on this. The principle of dynamic lifting is mentioned which combines the computation of the subdivision with the selection of the lifting. Use of symmetry makes the dynamic lifting more efficient.
In order to classify symmetric lifting functions (Definition 4.3) we define the orbit of $\mathbf{q} \in \mathcal{A}^{(i)}$ more precisely

$$\mathcal{O}_{(q,i)} = \{(\mathbf{a}, j) |\, \mathbf{a} = D^3(g)\mathbf{q} \in \mathcal{A}^{(j)}, j = \tilde{D}^4(g^{-1})i\}, \tag{41}$$

where $D^3(g)$ describes a permutation of vector components and $\tilde{D}^4(g^{-1})$ a permutation of indices expressing the permutation of supports $\mathcal{A}^{(i)}$ by $D^4(g^{-1})$.
The isotropy subgroup can then be described as

$$G_{(q,i)} = \{g \in G |\, D^3(g)\mathbf{q} = \mathbf{q} \text{ and } \tilde{D}^4(g^{-1})i = i\}. \tag{42}$$

Lemma 5.1 *A lifting ω is (G, D^3, D^4)-symmetric, if and only if for all orbits $\mathcal{O}$ holds: all tuples $(\mathbf{q}, i)$ in $\mathcal{O}$ have the same lifting value $\omega^{(i)}(\mathbf{q})$.*

The following algorithm makes sure that points in the same orbit receive the same lifting value. The implementation of the lifting process consists of two steps: first the points will be classified into orbits, then each point will be lifted.

Algorithm 5.2 Determination of a symmetric lifting function:

input: $\mathcal{A} = (\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \ldots, \mathcal{A}^{(r)})$ — *supports*
G, D^3, D^4 — *symmetry group with representations*
output: $\mathcal{O}$, $\hat{\mathcal{A}}$ — *orbits and lifted supports*

$m := 0$; — *initialize counter for orbits*
$\mathcal{O} := \emptyset$; — *initialize set of all orbits*
$\mathcal{R} := \mathcal{A}$; — *points to be processed*
while $\mathcal{R} \neq (\emptyset, \emptyset, \ldots, \emptyset)$ do
  $\mathbf{q} \in \mathcal{R}^{(i)}$; — *pick out an element*
  $m := m + 1$; — *update orbit counter*
  $\mathcal{O}_{(q,i)} := orbit(\mathbf{q}, i, \mathcal{R}, D^3, D^4)$; — *compute the orbit $\mathcal{O}_q$*
  $\mathcal{O} := \mathcal{O} \cup \mathcal{O}_{(q,i)}$; — *update $\mathcal{O}$*
  $\mathcal{R} := \mathcal{R} \setminus \mathcal{O}_{(q,i)}$; — *update $\mathcal{R}$*
end while;

second step of the algorithm
$w \in \mathbb{Z}^m$; — *choose m random integers*
$\hat{\mathcal{A}} := lift(\mathcal{O}, \omega)$; — *perform the lifting*
with routines:

$lift(\mathcal{O}, \omega)$
for $k = 1, 2, \ldots, \#\mathcal{O}$ do
  for each $(\mathbf{a}, j) \in \mathcal{O}_{(q,i)}$ do
    $\hat{\mathcal{A}}^{(j)} := \hat{\mathcal{A}}^{(j)} \cup \{(\mathbf{a}, \omega_k)^t\}$; — *lift the point*
  end for;
end for;
return $\hat{\mathcal{A}}$.

$orbit(\mathbf{q}, i, \mathcal{R}, D^3, D^4)$ — *compute the orbit*
$H := isotropy(\mathbf{q}, i, \mathcal{R}, D^3, D^4)$;
$\mathcal{O}_{(q,i)} :=$ for each left coset $g + H \in G/H$ collect
  $\mathbf{a} := D^3(g)\mathbf{q} \in \mathcal{R}^{(j)}$, $j = \tilde{D}^4(g^{-1})i$
  $(\mathbf{a}, j)$.

$isotropy(\mathbf{q}, i, \mathcal{R}, D^3, D^4)$ — *compute isotropy group*
$H :=$ for each $g \in G$ collect
  if $D^3(g)\mathbf{q} = \mathbf{q}$ and $D^4(g^{-1})i = i$ then g.

So far no algorithms have been presented for the efficient construction of the subdivision by the exploitation of the symmetry relations between the cells. Therefore, the lifting process has to be made more sophisticated. Here we present only the key idea, in order not to overload this paper with technical details. These will be presented in a subsequent paper.

The algorithms presented above is what can be defined as a *static lifting*, i.e. first the points are lifted and then the lower hull, which defines the subdivision, is computed.

Instead of this approach, the subdivision can be computed iteratively, by adding each time to the yet computed cells in the subdivision a new point and computing the new cells spanned by the new point and other cells. As this new point only is lifted on the time of addition, the process is a *dynamic lifting*. Points which belong to the interior of a cell will be lifted out. The process makes also use of efficient techniques in dynamic convex hull construction methods.

This concept of dynamic lifting is flexible enough for the immediate exploitation of the symmetry relations. Each time a new cell has been computed, also its conjugate cells are known. The two conclusions of Section 4 have to be taken into account. This leads to an efficient construction of the symmetric subdivision.

To ensure a good condition of the solution paths, while path tracking, the coefficients of the start systems should be chosen randomly, of course with the preservation of the symmetric structure. The following algorithm provides a possible implementation for generating random coefficients for a start system F'. The notation $j = \tilde{D}^2(g)i$ has the same meaning as the $\tilde{D}^4$ used above.

Algorithm 5.3 Generating a symmetric random coefficient start system:

input: $\mathcal{A}$ — *supports*
$\quad G, D^1, D^2$ — *symmetry group with permutation representations*
output: $F' = (f'_1, f'_2, \ldots, f'_n)$ — *randomized system, w.r.t. G*

$\mathcal{R} := \mathcal{A}$; — *points to be processed*
$\mathcal{F} := (\emptyset, \emptyset, \ldots, \emptyset)$; — *sets of terms*
for $i = 1, 2, \ldots, n$ do
$\quad$ while $\mathcal{R}^{(i)} \neq \emptyset$ do
$\quad\quad$ $\mathbf{q} \in \mathcal{R}^{(i)}$; — *pick out a point*
$\quad\quad$ $c_q \in \mathbb{C}$; — *generate random coefficient*
$\quad\quad$ for each $g \in G/\text{isotropy}(\mathbf{q}, i)$ do — *compute its orbit*
$\quad\quad\quad$ $j = \tilde{D}^2(g^{-1})i$;
$\quad\quad\quad$ $\mathcal{F}^{(j)} := \mathcal{F}^{(j)} \cup \{c_q \mathbf{x}^{D^1(g)q}\}$;
$\quad\quad\quad$ $\mathcal{R}^{(j)} := \mathcal{R}^{(j)} \setminus \{D^1(g)\mathbf{q}\}$; — *update $\mathcal{R}$*
$\quad\quad$ end for;
$\quad$ end while;
end for;
for $i = 1, 2, \ldots, n$ do
$\quad$ $f'_i := \displaystyle\sum_{c_q \mathbf{x}^q \in \mathcal{F}^{(i)}} c_q \mathbf{x}^q$;
end for;

In this algorithm we restrict to the case of permutation representations which immediately transports to the supports. Sign symmetries such as $Z_2(s_2)$ in Example 3.2 are automatically fulfilled.

Example 5.4 (Example 4.14 continued.) As explained in Example 3.5, there is only one subdivision of $\mathcal{A} = (\mathcal{A}^{(1)})$ suitable for solving the system $F^{(2)}$, i.e. $\triangle_2$, with lifting defined by $\omega_2 = [7, 1, 1, 8, 7]$, for $n = 2$. This lifting will now be generalized to general dimensions.

First we discuss $n = 3$ and use that $\mathcal{A}^{(1)} = \mathcal{A}^{(2)} = \mathcal{A}^{(3)}$. According to the term order of equation (40), choose $\omega_3 = [7, 1, 1, 1, 8, 7]$, which lifts $(1, 1, 1)^t$ out. The induced subdivision yields the following two initial form systems:

$$F_{\gamma_1}(\mathbf{x}) = \begin{pmatrix} 3x_1^2 + x_2^2 + x_3^2 + 5 \\ x_1^2 + 3x_2^2 + x_3^2 + 5 \\ x_1^2 + x_2^2 + 3x_2^2 + 5 \end{pmatrix} = 0 \quad F_{\gamma_2}(\mathbf{x}) = \begin{pmatrix} x_1^2x_2^2x_3^2 + 3x_1^2 + x_2^2 + x_3^2 \\ x_1^2x_2^2x_3^2 + x_1^2 + 3x_2^2 + x_3^2 \\ x_1^2x_2^2x_3^2 + x_1^2 + x_2^2 + 3x_3^2 \end{pmatrix} = 0. \quad (43)$$

Since the lifting function generalizes to a S_3-symmetric lifting function $\omega = (\omega_3, \omega_3, \omega_3)$, the two initial form systems are (S_3, D^1, D^1)-symmetric. While S_3 describes the symmetry of the polytopes, $F^{(3)}$ has another symmetry which does not operate on the polytopes, but automatically inherits to F_{γ_1} and F_{γ_2}. $F^{(3)}$ consists of polynomials which are invariant under the reflections $D^1(s_1) : \mathbf{x} \mapsto (-x_1, -x_2, x_3)$, $D^1(s_2) : \mathbf{x} \mapsto (x_1, -x_2, -x_3)$ and $D^1(s_1s_2) : \mathbf{x} \mapsto (-x_1, x_2, -x_3)$. These reflections together build the group isomorphic to the Kleinian group, denoted by $V_4 = Z_2(s_1) \times Z_2(s_2)$. Thus the symmetry group of $F^{(3)}$ is $S_3 \times Z_2^2$. The V_4-symmetry is inherited to the initial form systems. Additionally, $F_{\gamma_1}, F_{\gamma_2}$ consists of even polynomials invariant w.r.t. $D^1(s_3) : \mathbf{x} \mapsto -\mathbf{x}$ which is not true for $F^{(3)}$ itself. The first system F_{γ_1} has exactly 8 solutions. Since the symmetry group of F_{γ_1} is $S_3 \times Z_2^3$ these 8 solutions form one $S_3 \times Z_2^3$-orbit generated by one $\mathbf{x}$ with isotropy S_3. Because the symmetry group of $\mathcal{H}(\mathbf{x}, t)$ is $S_3 \times Z_2^2$, a proper subgroup of $S_3 \times Z_2^3$, this orbit breaks into two $S_3 \times Z_2^2$-orbits during continuation. Two pathfollowings have to be done starting with $\mathbf{x}$ and $-\mathbf{x}$. The second system F_{γ_2} has 16 solutions, generated by 4 solutions with isotropy S_3. The 16 solutions are arranged in two $S_3 \times Z_2^3$-orbits giving rise to 4 $S_3 \times Z_2^2$-orbits of solutions of $F^{(3)}$. Altogether only $6 = 2 \cdot 3 = 2 + 4$ instead of 24 continuation paths need to be traced.

For general n we state that the symmetry group of $F^{(n)}$ is $S_n \times Z_2^{n-1}$. The liftings ω_2, ω_3 can be generalized into $\omega_n = [7, 1, \ldots, 1, 8, 7]$. The initial form systems (43) generalize to two systems with symmetry $S_n \times Z_2^{n-1}$, where $Z_2(s_i)$ is given by the reflection

$$D^1(s_i) : \mathbf{x} \mapsto (x_1, \ldots, x_{i-1}, -x_i, -x_{i+1}, x_{i+1}, \ldots, x_n), \quad i = 1, \ldots, n-1.$$

For n even the reflection $\mathbf{x} \mapsto -\mathbf{x}$ is an element of Z_2^{n-1}. For n odd this is an additional sign-symmetry of F_{γ_1} and F_{γ_2} which is not valid for $F^{(n)}$ itself. By restriction to fix point spaces we see that for n even the solutions of F_{γ_1} are two $S_n \times Z_2^{n-1}$-orbits: one is generated by a solution with isotropy S_n and the other by a solution of type $(-x_1, x_1, \ldots, x_1)$. For n odd F_{γ_1} has two $S_n \times Z_2^{n-1}$-orbits as solutions, generated by $\mathbf{x}$ and $-\mathbf{x}$ with isotropy S_n. This is one $S_n \times Z_2^n$-orbit. Analogously, F_{γ_2} has $2(n-1)$ $S_n \times Z_2^{n-1}$-orbits as solutions. Altogether this means that the $n2^n$ solutions of $F^{(n)}$ are generated by $2n$ solutions. ■

6 Applications

The polynomial systems considered here are all coming out of the literature. The focus lies on symmetric systems which could not be treated well by homotopies with symmetric random product systems, see [21].

6.1 The system of E.R. Speer

The following system has been given by E.R. Speer and in [10] the Gröbner basis has been computed.

$$F(\mathbf{x}) = \begin{pmatrix} 4\beta(n + 2a_1 - 8x_1)(a_2 - a_3) - x_2x_3x_4 + x_2 + x_4 \\ 4\beta(n + 2a_1 - 8x_2)(a_2 - a_3) - x_1x_3x_4 + x_1 + x_3 \\ 4\beta(n + 2a_1 - 8x_3)(a_2 - a_3) - x_1x_2x_4 + x_2 + x_4 \\ 4\beta(n + 2a_1 - 8x_4)(a_2 - a_3) - x_1x_2x_3 + x_1 + x_3 \end{pmatrix} = 0, \tag{44}$$

where $a_1 = x_1+x_2+x_3+x_4,$, $a_2 = x_1x_2x_3x_4$, $a_3 = x_1x_2+x_2x_3+x_3x_4+x_4x_1$ and where β and n are parameters to the system. Here we consider them as complex constants to be chosen at random. The system is (D_4, D^1, D^1)-symmetric, where $D_4 = \{id, r, r^2, r^3, s, sr, sr^2, sr^3\}$ represented by $D^1(r)\mathbf{x} = (x_4, x_1, x_2, x_3)$ and $D^1(s)\mathbf{x} = (x_2, x_1, x_4, x_3)$. We restrict to $Z_2(s) \times Z_2(sr^2) = \{id, s, sr^2, r^2\} \subset D_4$, where $D^1(r^2)\mathbf{x} = (x_3, x_4, x_1, x_2), D^1(sr^2)\mathbf{x} = (x_4, x_3, x_2, x_1)$.

Application of Bézout's Theorem gives 625 as the total degree of this system. This means that with the traditional homotopy, 625 continuation paths have to be traced. The lowest m-homogeneous Bézout number ([22]) is obtained with a 4-homogenization, yielding $B_Z = 384$. In [21], a $Z_2(s) \times Z_2(sr^2)$-symmetric homotopy could be constructed. Since one restricted to non-conjugates, there were only 83 of a total of 271 paths to trace. Theorem 1.3 gives the sharper BKK bound 96.

A symmetric lifting could be chosen, but the system has a lot of terms, which makes the construction of the mixed subdivision quite lengthy. Therefore it is better to exploit the product structure in the system. So we consider the start system

$$F'(\mathbf{x}) = \begin{pmatrix} (1 + 2b_1(D^1(id)\mathbf{x}) - 8x_1)(a_2 - b_3(D^1(id)\mathbf{x})) \\ (1 + 2b_1(D^1(s)\mathbf{x}) - 8x_2)(a_2 - b_3(D^1(s)\mathbf{x})) \\ (1 + 2b_1(D^1(sr^2)\mathbf{x}) - 8x_3)(a_2 - b_3(D^1(sr^2)\mathbf{x})) \\ (1 + 2b_1(D^1(r^2)\mathbf{x}) - 8x_4)(a_2 - b_3(D^1(r^2)\mathbf{x})) \end{pmatrix} = 0, \tag{45}$$

where $b_1(\mathbf{x}) = c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4$
and $b_3(\mathbf{x}) = c_{31}x_1x_2 + c_{32}x_2x_3 + c_{33}x_3x_4 + c_{33}x_4x_1 + c_{35}$,
with all coefficients c_{kl} randomly chosen complex constants. Observe that F' is $(Z_2(s) \times Z_2(sr^2), D^1, D^1)$-symmetric. Since the Newton polytopes of F are contained in those of F', one can be sure that the BKK bound of F' is greater or equal to the BKK bound of F. That means that the useful ideas in Section 2 (i), (iii), (iv) are applied in a symmetry preserving way.

Because of the product structure the solution of (45) simplifies to the solution of subsystems. The subsystems are here nonlinear, but a lot sparser than the original system, which means that the BKK bound can be computed faster. The BKK bound of F' is the sum of the BKK bounds of the subsystems yielding 97. The augmentation of the BKK bound with 1 is due to the addition of the point $(0, 0, 0, 0)^t$ to the Newton polytopes of F in order to exploit its product structure. There are 16 subsystems, which can be divided into five groups, according to their type. A subsystem is said to be of type k if it contains k nonlinear equations. Only 7 subsystems need to be considered, due to symmetry. Table 2 lists the characteristics of the subsystems.

Note that the subsystems should be considered as fully mixed, in order to have non-degenerate initial form systems which correspond to the cells in their induced subdivision. The last subsystem deserves some special attention. Although its BKK bound equals

type	#systems in F'	#generating systems	BKK bound	#generated solutions
0	1	1	1	1
1	4	1	4	16
2	6	3	3×8	48
3	4	1	8	32
4	1	1	0	0
Total:	16	7	37	97

Table 2: Solving the subsystems of (45): The number of nonlinear equations, number of systems, the BKK bound of the generating system, and the number of generated solutions.

zero, it has solutions in $\mathbb{C}_0^n$, due to the symmetric choice of the coefficients. The 4-homogeneous Bézout bound equals 24. These 3×8 solutions of the system can be found by considering the restriction to the following fixed point spaces: (x_1, x_1, x_3, x_3), (x_1, x_2, x_1, x_2) and (x_1, x_2, x_2, x_1), as each restriction yields 8 solutions. Hence, the last subsystem only contributes to solutions in fixed point spaces, which can be computed seperately, by directly putting the restrictions on the original system F. So, by considering the generating solutions of F' which lie not in any fixed point space, only 10 instead of 96 paths need to be traced.

6.2 The cyclic n-roots problem

The following system belongs to a family of systems, the so-called cyclic n-roots problem, given in [5]:

$$F(\mathbf{x}) = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 + x_5 \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 \\ x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 \\ x_1x_2x_3x_4x_5 - 1 \end{pmatrix} = 0. \quad (46)$$

It is a notorious test problem for Gröbner basis computation although the Buchberger algorithm is not able to exploit the symmetry. The system remains invariant when the unknowns are permuted in a cyclic way and when they are read backwards. More precisely, the system is D_5-invariant, where D_5 is the dyhedral group spanned by r and s, $D_5 = \{id, r, r^2, r^3, r^4, s, rs, rs^2, rs^3, rs^4\}$. The generators of D_5 can be represented as follows:

$$D^1(r): \mathbb{C}^5 \to \mathbb{C}^5 : \mathbf{x} \mapsto (x_2, x_3, x_4, x_5, x_1) \quad (47)$$

and

$$D^1(s): \mathbb{C}^5 \to \mathbb{C}^5 : \mathbf{x} \mapsto (x_5, x_4, x_3, x_2, x_1). \quad (48)$$

So the name n-cyclic is misleading ignoring part of the symmetry. The total degree equals 120. A generalized Bézout number based on a set structure, see [22], equals 108. The BKK bound is 70, which matches the number of finite solutions.

The exploitation of the symmetry requires the construction of a D_5-invariant homotopy. A problem occurs when the lifting algorithm is applied to F, as all lifting values for one polynomial are the same and no proper mixed subdivision is obtained. In order to exploit this kind of symmetry we add points which are invariant w.r.t. D_5 and do not change the mixed volume, see Section 2. For the kth polynomial $(x_1x_2x_3x_4x_5)^{\frac{k}{5}}$ is added, $k = 1, 2, 3, 4$. Because the Newton polytopes are left unchanged, the BKK bound of this new system will remain the same. However, the new system is not a polynomial system anymore. The transformation $y_k = x_k^5$, $k = 1, 2, \ldots, 5$, is not used because it introduces new solutions and blows up the BKK bound.

A special trick is used which exploits the special structure of the system. From the last equation $x_1x_2x_3x_4x_5 = 1$ one obtains $(x_1x_2x_3x_4x_5)^{\frac{k}{5}} = \gamma$, $k = 1, 2, 3, 4$, where γ is the fifth root of unity. Replacing the added terms $(x_1x_2x_3x_4x_5)^{\frac{k}{5}}$ by γ^k fortunately leaves the BKK bound unchanged.
The start system for the cyclic n-roots problem is then

$$F'(\mathbf{x}) = \left(\begin{array}{rcl} x_1 + x_2 + x_3 + x_4 + x_5 & - & \gamma \\ x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 & - & \gamma^2 \\ x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 & - & \gamma^3 \\ x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 & - & \gamma^4 \\ x_1x_2x_3x_4x_5 & - & 1 \end{array} \right) = 0, \tag{49}$$

where γ is not treated as a variable, but as a constant. This can be interpreted as adding $(0, 0, 0, 0, 0)$ to the supports $\mathcal{A}^{(k)}$, $k = 1, 2, 3, 4$ leaving the mixed volume unchanged.

For the start system a D_5-invariant mixed subdivision can be constructed. The static lifting has been taken as follows: [17 17 17 17 17 1] for the terms is the first four equations (1 for the constant term) and [13 7] for the last equation. The resulting mixed subdivision contains 40 cells and is D_5-invariant. Table 3 lists the generating inner normals, together with the volume of the cell and the number of generated solutions.

A D_5-invariant homotopy is constructed. In order to know the solutions of F, it is sufficient to follow the paths in the homotopy $\mathcal{H}(\mathbf{x}, t) = (1 - t)F'(\mathbf{x}) + tF(\mathbf{x})$, for $t : 0 \to 1$, starting at the representatives of orbits of solutions of F'. One orbit, the classical solution $(1, \gamma, \gamma^2, \gamma^3, \gamma^4)$ is already known. By a change of the right hand side of F' into $(1, \gamma, \gamma^3, \gamma, 1)$, we have the subsystem associated with the first cell equal to

$$\left(\begin{array}{rcl} x_1 & - & 1 \\ x_1x_2 & - & \gamma \\ x_1x_2x_3 & - & \gamma^3 \\ x_1x_2x_3x_4 & - & \gamma \\ x_1x_2x_3x_4x_5 & - & 1 \end{array} \right) = 0, \tag{50}$$

which has already this classical solution. Hence, by exploitation of the symmetry, only 6 instead of 70 continuation paths need to be traced.

generating normal	vol(C)	$\#\mathcal{O}_C$	vol(C)$\cdot\#\mathcal{O}_C$
$(-16, 10, 0, 0, 0, 1)$	1	10	10
$(-16, 0, 10, 0, 0, 1)$	3	10	30
$(-16, 10, -10, 10, 0, 1)$	1	10	10
$(-16, 13, -3, -13, 13, 1)$	2	10	20
Total:	7		70

Table 3: The n-cyclic root problem for $n = 5$. The normals of 4 cells are given, one for each orbit of cells of the D_5-invariant subdivision. Secondly, their volume, order of the orbit, and the number of generated initial solutions are contained.

6.3 A symmetrized four-bar mechanism

The following application [19] leads to a system for which the supports are symmetrical, but the system is not: $F = (f_1, f_2, f_3, f_4)$ with

$$\begin{array}{rcl} f_l(\mathbf{x}) & = & a_{l1}x_1^2x_3^2 + a_{l2}x_1^2x_3x_4 + a_{l3}x_1^2x_3 + a_{l4}x_1^2x_4^2 + a_{l5}x_1^2x_4 \\ & & +a_{l6}x_1^2 + a_{l7}x_1x_2x_3^2 + a_{l8}x_1x_2x_3x_4 + a_{l9}x_1x_2x_3 + a_{l10}x_1x_2x_4^2 \\ & & +a_{11}x_1x_2x_4 + a_{l12}x_1x_3^2 + a_{l13}x_1x_3x_4 + a_{l14}x_1x_3 + a_{l15}x_1x_4^2 \\ & & +a_{l16}x_1x_4 + a_{l17}x_2^2x_3^2 + a_{l18}x_2^2x_3x_4 + a_{l19}x_2^2x_3 + a_{l20}x_2^2x_4^2 \\ & & +a_{l21}x_2^2x_4 + a_{l22}x_2^2 + a_{l23}x_2x_3^2 + a_{l24}x_2x_3x_4 + a_{l25}x_2x_3 \\ & & +a_{l26}x_2x_4^2 + a_{l28}x_2x_4 + a_{l28}x_3^2 + a_{l29}x_4^2, \qquad l = 1, \ldots, 4. \end{array}$$

This polynomial system expresses the geometric constrains of the precision-point problem, described in [19]. The total degree equals 256, the BKK bound equals 80. By the fact that all Newton polytopes are the same, the BKK bound has been computed efficiently, see [23]. For general choices of the parameters of the system, there are only 36 finite nonsingular solutions and two planes of solutions which are not at infinity.

The polytopes are D_2-symmetric, $D_2 = \{id, s, r, rs\}$, represented by

$$D^1(s)\mathbf{x} = (x_3, x_4, x_1, x_2) \quad \text{and} \quad D^1(r)\mathbf{x} = (x_4, x_3, x_2, x_1). \tag{51}$$

A symmetric start system F' has been chosen with the first polynomial

$$f_1'(\mathbf{x}) = a_{11}'x_1^2x_3^2 + a_{12}'x_1^2x_4^2 + a_{13}'x_2^2x_3^2 + a_{14}'x_2^2x_4^2 + a_{15}'x_1^2 + a_{15}'x_2^2 + a_{16}'x_3^2 + a_{16}'x_4^2 = 0. \tag{52}$$

By application of the group actions, the other equations of F' can be constructed. Note that the monomials whose corresponding vector of exponents belongs to the interior of the polytope have been left out. They have been lifted out. The system F' has as many regular solutions as the BKK bound. Not only is F' sparser and hence easier to solve, but there is also the additional sign symmetry which makes it easier to solve F'. In other words, the group Z_2^4 generated by the reflections which change the sign of one variable is acting on F', but not on F. For the solution of F', it is helpful to consider the polytopes to be semi-mixed, $r = 2$, with $k_1 = k_2 = 2$, so that the cells in the induced subdivision lead to nondegenerate initial form systems. By the exploitation of all the symmetry, only 3 paths were to follow to solve F'. During continuation of the paths defined by $\tilde{\mathcal{H}}(\mathbf{x}, t) = (1 - t)F'(\mathbf{x}) + tF(\mathbf{x})$ all 80 solution paths need to be followed, as only the supports of F have D_2-symmetry and the system F is not symmetric.

6.4 An application from neurofysiology

This system has been posted by Sjirk Boon to the newsgroups sci.math.num-analysis and sci.math.symbolic:

$$F(\mathbf{x}) = \begin{pmatrix} x_1^2 + x_3^2 - 1 \\ x_2^2 + x_4^2 - 1 \\ x_5 x_3^3 + x_6 x_4^3 - c_1 \\ x_5 x_1^3 + x_6 x_2^3 - c_2 \\ x_5 x_3^2 x_1 + x_6 x_4^2 x_2 - c_3 \\ x_5 x_3 x_1^2 + x_6 x_4 x_2^2 - c_4 \end{pmatrix} = 0. \tag{53}$$

The total degree equals 1024, whereas the BKK bound equals only 20. The parameters of the system are c_k, $k = 1, \ldots, 4$. In [23], it has been observed that for general values of the parameters c_k, only 8 finite solutions were found. This deficiency is due to the fact that the initial form system F_γ with inner normal $\gamma = (-1,-1,-1,-1,-1,-1)^t$ for a face of the Newton polytopes

$$F_\gamma(\mathbf{x}) = \begin{pmatrix} x_1^2 + x_3^2 \\ x_2^2 + x_4^2 \\ x_5 x_3^3 + x_6 x_4^3 \\ x_5 x_1^3 + x_6 x_2^3 \\ x_5 x_3^2 x_1 + x_6 x_4^2 x_2 \\ x_5 x_3 x_1^2 + x_6 x_4 x_2^2 \end{pmatrix} = 0 \tag{54}$$

has BKK bound equal to zero, but has solutions $(y_1, y_2, iy_1, iy_2, y_3, -y_3)$, $y_1, y_2, y_3 \in \mathbb{C}$, $i = \sqrt{-1}$. This is an application of Theorem 2.9, analogue to Example 3.6.

We consider here the symmetric version of this problem. By taking $c_1 = c_2$ and $c_3 = c_4$, the system F becomes $D_2(s_1, r_1) \times D_2(s_2, r_2)$-symmetric, with $D_2(s_k, r_k) = \{id, s_k, r_k, r_k s_k\}$, $k = 1, 2$ represented by

$$D^1(s_1)\mathbf{x} = (x_2, x_4, x_1, x_3, x_6, x_5) \quad D^1(r_1)\mathbf{x} = (x_3, x_1, x_4, x_2, x_5, x_6), \tag{55}$$

and

$$D^1(s_2)\mathbf{x} = (-x_1, x_2, -x_3, x_4, -x_5, x_6) \quad D^1(r_2)\mathbf{x} = (x_1, -x_2, x_3, -x_4, x_5, -x_6). \tag{56}$$

Application of the symmetric lifting algorithm leads to 6 orbits. The symmetric subdivision has three cells with respective volumes 12, 4 and 4. The last two are conjugates, so at most 16 paths are to be followed. By exploitation of the symmetry, only 3 paths need to be computed for solving the randomized system F'. It turns out that the first subsystem has no finite solutions, while the second one has one generating solution. This generating solution can then be used to compute the generating solution of F.

To solve the original non-symmetrical problem, one can use the symmetric system as start system. In this sense, a parameter-homotopy has been constructed.

6.5 Computational Experiences

The algorithms described above have been implemented in Ada, compiled and executed on a DECstation 5000/240. Table 4 summerizes all characteristical figures for the applications. The execution times listed in Table 5 only have a relative meaning, only meant to compare the advantages of exploiting the symmetry.

Acknowledgments. The authors wish to thank Bernd Sturmfels for explaining the work in [14] to the authors. The first author wishes to thank Pierre Verlinden for helpful discussions.

	characteristics				
system	n	d	B	BKK	NP
Speer	4	625	271	96	10
cyclic	5	120	108	70	7
four-bar	4	256	96	80	3
Boon	6	1024	216	20	1

Table 4: Characteristics for the applications. For each system, the dimension n, total degree d, the Bézout bound B and the BKK bound is listed. The last column contains the number of paths to be followed for solving the start system F'.

	execution times (cpu sec.)				
system	method	Subdiv.	Solve F'	Solve F	total
Speer	symmet.	109.9	51.2	67.9	229.0
	no sym.	108.5	6768.6	448.4	7325.5
cyclic	symmet.	84.7	10.6	1.1	96.4
	no sym.	16.4	377.4	48.7	442.5
four-bar	symmet.	2.5	4.4	495.6	502.5
	no sym.	0.1	39.1	495.6	534.8
Boon	symmet.	1.6	3.6	0.2	5.4
	no sym.	1.5	17.3	110.1	128.9

Table 5: Execution times for the applications. Each system has been solved twice, once with (symmet.) and once without (no sym.) the exploitation of the symmetry. In the first case, F' is a system with random coefficients to be used as start system. The timings are given for the construction of the subdivision (Subdiv.), the solution of F' and F. The last column contains the sum of these timings.

References

[1] W.A. Adkins and S.H. Weintraub. *Algebra, an Approach via Module Theory.* Springer–Verlag, New York, 1992.

[2] E. Allgower and K. Georg. *Numerical Continuation Methods, an Introduction*, volume 13 of *Springer Ser. in Comput. Math.* Springer–Verlag, Berlin Heidelberg New York, 1990.

[3] D.N. Bernshteĭn. The number of roots of a system of equations. *Functional Anal. Appl.*, 9(3):183–185, 1975. Translated from *Funktsional. Anal. i Prilozhen.*, 9(3):1–4,1975.

[4] U. Betke. Mixed volumes of polytopes. *Arch. Math.*, 58:388–391, 1992.

[5] G. Björk and R. Fröberg. A faster way to count the solutions of inhomogeneous systems of algebraic equations, with applications to cyclic n-roots. *J. Symbolic Computation*, 12(3):329–336, 1991.

[6] J. Canny and J.M. Rojas. An optimal condition for determining the exact number of roots of a polynomial system. In *Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation*, pages 96–101. ACM, 1991.

[7] J.F. Canny and I. Emiris. An Efficient Algorithm for the Sparse Mixed Resultant. In *Proceedings of the 10th International Symposion on Applied Algebra, Algebraic Algorithms and Error-Correcting Codes*, pages 89–104. Springer–Verlag, 1993.

[8] P. Deuflhard, B. Fiedler, and P. Kunkel. Efficient Numerical Pathfollowing Beyond Critical Points. *SIAM J. Numer. Anal.*, 24:912–927, 1987.

[9] I. Emiris and J. Canny. A Practical Method for the Sparse Resultant. To appear in the Proceedings of ISSAC–93.

[10] K. Gatermann. Symbolic solution of polynomial equation systems with symmetry. In Sh. Watanabe and M. Nagata, editors, *Proceedings of ISSAC-90 (Tokyo, Japan, August 20–24, 1990)*, pages 112–119. ACM, New York, 1990.

[11] K. Gatermann. Computation of Bifurcation Graphs. In E. Allgower, K. Georg, and R. Miranda, editors, *Exploiting Symmetry in Applied and Numerical Analysis*, AMS Lectures in Applied Mathematics Vol. 29, pages 187–201. AMS, 1993.

[12] K. Gatermann and A. Hohmann. Symbolic Exploitation of Symmetry in Numerical Pathfollowing. *Impact of Computing in Science and Engineering*, 3:330–365, 1991.

[13] I.M. Gel'fand, A.V. Zelevinskiĭ, and M.M. Kapranov. Discriminants of polynomials in several variables and triangulations of Newton polyhedra. *Leningrad Math. J.*, 2(3):449–505, 1991.

[14] B. Huber and B. Sturmfels. A Polyhedral Method for Solving Sparse Polynomial Systems. Submitted for publication.

[15] A.G. Khovanskiĭ. Newton polyhedra and the genus of complete intersections. *Functional Anal. Appl.*, 12(1):38–46, 1978. Translated from *Funktsional. Anal. i Prilozhen.*, 12(1),51–61,1978.

[16] A.G. Kushnirenko. Newton Polytopes and the Bézout Theorem. *Functional Anal. Appl.*, 10(3):233–235, 1976. Translated from *Funktsional. Anal. i Prilozhen.*, 10(3),82–83,1976.

[17] C.W. Lee. Regular triangulations of convex polytopes. In P. Gritzmann and B. Sturmfels, editors, *Applied Geometry and Discrete Mathematics - The Victor Klee Festschrift*, volume 4 of *DIMACS Series*, pages 443–456. AMS, Providence, R.I., 1991.

[18] A. Morgan. *Solving polynomial systems using continuation for engineering and scientific problems*. Prentice-Hall, Englewood Cliffs, N.J., 1987.

[19] A.P. Morgan and C.W. Wampler. Solving a planar four-bar design problem using continuation. *ASME J. of Mechanical Design*, 112:544–550, 1990.

[20] B.L. van der Waerden. *Einführung in die algebraische Geometrie*. Springer–Verlag, Berlin, 1939.

[21] J. Verschelde and R. Cools. Symmetric homotopy construction. To appear in *J. Comput. Appl. Math.*

[22] J. Verschelde and R. Cools. Symbolic Homotopy Construction. *Applicable Algebra in Engineering, Communication and Computing*, 4(3):169–183, 1993.

[23] J. Verschelde, P. Verlinden, and R. Cools. Homotopies Exploiting Newton Polytopes for Solving Sparse Polynomial Systems. To appear in *SIAM J. Numer. Anal.*

[24] G. Ziegler. Lectures on Polytopes. TR 93-6, Konrad-Zuse-Zentrum, Berlin, 1993.