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The Multiperiod Network Design Problem: Lagrangian-based Solution Approaches[2]

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Abstract

We present and prove a theorem which gives the optimal dual vector for which a Lagrangian dual problem in the Single Period Design Problem (SPDP) is maximized. Furthermore we give a straightforward generalization to the Multi-Period Design Problem (MPDP). Based on the optimal dual values derived we compute the solution of the Lagrangean relaxation and compare it with the lagrangean relaxation and optimal IP values.

I. Model and Notation

We consider a network represented by a strongly connected digraph $G = (V, A)$ over a finite time horizon $\mathcal{T} = \{1, \ldots, T\}$. We are given a finite set $\mathcal{C} = \{c_1, \ldots, c_M\}$ with $\mathcal{M} = \{1, \ldots, M\}$, where $c_1 \leq c_2 \leq \ldots \leq c_M$ and $c_m \in \mathbb{Z}_+$, $\forall m \in \mathcal{M}$ hold. For each time period $t \in \mathcal{T}$ and each arc $a \in A$ we relate a total of M integer design variables $\mathbf{x}_a^{(t)} = \{x_{a,1}^{(t)}, \ldots, x_{a,M}^{(t)}\}$, each with a cost $\kappa_m^{(t)} \in \mathbb{R}_+$, $\forall m \in \mathcal{M}$. Thus $x_{a,m}^{(t)} \in \mathbb{Z}_+$, $m \in \mathcal{M}$ is interpreted as the number of capacities of size c_m used on arc a at a given time period t.

Also, for each time period, a set of commodities denoted by $\mathcal{P}$ should be routed throughout the network. Each $p \in \mathcal{P}$ is related to a fixed demand per time period that should be routed from a single source node $s(p) \in V$ to a single destination node $t(p) \in V$ and w.l.o.g we consider $s(p) \neq t(p)$, $\forall p$. The flow of commodity p over arc a at a given time period t is denoted by $f_{a,p}^{(t)} \in \mathbb{R}_+$. We make use of the following additional notation and assumptions:

- $\mathbf{d}^{(t)}$: $|\mathcal{P}|$-dimensional real vector of the demands at a given time period t. It is possible that a commodity has a zero entry at period t and a positive one at $t+1$, meaning that the demand may appear at a later time period. We always denote the set of commodities - including the ones with possibly zero demand - by $\mathcal{P}$. It is important to note that the current model only considers non-decreasing demands, that is

$$d_p^{(t)} \leq d_p^{(t+1)} \Rightarrow \Delta d_p^{(t)} := d_p^{(t+1)} - d_p^{(t)}, \quad d_p^{(0)} := 0, \quad \forall t \in \mathcal{T}, \quad \forall p \in \mathcal{P} \tag{1}$$

- U_m: Upper bound for each $x_{a,m}^{(t)}$ which satisfies the total demand of all source destination pairs for the last time period T, that is

$$\mathrm{U}_m := \left\lceil \frac{\sum_{p \in P} d_p^T}{c_m} \right\rceil, \quad \forall m \in \mathcal{M} \tag{2}$$

- $r_m^{(t)}$: Unit cost per capacity type at a given period t, i.e $r_m^{(t)} := \frac{\kappa_m^{(t)}}{c_m}$, $\forall m \in \mathcal{M}$. For simplicity of notation and w.l.o.g we assume the following ordering:

$$0 < r_M^{(t)} \leq \ldots \leq r_2^{(t)} \leq r_1^{(t)}, \quad \forall t \in \mathcal{T} \tag{3}$$

- $\Delta r_M^{(t)}$: The difference in unit cost for c_M from period t to period $t+1$, that is

$$\Delta r_M^{(t)} := r_M^{(t)} - r_M^{(t+1)}, \quad r_M^{(T+1)} := 0, \quad \forall t = 1, \dots, T-1 \tag{4}$$

- We assume that the costs of each capacity type are non-increasing over time, that is

$$\kappa_m^{(t)} \geq \kappa_m^{(t+1)}, \quad \forall t = 1, \dots, T-1, \quad \forall m \in \mathcal{M} \tag{5}$$

- $\boldsymbol{\mu}$: $|A| \cdot |\mathcal{T}|$ dimensional Lagrangian dual vector. Every entry of the vector is denoted by $\mu_a^{(t)}$ for every $a \in A$ and every $t \in T$.
- $\boldsymbol{\rho}$: Rearranged $|A| \cdot |\mathcal{T}|$ dimensional Lagrangian dual vector. Each entry of the vector is a sum of $\mu_a^{(\tau)}$ from t to T, that is

$$\rho_a^{(t)} := \sum_{\tau=t}^{T} \mu_a^{(\tau)}, \quad \forall t \in \mathcal{T}, \quad \forall a \in A \tag{6}$$

- $Z(\boldsymbol{\rho})^{(t)}$: The contribution of period t to the objective function of the Lagrangian problem, that is

$$Z(\boldsymbol{\rho})^{(t)} := \sum_{a \in A} \sum_{m \in \mathcal{M}} \left(\kappa_{a,m}^{(t)} - c_m \rho_a^{(t)} \right) \cdot x_{a,m}^{(t)} + \sum_{p \in \mathcal{P}} \sum_{a \in A} \rho_a^{(t)} f_{a,p} \tag{7}$$

- Asterisk superscripts denote optimal values or vectors.

The model under study considers *persistent routing*. This means that if a portion of a commodity's demand is routed through a specific edge set at time period t, then this will remain true for future time periods. In particular, re-routing is not allowed.

A. Problem Statement and Lagrangian Relaxation

We can now provide the multicommodity flow formulation of the Multi-Period Design Problem (MPDP) with incremental flow and persistent routing:

$$\begin{array}{llll}
\textbf{min} & \sum\limits_{t \in \mathcal{T}} \sum\limits_{a \in A} \sum\limits_{m \in \mathcal{M}} \kappa_m^{(t)} x_{a,m}^{(t)} & & \textbf{[MPDP]} \\
\textbf{s.t} & \sum\limits_{a \in \delta^+(v)} f_{a,p}^{(t)} - \sum\limits_{a \in \delta^-(v)} f_{a,p}^{(t)} = \begin{cases} \Delta d_p^{(t)} & \text{if } v = s(p) \\ -\Delta d_p^{(t)} & \text{if } v = t(p) \\ 0 & \text{otherwise} \end{cases} & \forall v, t, p & \textbf{[FCONS]} \\
& \sum\limits_{p \in \mathcal{P}} \sum_{\tau=1}^{t} f_{a,p}^{(\tau)} \leq \sum\limits_{m \in \mathcal{M}} c_m \sum_{\tau=1}^{t} x_{a,m}^{(\tau)} & \forall a, t & \textbf{[CAPB]} \\
& f_{a,p}^{(t)} \in \mathbb{R}_+, \quad x_{a,m}^{(t)} \in \mathbb{Z}_+, \quad x_{a,m}^{(t)} \leq \mathrm{U}_m & \forall a, m, p, t &
\end{array} \tag{8}$$

The theory of Lagrangian Relaxation and its applications in solving discrete optimization problems can be found in several survey papers and textbooks, such as [Geo74], [Sha79], [Fis04], [BW05], [Wol98]. To derive a lower bound for the MPDP, we follow the approach of relaxing the CAPB constraints. For each arc and each time period, the CAPB constraint is related to a dual variable $\mu_a^{(t)} \geq 0$, and consequently $\rho_a^{(t)} \geq 0$ as defined above. After rearranging terms we have - for any vector $\boldsymbol{\rho}$ - the following Lagrangian problem:

$$
\begin{aligned}
Z(\boldsymbol{\rho}) = \mathbf{min} \quad & \sum_{t\in\mathcal{T}} \left(\sum_{a\in A} \sum_{m\in\mathcal{M}} \left(\kappa_m^{(t)} - c_m \rho_a^{(t)}\right) \cdot x_{a,m}^{(t)} + \sum_{p\in\mathcal{P}} \sum_{a\in A} \rho_a^{(t)} f_{a,p}^{(t)} \right) & & \textbf{[L-MPDP]} \\
\textbf{s.t.} \quad & \sum_{a\in\delta^+(v)} f_{a,p}^{(t)} - \sum_{a\in\delta^-(v)} f_{a,p}^{(t)} = \begin{cases} \Delta d_p^{(t)} & \text{if } v = s(p) \\ -\Delta d_p^{(t)} & \text{if } v = t(p) \\ 0 & \text{otherwise} \end{cases} & \forall v,t,p \quad & \textbf{[FCONS]} \\
& f_{a,p}^{(t)} \in \mathbb{R}_+, \quad x_{a,m}^{(t)} \in \mathbb{Z}_+, \quad x_{a,m}^{(t)} \leq \mathrm{U}_m & \forall a,m,p,t &
\end{aligned} \tag{9}
$$

The minimization of the Lagrangian over primary variables decomposes into two subproblems for each time period. The first is the Capacity Planning Problem (CPP), which further decomposes into one subproblem for each arc and each capacity type:

$$
\begin{aligned}
\mathbf{min} \quad & (\kappa_m^{(t)} - c_m \rho_a^{(t)}) \cdot x_{a,m}^{(t)} \qquad \textbf{[CPP-t-a-m]} \\
\mathbf{s.t} \quad & x_{a,m}^{(t)} \in \mathbb{Z}_+,\ x_{a,m}^{(t)} \leq \mathrm{U}_m,\ \forall a,m
\end{aligned} \tag{10}
$$

The second subproblem is the uncapacitated multicommodity network flow problem, which we state in the following:

$$
\begin{aligned}
\mathbf{min} \quad & \sum_{p\in\mathcal{P}} \sum_{a\in A} \rho_a^{(t)} f_{a,p}^{(t)} & & \textbf{[UNMCF-t]} \\
\mathbf{s.t} \quad & \sum_{a\in\delta^+(v)} f_{a,p}^{(t)} - \sum_{a\in\delta^-(v)} f_{a,p}^{(t)} = \begin{cases} \Delta d_p^{(t)} & \text{if } v = s(p) \\ -\Delta d_p^{(t)} & \text{if } v = t(p) \\ 0 & \text{otherwise} \end{cases} & \forall v,p \quad & \textbf{[FCONS-t]} \\
& f_{a,p}^{(t)} \in \mathbb{R}_+ & \forall a,p &
\end{aligned} \tag{11}
$$

Furthermore, since the remaining constraints do not couple the flow variables for the individual commodities, for each commodity $p \in \mathcal{P}$ a minimum cost flow problem should be solved. Altogether the solution from the minimization of the Lagrangian is summarized in the following

Theorem 1 *Given a Lagrangian dual vector $\boldsymbol{\mu}$, and consequently a vector $\boldsymbol{\rho}$ related to the CAPB constraints, the primal variables which minimize the Lagrangian take the following values:*

$$
x_{a,m}^{(t)}(\boldsymbol{\rho}) = \begin{cases} 0 & \frac{\kappa_m^{(t)}}{c_m} > \rho_a^{(t)} \\ \mathrm{U}_m & \frac{\kappa_m^{(t)}}{c_m} < \rho_a^{(t)} \\ [0, \mathrm{U}_m] & \frac{\kappa_m^{(t)}}{c_m} = \rho_a^{(t)} \end{cases} \quad \forall t,a,m \tag{12}
$$

whereas the optimal flow for each commodity $p \in \mathcal{P}$ is the solution of a shortest path problem

$$
\begin{aligned}
\textbf{\textit{min}} \quad & \sum_{a\in A} \rho_a^{(t)} f_{a,p}^{(t)} \\
\textbf{\textit{s.t}} \quad & \sum_{a\in\delta^+(v)} f_{a,p}^{(t)} - \sum_{a\in\delta^-(v)} f_{a,p}^{(t)} = \begin{cases} \Delta d_p^{(t)} & \textit{if } v = s(p) \\ -\Delta d_p^{(t)} & \textit{if } v = t(p) \\ 0 & \textit{otherwise} \end{cases} & \forall v,t \\
& f_{a,p}^{(t)} \in \mathbb{R}_+ & \forall a,t
\end{aligned} \tag{13}
$$

Since the above holds for any $\boldsymbol{\rho}$, we are interested in the particular vector that yields the tightest lower bound. This leads to the following Lagrangian dual problem $D(\boldsymbol{\rho})$:

$$\begin{array}{lll} \textbf{max} & Z(\boldsymbol{\rho}) & \textbf{[D-MPDP]} \\ \textbf{s.t} & \rho_a^{(t)} \geq 0 \quad \forall a, t & \end{array}$$

Theorem 2 *If Z_D is the solution of D-MPDP and Z_{MIP}, Z_{LP} the optimal solution of the MPDP and its solution considering a linear relaxation of the integer installation variables respectively, then it holds that*

$$Z_D = Z_{LP} \leq Z_{MIP} \tag{14}$$

The equality on the left hand side is due to the integrality property [Geo74], which states that the optimal value of the Lagrangian problem does not alter when we drop the integrality conditions on the variables $x_{a,m}^{(t)}$. The right hand side inequality results from the weak duality theorem [BW05, Th.4.8]. Near optimal Lagrangian dual vectors are in practise obtained algorithmically, and there are several well known methods that are widely used. In the next section we find in closed form an optimal Lagrangian dual vector for the MPDP.

II. Main Results

We will first give a proof for an optimal Lagrangian dual vector for the Single Period Design Problem (SPDP), then generalize this result for the MPDP. In order to recover an optimal Lagrangian dual vector for the SPDP we make use of the second proposition, and obtain the desired result as a consequence of the linear programming relaxation of the SPDP.

A. Single Period Results

Theorem 3 *The optimal solution of the linear programming relaxation of the SPDP equals*

$$x_{a,m}^* = \begin{cases} \frac{\sum_{p\in\mathcal{P}} f_{a,p}^*}{c_M} & \textit{if} \quad m = M \\ 0 & \textit{otherwise} \end{cases} \quad \forall a, m \tag{15}$$

where $f_{a,p}^*$ solves the UNMCF with $\rho_a = r_M$, $\forall a \in A$.

Proof: Consider any feasible flow $f_{a,p}$ $\forall a, p$ of the linear relaxation of the SPDP. Then

$$x_{a,m} = \begin{cases} \frac{\sum_{p\in\mathcal{P}} f_{a,p}}{c_M} & \text{if} \quad m = M \\ 0 & \text{otherwise} \end{cases} \quad \forall a, m$$

satisfies the CABP constraints with equality for every arc. Thus the objective function equals

$$\begin{aligned} \sum_{a\in A}\sum_{m\in\mathcal{M}} \kappa_m x_{a,m} &= \sum_{a\in A} \kappa_M x_{a,M} \\ &= \sum_{a\in A}\sum_{p\in\mathcal{P}} \frac{\kappa_M}{c_M} f_{a,p} \\ &= \sum_{a\in A}\sum_{p\in\mathcal{P}} r_M \cdot f_{a,p} \end{aligned} \tag{16}$$

Because of (3) we have that r_M is the smallest unit cost per capacity type. If any portion of the flow for any particular arc is satisfied by a capacity type other than that of size c_M, then it can be replaced by the appropriate ammount of capacity type c_M at a smaller cost in the objective function. Hence it is clear that for any given feasible flow, (16) is minimal. It is clear that the solution of the UNMCF, $\sum\limits_{p\in\mathcal{P}}\sum\limits_{a\in A}\rho_a f^*_{a,p}$ with $\rho_a = r_M$, $\forall a \in A$, is a feasible flow for the linear programming relaxation of SPDP. Then, $\sum\limits_{a\in A}\sum\limits_{p\in\mathcal{P}} r_M \cdot f^*_{a,p} \leq \sum\limits_{a\in A}\sum\limits_{p\in\mathcal{P}} r_M \cdot f_{a,p}$. ■

Theorem 4 *An optimal vector $\boldsymbol{\mu}^*$ for the Lagrangian dual of the SPDP equals $r_M \cdot \vec{1}$.*

Proof: We make use of Theorem 2. By the left hand side of (14), for any optimal vector $\boldsymbol{\mu}^*$, we have that

$$\begin{aligned}
\min_{(x_{a,m},f_{a,p})}\left(\sum_{a\in A}\sum_{m\in\mathcal{M}}(\kappa_m - c_m\mu^*_a)\cdot x_{a,m} + \sum_{p\in\mathcal{P}}\sum_{a\in A}\mu^*_a f_{a,p}\right) &= \sum_{a\in A}\sum_{p\in\mathcal{P}} r_M f^*_{a,p}\\
\min_{x_{a,m}}\sum_{a\in A}\sum_{m\in\mathcal{M}}(\kappa_m - c_m\mu^*_a)\cdot x_{a,m} \;+\; \min_{f_{a,p}}\sum_{p\in\mathcal{P}}\sum_{a\in A}\mu^*_a f_{a,p} &= \sum_{a\in A}\sum_{p\in\mathcal{P}} r_M f^*_{a,p}
\end{aligned}$$

Thus, in order to prove that a given vector $\boldsymbol{\mu}$ is optimal for the Lagrangian dual of the SPND it is sufficient to check whether the equation above holds with equality when $\boldsymbol{\mu}$ is used in the left hand side. Let $\boldsymbol{\mu} = r_M \cdot \vec{1}$. Then,

$$\kappa_m - c_m r_M \;=\; \kappa_m - \kappa_M \geq 0 \quad \forall m \in \mathcal{M} \tag{17}$$

As a consequence of Theorem 1, equation (12) and the above inequality (17), we have that

$$\min_{x_{a,m}}\sum_{a\in A}\sum_{m\in\mathcal{M}}(\kappa_m - c_m r_M)\cdot x_{a,m} \;=\; 0 \tag{18}$$

All we have left to prove is that,

$$\min_{f_{a,p}}\sum_{p\in\mathcal{P}}\sum_{a\in A} r_M f_{a,p} \;=\; \sum_{a\in A}\sum_{p\in\mathcal{P}} r_M f^*_{a,p} \tag{19}$$

holds with equality. But the left hand side is the solution of the UNMCF with $\rho_a = r_M$. Hence, due to Theorem 3, the equality holds. ■

B. Multi-Period Result

A natural question for the multi period case is the following:

When is the optimal solution of the MPDP the sum of periodwise optimal solutions?

- Given our assumptions on the monotonically decreasing capacity to price ratios it is clear that when only one capacity type is available, the optimal solution for the MPDP will be the sum of the optimal solutions for each period.
- When the optimal solution for each period fulfills each constraint with equality then it is also clear that the solution of the MPDP will be the sum of optimal solutions for each period.

To treat the more general case when the above do not hold, it is important to notice that the L-MPDP decomposes into time periods. The substitution $\rho^{(t)}_a := \sum_{\tau=t}^{T}\mu^{(\tau)}_a, \;\; \forall t \in \mathcal{T}, \;\; \forall a \in A$ is intended to make this easier to see. Thus for each time period the design and flow varibles depend only on $\boldsymbol{\rho}$. Hence for each time period t, we have the following Lagrangian problem:

$$
\begin{array}{llll}
\textbf{min} & \left(\sum_{a\in A}\sum_{m\in\mathcal{M}}\left(\kappa_m^{(t)} - c_m\rho_a^{(t)}\right)\cdot x_{a,m}^{(t)} + \sum_{p\in\mathcal{P}}\sum_{a\in A}\rho_a^{(t)} f_{a,p}^{(t)}\right) & & \textbf{[L-MPDP-t]} \\
\textbf{s.t.} & \sum_{a\in\delta^+(v)} f_{a,p}^{(t)} - \sum_{a\in\delta^-(v)} f_{a,p}^{(t)} = \begin{cases} \Delta d_p^{(t)} & \text{if } v = s(p) \\ -\Delta d_p^{(t)} & \text{if } v = t(p) \\ 0 & \text{otherwise} \end{cases} & \forall v, p & \textbf{[FCONS-t]} \\
& f_{a,p}^{(t)} \in \mathbb{R}_+, \quad x_{a,m}^{(t)} \in \mathbb{Z}_+, \quad x_{a,m}^{(t)} \leq \mathrm{U}_m & \forall a, m, p &
\end{array}
\tag{20}
$$

Theorem 5 *An optimal vector $\boldsymbol{\mu}^*$ for the D-MPDP is equal entrywise to $\mu_a^{(t)} = \boldsymbol{\Delta}\mathbf{r}_{\mathbf{M}}^{(\mathbf{t})}$, and as a result of telescoping sums $\rho_a^{(t)*} = r_M^{(t)}$, $\forall t \in \mathcal{T}$ and $\forall a \in A$.*

Proof: It is important to note that the theorem holds under the initial assumption (3) that the costs of each capacity type are non-increasing over the given time horizon. This allows each $\boldsymbol{\Delta}\mathbf{r}_{\mathbf{M}}^{(\mathbf{t})}$ to be nonnegative as required. Because of the decomposition of the L-MPDP into time periods which depend only on $\boldsymbol{\rho}$, it is sufficient to consider the optimal vector for the Lagrangian dual of each L-MPDP-t. But this is simply the optimal vector for the Lagrangian dual of the SPDP for each period t. Therefore by Theorem 4 we have that $\boldsymbol{\rho}^{*(t)} = r_M^{(t)} \cdot \vec{1}$. ∎

Theorem 6 *The optimal solution of the linear programming relaxation of the MPDP is given by the solution of $|\mathcal{P}| \cdot |\mathcal{T}|$ shortest path problems.*

Proof: This is an immediate consequence of the previous theorems. since a set of optimal Lagrangian duals has been provided in closed form, the CPP-t-a-m can be solved by inspection and the UNMCFs should be solved per period and per commodity, given the optimal vector $\boldsymbol{\mu}^*$. ∎

III. Computational Results

Since we have established that the Lagrangian relaxation of the CAPB constraints yields the same lower bound as the linear programming relaxation of the MPDP, it is natural to investigate the quality of this lower bound. Unfortunately this lower bound can be quite weak. The following example illustrates our point:

- Consider a connected two node network with only one arc. The tail of the arc indicates the source node, whereas the arrow indicates the sink. We have $c_1 = 100$, $\kappa_1 = 100$ and $d_1 = 1$. The objective function value of the linear programming relaxation of the SPDP is 1 whereas the optimal integer solution value is 100.

Next, we present some computations for the network topology in the picture below. Prices are kept fixed, and demands and capacity types are varied. As expected by the example given above when the demands are relatively small compared to the capacity types the gap between the linear relaxation and the optimal is large. This can be observed in the third row of the second table.

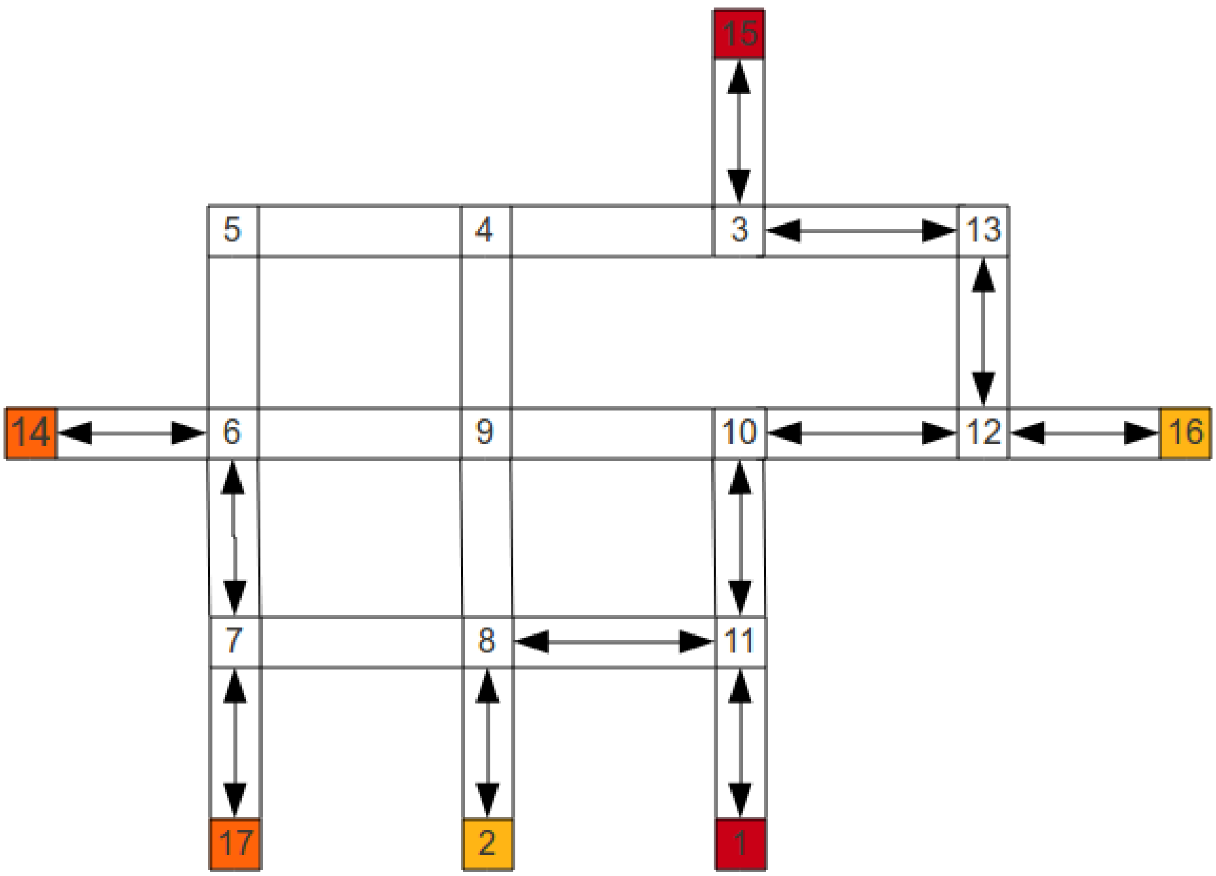

Fig. 1. Example Network

Periods	d1	d2	d3	c1	c2	c3	c4	p1	p2	p3	p4
t=1	1.0	1.0	1.0	2.0	5.0	10.0	15.0	3.0	6.0	11.0	14.0
t=2	2.0	2.0	2.0	2.0	5.0	10.0	15.0	2.5	5.0	9.5	12.0
t=3	3.0	3.0	3.0	2.0	5.0	10.0	15.0	2.0	4.0	8.0	10.0
t=4	3.5	3.5	3.5	2.0	5.0	10.0	15.0	1.5	3.0	6.5	8.0
t=5	4.0	4.0	4.0	2.0	5.0	10.0	15.0	1.0	2.0	5.0	6.0
t=1	5.0	12.0	15.0	2.0	5.0	10.0	15.0	3.0	6.0	11.0	14.0
t=2	10.0	24.0	30.0	2.0	5.0	10.0	15.0	2.5	5.0	9.5	12.0
t=3	15.0	36.0	45.0	2.0	5.0	10.0	15.0	2.0	4.0	8.0	10.0
t=4	17.5	42.0	52.5	2.0	5.0	10.0	15.0	1.5	3.0	6.5	8.0
t=5	20.0	48.0	60.0	2.0	5.0	10.0	15.0	1.0	2.0	5.0	6.0
t=1	2.0	3.0	4.0	20.0	27.0	45.0	59.0	3.0	6.0	11.0	14.0
t=2	4.0	6.0	8.0	40.0	27.0	45.0	59.0	2.5	5.0	9.5	12.0
t=3	6.0	9.0	12.0	20.0	27.0	45.0	59.0	2.0	4.0	8.0	10.0
t=4	7.0	10.5	14.0	20.0	27.0	45.0	59.0	1.5	3.0	6.5	8.0
t=5	8.0	12.0	16.0	20.0	27.0	45.0	59.0	1.0	2.0	5.0	6.0

Fig. 2. Table I: 3 examples of Demads, Capacities and Prices per period for the above network

Time Periods	LP Value	LR Value	Integer Value
T=5	715580.458	715578.341	802022.5
T=5	1049909.791	1044905.873	1128215.0
T=5	93395.063	93391.22	162615.0

Fig. 3. Table II: Comparison between the linear relaxation, lagrangean relaxation and actual integer solution