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On the Convergence of Cascadic Iterations for Elliptic Problems

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ON THE CONVERGENCE OF CASCADIC ITERATIONS FOR ELLIPTIC PROBLEMS

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Abstract. We consider nested iterations, in which the multigrid method is replaced by some simple basic iteration procedure, and call them *cascadic iterations*. They were introduced by Deuflhard, who used the conjugate gradient method as basic iteration (CCG method). He demonstrated by numerical experiments that the CCG method works within a few iterations if the linear systems on coarser triangulations are solved accurately enough. Shaidurov subsequently proved multigrid complexity for the CCG method in the case of H^2-regular two-dimensional problems with quasi-uniform triangulations. We show that his result still holds true for a large class of smoothing iterations as basic iteration procedure in the case of two- and three-dimensional $H^{1+\alpha}$-regular problems. Moreover we show how to use cascadic iterations in adaptive codes and give in particular a new termination criterion for the CCG method.

Key Words. Finite element approximation, cascadic iteration, nested iteration, smoothing iteration, conjugate gradient method, adaptive triangulations

AMS(MOS) subject classification. 65F10, 65N30, 65N55

Introduction. Let $\Omega \subset \mathbf{R}^d$ be a polygonal Lipschitz domain. We consider an elliptic Dirichlet problem on Ω in the weak formulation:

$$u \in H_0^1(\Omega): \qquad a(u,v) = (f,v)_{L^2} \quad \forall v \in H_0^1(\Omega).$$

Here $f \in H^{-1}(\Omega)$ and $a(\cdot,\cdot)$ is assumed to be a $H_0^1(\Omega)$-elliptic symmetric bilinear form. The induced energy-norm will be denoted by

$$\|u\|_a^2 = a(u,u) \qquad \forall u \in H_0^1(\Omega).$$

Given a nested family of triangulations $(\mathcal{T}_j)_{j=1}^{\ell}$ the spaces of linear finite elements are

$$X_j = \{u \in C(\bar{\Omega}) : u|_T \in P_1(T) \quad \forall T \in \mathcal{T}_j, \quad u|_{\partial\Omega} = 0\},$$

where $P_1(T)$ denotes the linear functions on the triangle T. We have

$$X_0 \subset X_1 \subset \ldots \subset X_\ell \subset H_0^1(\Omega).$$

The finite element approximations are given by

$$u_j \in X_j: \qquad a(u_j, v_j) = (f, v_j)_{L^2} \qquad \forall v_j \in X_j.$$

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For fine meshes $\mathcal{T}_\ell$ the direct computation of u_ℓ is a prohibitive expensive computational task, so one uses iterative methods. With the choice of some *basic iterative procedure* $\mathcal{I}$, the following *cascadic iteration* makes use of the nested structure of the spaces X_j:

$$\text{(1)} \qquad \begin{aligned} &\text{(i)} \quad u_0^* = u_0 \\ &\text{(ii)} \quad j = 1, \ldots, \ell: \quad u_j^* = \mathcal{I}_{j,m_j} u_{j-1}^*. \end{aligned}$$

Here $\mathcal{I}_{j,m_j}$ denotes m_j steps of the basic iteration applied on level j. This kind of iteration is known in the literature under different names, depending on the choice of the basic iteration and the parameter m_j:

- *Nested iteration*: the basic iteration is a multigrid-cycle, the parameters m_j are chosen *a priori* as a small constant number of iterations, cf. Hackbusch [5].
- *Cascade*: the basic iteration is a multilevel preconditioned cg-method, the m_j are chosen *a posteriori* due to certain termination criteria. This method was named and invented by Deuflhard, Leinen and Yserentant [4].
- *CCG-iteration*: the basic iteration is a *plain* cg-method, the m_j are chosen *a posteriori*. CCG stands for *cascadic conjugate gradient* method and was introduced by Deuflhard [3].

We call a cascadic iteration *optimal* for level ℓ, if we obtain *accuracy*

$$\|u_\ell - u_\ell^*\|_a \approx \|u - u_\ell\|_a,$$

i.e. if the iteration error is comparable to the approximation error, and if we obtain *multigrid complexity*

$$\text{amount of work } = O(n_\ell),$$

where $n_\ell = \dim X_\ell$. The optimality of the nested iteration and of Cascade are well known [5, 4], at least for certain situations. The optimality of the CCG method has been demonstrated by several numerical examples [3]. This has been considered as rather astonishing, since only a plain basic iteration is used.

Shaidurov [10] has recently shown for H^2-regular problems and quasi-uniform triangulations in two dimension, that the CCG method is optimal for a certain choice of the parameters m_j.

Since he shows in essence, that the cg-method has some smoothing properties, we were lead to consider rather general smoothers as basic iterations. The main result of Section 1 is as follows: For $H^{1+\alpha}$-regular problems with $0 < \alpha \leq 1$ and quasi-uniform triangulations it is possible to choose the parameters m_j, that

- for $d = 3$: the cascadic iteration is optimal for level ℓ.
- for $d = 2$: the cascadic iteration is accurate and has complexity $O(n_\ell \log n_\ell)$.

This result holds for a large class of smoothing iterations. In Section 2 we show how Shaidurov's result fits in our frame. Finally in Section 3 we show for adaptive grids how to choose the m_j *a posteriori* by certain termination criteria. Under some heuristically motivated assumptions on the adaptive grids we are able to show optimality for the cg-method as basic iteration.

Remark. With respect to the CCG method, we decided to call the algorithm (1) *cascadic iteration*. Since the interesting case here is the use of plain basic iterations, we had to choose a name different from *nested iteration*.

1. General smoothers as basic iteration. In this and the following section we consider *quasi-uniform* triangulations with meshsize parameter

$$\frac{1}{c} 2^{-j} \le h_j = \max_{T \in \mathcal{T}_j} \operatorname{diam} T \le c\, 2^{-j}.$$

In the following we use the symbol c for any positive constant, which only depends on the bilinear form $a(\cdot,\cdot)$, on Ω and the shape regularity as well as the quasi-uniformity of the triangulations. All other dependencies will be explicitly indicated.

A general assumption on the elliptic problem will be $H^{1+\alpha}$-regularity for some $0 < \alpha \le 1$, i.e.,

$$\|u\|_{H^{1+\alpha}} \le c\, \|f\|_{H^{\alpha-1}} \qquad \forall f \in H^{\alpha-1}(\Omega).$$

The approximation error of the finite element method is then given in energy norm as

$$\|u - u_j\|_a \le c\, h_j^\alpha \|f\|_{H^{\alpha-1}} \qquad j = 0, \ldots, \ell, \tag{2}$$

cf. [6, Lemma 8.4.9]. By the well-known Aubin-Nitsche lemma and an interpolation argument one gets the *approximation property*

$$\|u_j - u_{j-1}\|_{H^{1-\alpha}} \le c\, h_j^\alpha \|u_j - u_{j-1}\|_a \qquad j = 1, \ldots, \ell, \tag{3}$$

cf. [6, Theorem 8.4.14].

We consider the following type of basic iterations for the finite-element problem on level j started with $u_j^0 \in X_j$:

$$u_j - \mathcal{I}_{j,m_j} u_j^0 = \mathcal{S}_{j,m_j}(u_j - u_j^0)$$

with a linear mapping $\mathcal{S}_{j,m_j} : X_j \to X_j$ for the error propagation. We call the basic iteration a *smoother*, if it obeys the *smoothing property*

$$(4) \qquad \begin{array}{ll} \text{(i)} & \|\mathcal{S}_{j,m_j} v_j\|_a \le c\,\dfrac{h_j^{-1}}{m_j^{\gamma}} \|v_j\|_{L^2} \\ \text{(ii)} & \|\mathcal{S}_{j,m_j} v_j\|_a \le \|v_j\|_a \end{array} \qquad \forall v_j \in X_j,$$

with a parameter $0 < \gamma \le 1$. As is shown in [5] the symmetric Gauß-Seidel-, the SSOR- and the damped Jacobi-iteration are smoothers in the sense of (4) with parameter

$$\gamma = 1/2.$$

Lemma 1.1. *A smoother in the sense of* (4) *fulfills*

$$\|\mathcal{S}_{j,m_j} v_j\|_a \le c\,\frac{h_j^{-\alpha}}{m_j^{\alpha\gamma}} \|v_j\|_{H^{1-\alpha}} \qquad \forall v_j \in X_j.$$

Proof. This can be shown be an usual interpolation argument using discrete Sobolev norms like those introduced in [1] and their equivalence to the fractional Sobolev norms. □

We are now able to state and prove the main convergence estimate for the cascadic iteration (1).

Theorem 1.2. *The error of the cascadic iteration with a smoother as basic iteration can be estimated by*

$$\|u_\ell - u_\ell^*\|_a \le c \sum_{j=1}^{\ell} \frac{1}{m_j^{\alpha\gamma}} \|u_j - u_{j-1}\|_a \le c \sum_{j=1}^{\ell} \frac{h_j^{\alpha}}{m_j^{\alpha\gamma}} \|f\|_{H^{\alpha-1}}.$$

Proof. For $j = 1, \ldots, \ell$ we get by the linearity of $\mathcal{S}_{j,m_j}$

$$\begin{aligned} \|u_j - u_j^*\|_a &= \|\mathcal{S}_{j,m_j}(u_j - u_{j-1}^*)\|_a \\ &\le \|\mathcal{S}_{j,m_j}(u_j - u_{j-1})\|_a + \|\mathcal{S}_{j,m_j}(u_{j-1} - u_{j-1}^*)\|_a. \end{aligned}$$

The first term can be estimated by Lemma 1.1 and the approximation property (3):

$$\begin{aligned} \|\mathcal{S}_{j,m_j}(u_j - u_{j-1})\|_a &\le c\,\frac{h_j^{-\alpha}}{m_j^{\alpha\gamma}} \|u_j - u_{j-1}\|_{H^{1-\alpha}} \\ &\le c\,\frac{1}{m_j^{\alpha\gamma}} \|u_j - u_{j-1}\|_a. \end{aligned}$$

If we estimate the second term by property (4(ii)) of a smoother, we get

$$\|u_j - u_j^*\|_a \le \frac{c}{m_j^{\alpha\gamma}} \|u_j - u_{j-1}\|_a + \|u_{j-1} - u_{j-1}^*\|_a.$$

Using $u_0^* = u_0$ we get by induction

$$\|u_\ell - u_\ell^*\|_a \le c \sum_{j=1}^{\ell} \frac{1}{m_j^{\alpha\gamma}} \|u_j - u_{j-1}\|_a.$$

Galerkin orthogonality gives

$$\|u_j - u_{j-1}\|_a \le \|u - u_{j-1}\|_a,$$

so that the error estimate (2) yields the second assertion. $\square$

We choose m_j as the smallest integer, such that

$$m_j^\gamma \ge m^\gamma \cdot 2^{(\gamma d+1)(\ell-j)/2}. \tag{5}$$

The integer $m = m_\ell$ is therefore the number of iterations on level ℓ. With this choice the cascadic iteration can be shown to be optimal.

Lemma 1.3. *Let the number m_j of iterations on level j be given by (5). The cascadic iteration yields the error*

$$\|u_\ell - u_\ell^*\|_a \le c(\gamma)\, \frac{h_\ell^\alpha}{m^{\alpha\gamma}} \|f\|_{H^{\alpha-1}},$$

if $\gamma > 1/d$, and

$$\|u_\ell - u_\ell^*\|_a \le c\, \frac{h_\ell^\alpha (1 + |\log h_\ell|)}{m^{\alpha/d}} \|f\|_{H^{\alpha-1}}$$

if $\gamma = 1/d$.

Proof. By $2^{-j}/c \le h_j \le c\,2^{-j}$ we get

$$\sum_{j=1}^{\ell} \frac{h_j^\alpha}{m_j^{\alpha\gamma}} \le c\, m^{-\alpha\gamma} 2^{-(\gamma d+1)\alpha\ell/2} \sum_{j=1}^{\ell} 2^{(\gamma d-1)\alpha j/2}.$$

If $\gamma > 1/d$ this is a geometric sum which can be estimated by

$$\sum_{j=1}^{\ell} \frac{h_j^\alpha}{m_j^{\alpha\gamma}} \le c(\gamma)\, m^{-\alpha\gamma} 2^{-(\gamma d+1)\alpha\ell/2} 2^{(\gamma d-1)\alpha\ell/2} = c(\gamma) m^{-\alpha\gamma} 2^{-\alpha\ell} \le c(\gamma)\, m^{-\alpha\gamma} h_\ell^\alpha.$$

In the case $\gamma = 1/d$ the sum is equal to ℓ, such that

$$\sum_{j=1}^{\ell} \frac{h_j^\alpha}{m_j^{\alpha/d}} \le c\, m^{-\alpha/d} 2^{-\alpha\ell} \ell \le c\, m^{-\alpha/d} h_\ell^\alpha (1 + |\log h_\ell|).$$

Theorem 1.2 yields the assertion. $\square$

The complexity of the method is given by the following

Lemma 1.4. *Let the number m_j of iterations on level j be given by (5). If $\gamma > 1/d$ we get*

$$\sum_{j=1}^{\ell} m_j\, n_j \le c(\gamma)\, m\, n_\ell,$$

if $\gamma = 1/d$

$$\sum_{j=1}^{\ell} m_j\, n_j \le c\, m\, n_\ell (1 + \log n_\ell).$$

Proof. We have

$$2^{dj}/c \le n_j = \dim X_j \le c\, 2^{dj}.$$

Therefore we get

$$\sum_{j=1}^{\ell} m_j\, n_j \le c\, m 2^{(\gamma d+1)\ell/2\gamma} \sum_{j=1}^{\ell} 2^{(\gamma d-1)j/2\gamma}.$$

If $\gamma > 1/d$ this is a geometric sum which can be estimated by

$$\sum_{j=1}^{\ell} m_j\, n_j \le c(\gamma)\, m 2^{(\gamma d+1)\ell/2\gamma} 2^{(\gamma d-1)\ell/2\gamma} = c(\gamma) m 2^{d\ell} \le c(\gamma)\, m\, n_\ell.$$

In the case $\gamma = 1/d$ the sum is equal to ℓ, such that

$$\sum_{j=1}^{\ell} m_j\, n_j \le c\, m\, 2^{d\ell}\, \ell \le c\, m\, n_\ell (1 + \log n_\ell).$$

$\square$

In order that also in the case $\gamma = 1/d$ the cascadic iteration has an iteration error like the discretization error (2), we choose a special number of final iterations.

Lemma 1.5. *Let $\gamma = 1/d$. If we choose the number of iterations on level ℓ as the smallest integer m with*

$$m \ge m_* (1 + |\log h_\ell|)^{d/\alpha}$$

we get for the error of the cascadic iteration

$$\|u_\ell - u_\ell^*\|_a \le c\, \frac{h_\ell^\alpha}{m_*^{\alpha/d}} \|f\|_{H^{\alpha-1}},$$

and as complexity

$$\sum_{j=1}^{\ell} m_j\, n_j \le c\, m_* n_\ell (1 + \log n_\ell).$$

Proof. By observing that

$$c_0\, \ell \le (1 + |\log h_\ell|) \le c_1\, (1 + \log n_\ell) \le c_2\, \ell$$

the assertion is clear by Lemma 1.3 and Lemma 1.4. □

Our results show, that the cascadic iteration with a plain Gauss-Seidel-, SSOR- or damped Jacobi-iteration (all with $\gamma = 1/2$) as basic iteration is

- optimal for $d = 3$,
- accurate with complexity $O(n_\ell |\log n_\ell|)$ for $d = 2$.

2. Conjugate gradient method as basic iteration. When using the conjugate gradient method as the basic iteration we have to tackle with a problem: the result

$$\mathcal{I}_{j,m_j} u_j^0$$

of m_j steps of the cg-method is *not* linear in the starting value u_j^0. Thus, it seems that our frame up to now does not cover the cg-method. However, there is a remedy which uses results on the cg-method well known in the Russian literature [7, 9].

We have to fix some notation. Let $\langle \cdot, \cdot \rangle$ be the euclidean scalar product of the nodal basis in the finite element space X_j, the induced norm will be denoted by

$$|v_j|^2 = \langle v_j, v_j \rangle \qquad \forall v_j \in X_j.$$

We define the linear operator $A_j : X_j \to X_j$ by

$$\langle A v_j, w_j \rangle = a(v_j, w_j) \qquad \forall v_j, w_j \in X_j,$$

which is represented in the nodal basis by the usual *stiffness matrix*. The error of the cg-method applied to the stiffness matrix can be expressed by

$$\|u_j - \mathcal{I}_{j,m_j} u_j^0\|_a = \min_{\substack{p \in P_{m_j}, \\ p(0)=1}} \|p(A_j)(u_j - u_j^0)\|_a.$$

Here P_{m_j} denotes the set of polynomials p with $\deg p \le m_j$.

The idea is now, to find some polynomial $q_{j,m_j} \in P_{m_j}$ with $q_{j,m_j}(0) = 1$, such that

$$\mathcal{S}_{j,m_j} = q_{j,m_j}(A_j)$$

defines a smoother in the sense of (4). Since the error in energy of the cg-method is then *majorized* by this linear smoother, the results of Section 1 are immediately valid for the cg-method.

The choice of q_{j,m_j} depends on the following solution of a polynomial minimization problem.

Lemma 2.1. *Let $\lambda > 0$. The Chebyshev polynomial T_{2m+1} has the representation*

$$T_{2m+1}(x) = (-1)^m (2m+1)\, x\, \phi_{\lambda,m}(\lambda x^2)$$

with a unique $\phi_{\lambda,m} \in P_m$ and $\phi_{\lambda,m}(0) = 1$. The polynomial $\phi_{\lambda,m}$ solves the minimization problem

$$\max_{x \in [0,\lambda]} |\sqrt{x} p(x)| = \min!$$

over all polynomials $p \in P_m$ which are normalized by $p(0) = 1$. The minimal value is given by

$$\max_{x \in [0,\lambda]} |\sqrt{x} \phi_{\lambda,m}(x)| = \frac{\sqrt{\lambda}}{2m+1}.$$

Moreover we have

$$\max_{x \in [0,\lambda]} |\phi_{\lambda,m}(x)| = 1.$$

A proof may be found in the book of Shaidurov [9]. However, Shaidurov represents $\phi_{\lambda,m}$ by the expression

$$\phi_{\lambda,m}(x) = \prod_{k=1}^{m} (1 - x/\mu_k), \quad \mu_k = \lambda \cos^2((2k-1)\pi/2(2m+1)).$$

As a fairly easy consequence Shaidurov [10] proves the following

Theorem 2.2. *The linear operator*

$$\mathcal{S}_{j,m_j} = \phi_{\lambda_j,m_j}(A_j), \qquad \lambda_j = \max \sigma(A_j)$$

satisfies

$$\begin{aligned} &\text{(i)} \quad \|\mathcal{S}_{j,m_j} v_j\|_a \le \frac{\sqrt{\lambda_j}}{2m_j+1} |v_j| \\ &\text{(ii)} \quad \|\mathcal{S}_{j,m_j} v_j\|_a \le \|v_j\|_a \end{aligned} \qquad \forall v_j \in X_j.$$

A little finite element theory shows that we have found a majorizing smoother for the cg-method.

Corollary 2.3. *The linear operator*

$$\mathcal{S}_{j,m_j} = \phi_{\lambda_j,m_j}(A_j), \qquad \lambda_j = \max \sigma(A_j)$$

defines a smoother in the sense of (4) with parameter $\gamma = 1$.

Proof. The usual inverse inequality shows that the maximum eigenvalue of the stiffness matrix can be estimated by

$$\lambda_j \le c h_j^{d-2},$$

cf. [6, Theorem 8.8.6]. The euclidean norm with respect to the nodal basis is related to the L^2-norm by

$$\frac{1}{c} h_j^d |v_j|^2 \le \|v_j\|_{L^2}^2 \le c h_j^d |v_j|^2,$$

cf. [6, Theorem 8.8.1]. Thus Theorem 2.2 gives

$$\|\mathcal{S}_{j,m_j} v_j\|_a \le c \frac{h_j^{(d-2)/2}}{2m_j+1} |v_j| \le c \frac{h_j^{-1}}{2m_j+1} \|v_j\|_{L^2},$$

i.e., (4(i)) with $\gamma = 1$. Property (4(ii)) was already stated in Theorem 2.2. □

With the help of this majorizing smoother it is immediately clear, that Theorem 1.2, Lemma 1.3, Lemma 1.4 and Lemma 1.5 remain valid for the cascadic iteration with the cg-method as basic iteration, short the CCG-method of Deuflhard [3]. In particular the CCG method is optimal for $d = 2, 3$.

3. Adaptive control of the CCG-method. In this section we develop an adaptive control of the CCG method which is based on our theoretical considerations and some additional assumptions on the family of triangulation.

For adaptive triangulations we drop the assumption of quasi-uniformity. All constants in the sequel will not depend on the quasi-uniformity, but only on the shape regularity of the triangulations.

In order to bound the maximum eigenvalue of the involved matrix, we compute the stiffness matrix A_j with respect to the scaled nodal basis

$$h_{j,i}^{(2-d)/2}\psi_{j,i} \qquad i = 1, \ldots, n_j,$$

where $\{\psi_{j,i}\}_i$ is the usual nodal basis of X_j and

$$h_{j,i} = \operatorname{diam}\operatorname{supp}\psi_{j,i}.$$

In this section we denote by $\langle\cdot,\cdot\rangle$ the euclidean scalar product with respect to this *scaled* nodal basis. Xu [11] has shown the equivalence of norms

$$\frac{1}{c}|v_j|^2 \le \sum_{T\in\mathcal{T}_j} h_T^{-2}\|v_j\|_{L^2(T)}^2 \le c\,|v_j|^2, \qquad h_T = \operatorname{diam} T,$$

and the bound

$$\lambda_j = \max\sigma(A_j) \le c$$

for the maximum eigenvalue of A_j. Hence, we get for the majorizing smoothing iteration of the conjugate gradient method as defined in Section 2

$$\|\mathcal{S}_{j,m_j}v_j\|_a \le \frac{c}{m_j}\left(\sum_{T\in\mathcal{T}_j} h_T^{-2}\|v_j\|_{L^2(T)}^2\right)^{1/2}.$$

In order to turn this into a starting point for a theorem like Theorem 1.2 we make the following two assumptions on the family of triangulations:

$$\text{(6)}\qquad \begin{aligned} &\text{(i)} \quad h_T^{-2}\|u_j - u_{j-1}\|_{L^2(T)}^2 \le c\|u_j - u_{j-1}\|_{H^1(T)}^2, \qquad \forall T\in\mathcal{T}_j \\ &\text{(ii)} \quad \|u - u_j\| \le c\,n_j^{-1/d}\|f\|_{L^2}. \end{aligned}$$

This is heuristically justified for *adaptive* triangulations. The first assumption (i) means, that the finite element correction is locally of high frequency with respect to the finer triangulation. In other words, the refinement resolves changes *but not more*. Thus it is a statement of the *efficiency* of a triangulation. Note that quasi-uniform triangulations do not accomplish assumption (ii) for problems which are not H^2-regular. The second assumption is a statement of optimal *accuracy*, which is justified by results of nonlinear approximation theory like [8]. The same proof as for Theorem 1.2 gives the following

Lemma 3.1. *Assumption (6) implies for the error of the cascadic conjugate gradient iteration*

$$\|u_\ell - u_\ell^*\|_a \le c \sum_{j=1}^{\ell} \frac{1}{m_j} \|u_j - u_{j-1}\|_a \le c \sum_{j=1}^{\ell} \frac{1}{m_j\, n_j^{1/d}} \|f\|_{L^2}.$$

We can now extend Lemma 1.3 and Lemma 1.4 to the adaptive case. Here we have to use additionally, that the sequence of number of unknowns belongs to some kind of geometric progression:

$$n_j < \sigma_0 n_j \le n_{j+1} \le \sigma_1 n_j \qquad j = 0, 1, \dots.$$

With the choice of the iteration numbers as smallest integers for which

$$m_j \ge m \left(\frac{n_\ell}{n_j}\right)^{(d+1)/2d}, \tag{7}$$

we get for $d > 1$ under assumption (6) the final error

$$\|u_\ell - u_\ell^*\|_a \le \frac{c}{m\, n_\ell^{1/d}} \|f\|_{L^2}$$

and the complexity

$$\sum_{j=1}^{\ell} m_j\, n_j \le c\, m\, n_\ell.$$

However, in an adaptive algorithm we do *not* know at level j the number n_ℓ of nodal points at the final level. So far our iteration is not implementable. But with slight changes we can still operate with it. We define the final level ℓ as the one, for which the approximation error is below some user given tolerance TOL. Hence assumption (6) gives us the relation

$$\frac{\|u - u_j\|_a}{\mathrm{TOL}} \approx \left(\frac{n_\ell}{n_j}\right)^{1/d},$$

which leads us to replace (7) by the smallest integer with

$$m_j \ge m \left(\frac{\|u - u_j\|_a}{\mathrm{TOL}}\right)^{(d+1)/2}. \tag{8}$$

The actual approximation error $\|u - u_j\|_a$ is also not known, so we replace this expression by

$$\|u - u_j\|_a \approx \epsilon_{j-1} \left(\frac{n_{j-1}}{n_j}\right)^{1/d} \tag{9}$$

for some estimate ϵ_{j-1} of the previous approximation error $\|u-u_{j-1}\|_a$, cf. [2].

This algorithm is nearest to the a priori choice of the parameters m_j. In practice, the basic iteration can be accurate enough much earlier than stated in theory. The crucial relation we used for the algorithm was

$$\|u_j - u_j^*\|_a \leq \frac{c}{m_j}\|u-u_{j-1}\|_a + \|u_{j-1}-u_{j-1}^*\|_a,$$

so we turn it into a termination criterion for the basic iteration by using (8). We terminate the iteration with

$$\|u_j - u_j^*\|_a \leq \rho\left(\frac{\mathrm{TOL}}{\|u-u_j\|_a}\right)^{(d+1)/2} + \|u_{j-1}-u_{j-1}^*\|_a. \tag{10}$$

Here $0<\rho<1$ is some safety factor. Of course we replace $\|u-u_j\|_a$ by the estimate (9) and $\|u_j-u_j^*\|_a$, $\|u_{j-1}-u_{j-1}^*\|_a$ by some estimate of the iteration error.

Despite the fact, that our termination criterion is different from the original one of Deuflhard [3] we get comparable numerical results for the CCG method. They share the essential feature that the iteration has to be more accurate on coarser triangulations. However, the basis for (10) seems to be a sound combination of theory and heuristics.

Example. We applied the adaptive CCG method with $\mathrm{TOL}=10^{-4}$ to the elliptic problem

$$-\Delta u = 0, \qquad u|_{\Gamma_1}=1, \quad u|_{\Gamma_2}=0, \quad \partial_n u|_{\Gamma_3}=0$$

on a domain Ω which is a unit square with slit

$$\Omega = \{x\in\mathbf{R}^2 : |x|_\infty \leq 1\} \cap \{x\in\mathbf{R}^2 : |x_2| \geq 0.03x_1\}.$$

The boundary pieces are

$$\Gamma_1 = \{x\in\Omega : x_1=1, x_2\geq 0.03\}, \quad \Gamma_2=\{x\in\Omega : x_1=1, x_2\leq -0.03\},$$

and $\Gamma_3 = \partial\Omega \setminus (\Gamma_1\cup\Gamma_2)$.

Figure 1 shows a typical triangulation and isolines of the solution. In Figure 2 we have drawn the number of iterations m_j which were used for the corresponding number of unknowns n_j. We compared three different implementations:

- CCG1 : the CCG method with termination criterion (10).

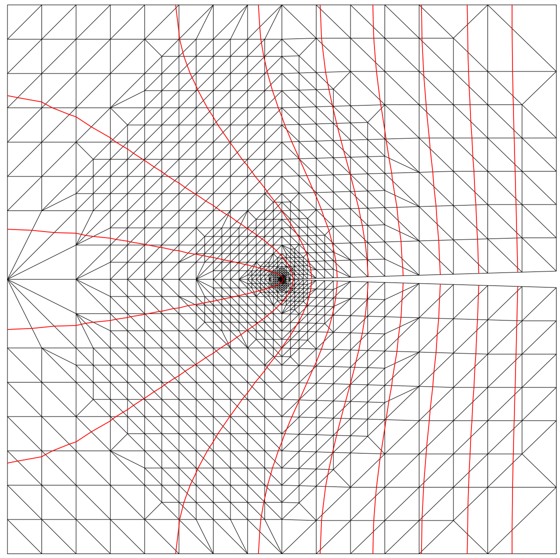

FIG. 1. *Typical adaptive triangulation with isolines of solution*

- CCG2 : the CCG method with Deuflhard's termination criterion [3].
- CSGS : an adaptive cascadic iteration using symmetric Gauss-Seidel as basic iteration. We implemented the termination criterion (10) in this case also.

We observe that the CCG1 and CCG2 implementations are comparable, the CCG1 needs one adaptive step less in order to achieve the tolerance TOL.

The performance of the CSGS implementation is a little bit smoother and gives completely satisfactory results. In fact, if one looks at the real CPU-time needed for the *achieved accuracy* $\|u - u_j^*\|_a$, one gets nearly indistinguishable curves for all three implementations!

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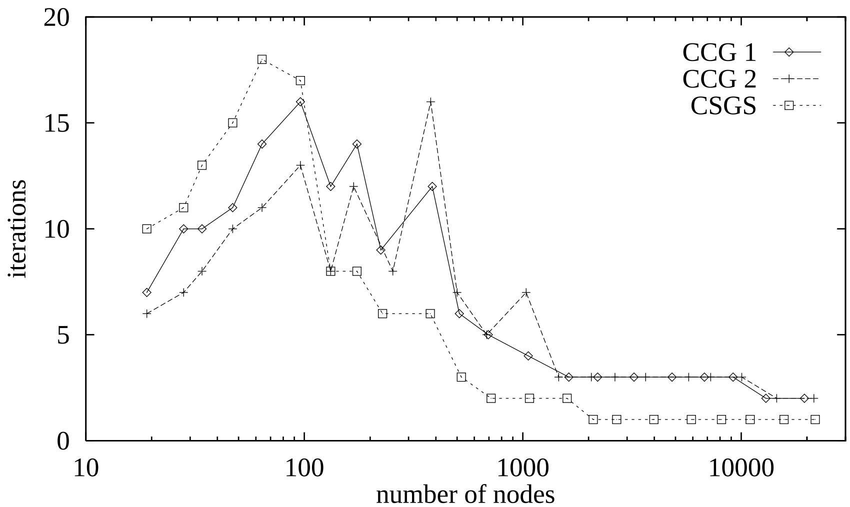

FIG. 2. *Number of iterations vs. number of unknowns*

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