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ZIB-Report 11-43 (November 2011)

Herausgegeben vom
Konrad-Zuse-Zentrum für Informationstechnik Berlin
Takustraße 7
D-14195 Berlin-Dahlem

Telefon: 030-84185-0
Telefax: 030-84185-125

e-mail: bibliothek@zib.de
URL: http://www.zib.de

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

Workload and busy period for $M/GI/1$ with a general impatience mechanism

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Abstract

The paper deals with the workload and busy period for the $M/GI/1$ system under FCFS discipline, where the customers may become impatient during their waiting for service with generally distributed maximal waiting times and also during their service with generally distributed maximal service times depending on the time waited for service. This general impatience mechanism, originally introduced by Kovalenko (1961) and considered by Daley (1965), too, covers the special cases of impatience on waiting times as well as impatience on sojourn times, for which Boxma et al. (2010), (2011) gave new results and outlined special cases recently. Our unified approach bases on the vector process of workload and busy time. Explicit representations for the LSTs of workload and busy period are given in case of phase-type distributed impatience.

Mathematics Subject Classification (MSC 2000): 60K25, 68M20, 90B22.

Keywords: $M/GI/1+GI$; $M/GI/1+PH$; impatient customers; workload; busy period; waiting time dependent service.

1 Introduction and model description

In this paper we deal with the workload (virtual waiting time) and busy period for the $M/GI/1+GI$ system, where the customers are served under FCFS discipline with generally distributed service times and where the customers may become impatient during their waiting for service with generally distributed maximal waiting times but also during their service with generally distributed maximal service times depending on the time waited for service. More precisely: at a single server with a waiting room of infinite capacity there arrive customers according to a Poisson process of intensity λ. The distribution $B(y)$ of their required service times S has finite expectation. Each customer waits at most the time I for service, else he leaves the system by impatience after having waited the time I. The distribution $C(u) := P(I \leq u)$ has finite expectation. If the service of a customer starts after having waited for service the time W, then the customer waits for service completion at most the time J, else he leaves the system by impatience after having waited the time J during service. The distribution $G_w(x) := P(J \leq x \,|\, W = w)$ may be defective. Note that the system is stable as the workload process is dominated by the workload process in the corresponding infinite server system with required service time $I + S$. For technical reasons we assume that $B(y)$, $C(u)$, and $G_w(x)$ are continuous functions and that $C(u) < 1$, $u \in \mathbb{R}_+$. For notational convenience let $\bar{B}(y) := 1-B(y)$, $\bar{C}(u) := 1-C(u)$, and $\bar{G}_w(x) := 1-G_w(x)$.

We refer to the above impatience mechanism as Kovalenko's impatience mechanism since this general impatience mechanism was first considered – for our best knowledge – by Kovalenko [Kov], who derived the Volterra equation for the density of the stationary workload distribution, cf. Section 2.2 below, expressed several performance measures of the system in terms of it, and outlined the special case of mixed deterministic impatience times. Later [Dal] investigated the system in more detail.

The general impatience mechanism covers well known impatience mechanisms as special cases:

(i) If $\bar{G}_w(x) \equiv 1$, then the customers may become impatient only during their *waiting time*. This model is denoted by $M/GI/1 + GI^w$, where w refers to waiting time, as proposed in [BPS].

(ii) In case of $\bar{G}_w(x) = \bar{C}(x + w)/\bar{C}(w)$, i.e.

$$P(J > x \,|\, W = w) = P(I > x{+}w \,|\, I > w),$$

the customer's overall impatience time d.f. is $C(u)$, i.e., the impatience refers to the *sojourn time*, and his real sojourn time is the minimum of his maximal sojourn time with d.f. $C(u)$ and his waiting plus service time. This model is denoted by $M/GI/1 + GI^s$, where s refers to sojourn time, as proposed in [BPS].

(iii) In [Kov] a mixed scheme of (i) and (ii) is proposed by choosing

$$\bar{G}_w(x) = \frac{\bar{C}(\alpha x{+}w)}{\bar{C}(w)}\,, \quad x \in \mathbb{R}_+, \tag{1.1}$$

for $\alpha \in \mathbb{R}_+$. In this paper we consider also the mixed scheme

$$\bar{G}_w(x) = \left(\frac{\bar{C}(x+w)}{\bar{C}(w)}\right)^\alpha, \quad x \in \mathbb{R}_+, \tag{1.2}$$

for $\alpha \in \mathbb{R}_+$. The mixed scheme (1.2) has the advantage that there exists a simple relation between the workload distributions for different values of α, cf. Section 2.2 below. Note that the schemes (1.1) and (1.2) coincide for $\alpha = 0$ and $\alpha = 1$, where we obtain (i) and (ii), respectively. Moreover, the schemes (1.1) and (1.2) coincide for exponential maximal waiting times, i.e. for $\bar{C}(u) = e^{-\gamma u}$, $u \in \mathbb{R}_+$, where we have $\bar{G}_w(x) = e^{-\alpha\gamma x}$, $x \in \mathbb{R}_+$, cf. Section 3.1 below.

Thus Kovalenko's general impatience mechanism provides a unified approach to different particular impatience mechanisms and does not require to handle them by a separate mathematical analysis, cf. e.g. [BPS].

There is a huge literature concerning queueing models with impatient customers. A good overview of results for workload, busy period, number of customers in the system, and others for various particular models, including the systems $M/GI/1+GI^w$ and $M/GI/1+GI^s$, and for various impatience distributions (M, D, E_k) as well as model variants including the observable (balking) and unobservable (reneging) case, is given in [PSZ], recently in [BPS], and in the references therein. Thus in this paper we will only refer to papers which are directly connected to our results.

For the systems $M/GI/1+GI^w$ and $M/GI/1+GI^s$ in [BPS, Section 5] an expression is derived for the LST of the busy period in terms of the solution of a non linear integral equation for the LST of the work load to move from a given level down to another given level. For the $M/GI/1 + D^w$ system in [KBL] an explicit representation for the LST of the busy period length is given starting from an initial workload not greater than the constant maximal impatience time. Note that in their framework the busy period length is zero if the initial work load is zero. In [BPSZ] results are outlined for the busy period in the systems $M/M/1+GI^w$ and $M/GI/1+GI^w$ with a discrete impatience time distribution. Concerning results for the busy period in case of exponentially distributed impatience times, cf. Section 3.1 below. For results concerning the workload, cf. Section 2.2 below.

The paper is organized as follows. In Section 2, a Fredholm integral equation is derived for the density of the vector process of workload and busy time. Its solution is given in terms of the solutions of two Volterra equations and thus by two Neumann series, cf. Theorem 2.1. In Section 2.1, the LSTs of workload and busy period are represented in terms of the Laplace transforms of these Neumann series. In Section 2.2, the workload distribution for any $\alpha \in \mathbb{R}_+$ within the scheme (1.2) is given by the workload distribution for $\alpha = 0$, cf. Theorem 2.2. In particular, the workload distribution in $M/GI/1 + GI^s$ is given by the workload distribution in $M/GI/1 + GI^w$. In Section 3, we analyze the special case of phase-type distributed impatience times for the mixed scheme (1.1). In the very special case of exponentially distributed impatience times, the system is easily reduced to the well known $M/GI/1 + M^w$ system with a modified service time distribution, cf. Section 3.1. In case of generally phase-type distributed impatience times, explicit representations for the Laplace transforms of the two Neumann series mentioned above are derived in Sections 3.2 and 3.3, which imply explicit representations for the LSTs of workload and busy period in case of generally phase-type distributed impatience times within the mixed scheme (1.1), generalizing corresponding results for $M/GI/1 + M^w$ and $M/GI/1 + M^s$.

2 The vector process of workload and busy time

Let V_t be the workload (virtual waiting time) at time t, i.e. the duration a virtual customer without impatience arriving at time t would have to wait for service, and let U_t be the age of the busy period at time t if $V_t > 0$, else $U_t := 0$. Note that (V_t, U_t), $t \in \mathbb{R}$, is a Markov process, where the sample paths are right continuous almost surely. In the following we assume that the system is in steady state, i.e., that the process is stationary and ergodic.

Remark 2.1 *The process (V_t, U_t), $t \in \mathbb{R}$, is equal in distribution to the corresponding process in the $M/GI/1$ system without impatience where the required service time S depends on the time W waited for service such that*

$$P(S > y \mid W = w) := \bar{C}(w)\bar{G}_w(y)\bar{B}(y) = K(y+w, w), \quad w, y \in \mathbb{R}_+,$$

cf. (2.3) *below. In this system, V_0 is equal in distribution to the waiting time W due to PASTA.*

The results of this section can be generalized to corresponding results for the waiting time and busy period in general $M/GI/1$ systems where the required service time depends on the time waited for service, cf. [BKNN] and the references therein for such systems. In view of (2.15) *below, it suffices to assume that $P(S > y \mid W = w)$, $w, y \in \mathbb{R}_+$, is continuous and that* $\max_{w \in [0,y]} P(S > y - w \mid W = w)$ *is integrable over $y \in \mathbb{R}_+$.*

We want to analyze the expectation

$$E[\,\mathbb{I}\{V_0\!>\!x\}e^{-s(U_0+x)}], \quad s,x\in\mathbb{R}_+. \tag{2.1}$$

Note that in case of $V_0 > x$ the duration of the busy period running at time $t = 0$ is at least $U_0 + x$. Taking into account the dynamics of the system during the interval $[-h, 0]$, the balance equation for (2.1) may be written as

$$\begin{aligned}E[\,\mathbb{I}\{V_0\!>\!x\}e^{-s(U_0+x)}] &= (1\!-\!\lambda h)E[\,\mathbb{I}\{V_{-h}\!>\!x\!+\!h\}e^{-s(U_{-h}+x+h)}]\\ &+ \lambda h\bar{G}_0(x\!+\!h)\bar{B}(x\!+\!h)E[\,\mathbb{I}\{V_{-h}\!=\!0\}e^{-s(x+h)}]\\ &+ \lambda h\int_h^{x+h}\bar{C}(\xi)\bar{G}_\xi(x\!+\!h\!-\!\xi)\bar{B}(x\!+\!h\!-\!\xi)\\ &\qquad\qquad \mathrm{d}_\xi E[\,\mathbb{I}\{V_{-h}\!\le\!\xi\}e^{-s(U_{-h}+x+h)}]\\ &+ \lambda h E[\,\mathbb{I}\{V_{-h}\!>\!x\!+\!h\}e^{-s(U_{-h}+x+h)}] + o(h)\end{aligned} \tag{2.2}$$

for sufficiently small $h > 0$. Using the stationarity of (V_t, U_t), $t \in \mathbb{R}$, and introducing the kernel

$$K(x,\xi) := \bar{C}(\xi)\bar{G}_\xi(x\!-\!\xi)\bar{B}(x\!-\!\xi), \quad 0\le\xi\le x, \tag{2.3}$$

(2.2) is equivalent to

$$\begin{aligned}0 = \frac{\partial}{\partial x}\,E[\,\mathbb{I}\{V_0\!>\!x\}e^{-s(U_0+x)}] &+ \lambda\int_0^x K(x,\xi)\\ &\frac{\partial}{\partial\xi}\,(-E[\,\mathbb{I}\{V_0\!>\!\xi\}e^{-s(U_0+x)}])\mathrm{d}\xi + \lambda p(0)K(x,0)e^{-sx},\end{aligned} \tag{2.4}$$

where $p(0) := P(V_0 = 0)$ is the empty probability of the system.

Now we want to transform (2.4) into an integral equation. In view of $\lim_{x\to\infty} E[\,\mathbb{I}\{V_0 > x\}e^{-s(U_0+x)}] = 0$, we obtain

$$\begin{aligned}&\frac{\partial}{\partial\xi}\,(-E[\,\mathbb{I}\{V_0\!>\!\xi\}e^{-s(U_0+x)}])\\ &\quad= \frac{\partial}{\partial\xi}\left(e^{-s(x-\xi)}\int_\xi^\infty\frac{\partial}{\partial\eta}\,(E[\,\mathbb{I}\{V_0\!>\!\eta\}e^{-s(U_0+\eta)}])\mathrm{d}\eta\right)\\ &\quad= e^{-s(x-\xi)}\biggl(\frac{\partial}{\partial\xi}\,(-E[\,\mathbb{I}\{V_0\!>\!\xi\}e^{-s(U_0+\xi)}])\\ &\qquad\quad - s\int_\xi^\infty\frac{\partial}{\partial\eta}\,(-E[\,\mathbb{I}\{V_0\!>\!\eta\}e^{-s(U_0+\eta)}])\mathrm{d}\eta\biggr).\end{aligned}$$

Using the notation

$$\varphi(s,x) := \frac{1}{\lambda p(0)}\frac{\partial}{\partial x}\left(-E[\mathbb{I}\{V_0 > x\}e^{-s(U_0+x)}]\right), \quad s,x \in \mathbb{R}_+, \tag{2.5}$$

thus (2.4) is equivalent to the Fredholm integral equation

$$\varphi(s,x) = \lambda\int_0^x K(x,\xi)e^{-s(x-\xi)}\left(\varphi(s,\xi) - s\int_\xi^\infty \varphi(s,\eta)\mathrm{d}\eta\right)\mathrm{d}\xi \\ + K(x,0)e^{-sx}, \quad x \in \mathbb{R}_+. \tag{2.6}$$

Theorem 2.1 *For fixed $s \in \mathbb{R}_+$, the integral equation* (2.6) *has a uniquely determined solution $\varphi(s,\cdot) \in C(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$, which is given by*

$$\varphi(s,x) := \varphi_1(s,x) - \alpha(s)\varphi_2(s,x), \quad x \in \mathbb{R}_+, \tag{2.7}$$

where for $j = 1,2$

$$\varphi_j(s,x) := a_j(s,x) + \sum_{n=1}^{\infty}\lambda^n\int_0^x K_n(s,x,\xi)a_j(s,\xi)\mathrm{d}\xi, \quad x \in \mathbb{R}_+, \tag{2.8}$$

$$a_1(s,x) := K(x,0)e^{-sx}, \quad a_2(s,x) := s\int_0^x K(x,\eta)e^{-s(x-\eta)}\mathrm{d}\eta, \\ x \in \mathbb{R}_+, \tag{2.9}$$

$$K_1(s,x,\xi) := K(x,\xi)e^{-s(x-\xi)} + s\int_\xi^x K(x,\eta)e^{-s(x-\eta)}\mathrm{d}\eta, \quad 0 \le \xi \le x, \tag{2.10}$$

and for $n = 2,3,\ldots$

$$K_n(s,x,\xi) := \int_\xi^x K_1(s,x,\eta)K_{n-1}(s,\eta,\xi)\mathrm{d}\eta, \quad 0 \le \xi \le x, \tag{2.11}$$

$$\alpha(s) := \left(\lambda\int_{\mathbb{R}_+}\varphi_1(s,\xi)\mathrm{d}\xi\right)\Big/\left(1+\lambda\int_{\mathbb{R}_+}\varphi_2(s,\xi)\mathrm{d}\xi\right). \tag{2.12}$$

Proof. For fixed $s \in \mathbb{R}_+$ let $\varphi_j(s,\cdot)$, $j \in \{1,2\}$, be the solution of the Volterra integral equation

$$\varphi_j(s,x) = \lambda\int_0^x K(x,\xi)e^{-s(x-\xi)}\left(\varphi_j(s,\xi) + s\int_0^\xi \varphi_j(s,\eta)\mathrm{d}\eta\right)\mathrm{d}\xi \\ + a_j(s,x), \quad x \in \mathbb{R}_+, \tag{2.13}$$

or equivalently by using Fubini's Theorem

$$\varphi_j(s,x) = \lambda \int_0^x K_1(s,x,\xi)\varphi_j(s,\xi)\mathrm{d}\xi + a_j(s,x), \quad x \in \mathbb{R}_+. \tag{2.14}$$

Note that

$$0 \le K_1(s,x,\xi) \le \max_{\eta\in[0,x]} K(x,\eta) \le \bar{C}(x/2)+\bar{B}(x/2), \quad 0 \le \xi \le x. \tag{2.15}$$

The first inequality on the r.h.s. follows by using $K(x,\eta) \le \max_{\xi\in[0,x]} K(x,\xi)$ in (2.10) and the last inequality is a consequence of (2.3) and of the facts that $\bar{C}(\xi)$ and $\bar{B}(\xi)$ are non negative decreasing functions and that $\bar{C}(\xi)$, $\bar{B}(\xi)$, $\bar{G}_\xi(x)$ are bounded by 1. In view of $a_1(s,x) + a_2(s,x) = K_1(s,x,0)$, $x \in \mathbb{R}_+$, from (2.15) for $j = 1,2$ we find

$$0 \le a_j(s,x) \le \bar{C}(x/2)+\bar{B}(x/2), \quad x \in \mathbb{R}_+. \tag{2.16}$$

The existence and uniqueness of a solution of (2.14) in $C([0,b])$ follows from Banach's fixed point theorem using the norm $||\varphi|| := \sup_{x\in[0,b]} |e^{-3\lambda x}\varphi(x)|$, where the contraction of the corresponding linear operator

$$(\mathbb{A}\varphi)(x) := \lambda \int_0^x K_1(s,x,\xi)\varphi(\xi)\mathrm{d}\xi, \quad x \in [0,b],$$

follows from (2.15). Considering a sequence of intervals $[0,b]$, $b \to \infty$, one finds that existence and uniqueness even holds in $C(\mathbb{R}_+)$.

The $\varphi_j(s,x)$ are given by the corresponding Neumann series (2.8). From (2.15) by induction on $n = 1,2,\dots$ it follows that

$$0 \le K_n(s,x,\xi) \le (\bar{C}(x/2)+\bar{B}(x/2)) \frac{1}{(n-1)!}\left(\int_\xi^x (\bar{C}(\eta/2)+\bar{B}(\eta/2))\mathrm{d}\eta\right)^{n-1}, \quad 0 \le \xi \le x. \tag{2.17}$$

Taking into account (2.16), from (2.17) for $n = 1,2,\dots$ and $j = 1,2$ we find

$$0 \le \int_0^x K_n(s,x,\xi)a_j(s,\xi)\mathrm{d}\xi$$

$$\le (\bar{C}(x/2)+\bar{B}(x/2))\frac{1}{n!}\left(\int_0^x (\bar{C}(\xi/2)+\bar{B}(\xi/2))\mathrm{d}\xi\right)^n \tag{2.18}$$

$$\le (\bar{C}(x/2)+\bar{B}(x/2))\frac{(2EI+2ES)^n}{n!}, \quad x \in \mathbb{R}_+. \tag{2.19}$$

Because of (2.8), (2.16), and (2.19), for $j=1,2$ it holds

$$0 \le \varphi_j(s,x) \le e^{2\lambda(EI+ES)}(\bar{C}(x/2)+\bar{B}(x/2)), \quad x \in \mathbb{R}_+, \tag{2.20}$$

which implies $\varphi_j(s,\cdot) \in L_1(\mathbb{R}_+)$.

From (2.13), (2.9), and (2.12) we find that the r.h.s. of (2.7) is a solution of (2.6). On the other hand, let $\varphi(s,\cdot) \in C(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$ be an arbitrary solution of (2.6). Then $\varphi(s,x) + \lambda \int_{\mathbb{R}_+} \varphi(s,\xi)\mathrm{d}\xi\, \varphi_2(s,x)$ satisfies (2.13) for $j=1$, and thus it holds $\varphi(s,x)+\lambda \int_{\mathbb{R}_+} \varphi(s,\xi)\mathrm{d}\xi\, \varphi_2(s,x) = \varphi_1(s,x)$, $x \in \mathbb{R}_+$. Integration over $x \in \mathbb{R}_+$ and (2.12) provide $\lambda \int_{\mathbb{R}_+} \varphi(s,\xi)\mathrm{d}\xi = \alpha(s)$, and therefore $\varphi(s,x)$ is the solution of (2.6) given by (2.7). $\square$

Let $s \in \mathbb{R}_+$ and $j \in \{1,2\}$ be fixed. Because of (2.8), (2.16), and (2.19), for the Laplace transform $\Phi_j(s,\theta)$, $\theta \in \mathbb{C}_+ := \{z \in \mathbb{C} : \Re z \ge 0\}$, of $\varphi_j(s,\cdot)$ it holds

$$\Phi_j(s,\theta) := \int_{\mathbb{R}_+} e^{-\theta x}\varphi_j(s,x)\mathrm{d}x = \sum_{n=0}^{\infty} A_{n,j}(s,\theta)\lambda^n, \quad \theta \in \mathbb{C}_+, \tag{2.21}$$

where

$$A_{0,j}(s,\theta) := \int_{\mathbb{R}_+} e^{-\theta x} a_j(s,x)\mathrm{d}x, \quad \theta \in \mathbb{C}_+, \tag{2.22}$$

is the Laplace transform of $a_j(s,\cdot)$, and for $n=1,2,\dots$

$$A_{n,j}(s,\theta) := \int_{\mathbb{R}_+} e^{-\theta x} \int_0^x K_n(s,x,\xi)a_j(s,\xi)\mathrm{d}\xi\mathrm{d}x, \quad \theta \in \mathbb{C}_+. \tag{2.23}$$

Note that (2.16) and (2.18) imply

$$|A_{n,j}(s,\theta)| \le \frac{(2EI+2ES)^{n+1}}{(n+1)!}, \quad \theta \in \mathbb{C}_+,\ n \in \mathbb{Z}_+. \tag{2.24}$$

Due to (2.7) and (2.12), for the Laplace transform $\Phi(s,\theta)$, $\theta \in \mathbb{C}_+$, of $\varphi(s,\cdot)$ we have the representation

$$\begin{aligned}\Phi(s,\theta) &:= \int_{\mathbb{R}_+} e^{-\theta x}\varphi(s,x)\mathrm{d}x \\ &= \Phi_1(s,\theta) - \frac{\lambda\Phi_1(s,0)}{1+\lambda\Phi_2(s,0)}\,\Phi_2(s,\theta), \quad s \in \mathbb{R}_+,\ \theta \in \mathbb{C}_+. \end{aligned} \tag{2.25}$$

Note that

$$\Phi_2(0,\theta) = 0, \quad \theta \in \mathbb{C}_+, \tag{2.26}$$

because of (2.21), (2.22), (2.23), and (2.9). Thus from (2.25) it follows

$$\Phi(0,\theta) = \Phi_1(0,\theta), \quad \theta \in \mathbb{C}_+. \tag{2.27}$$

2.1 Performance measures

In view of $\lim_{x\to\infty} E[\mathbb{I}\{V_0 > x\}e^{-s(U_0+x)}] = 0$, from (2.5) we obtain

$$E[\mathbb{I}\{V_0 > x\}e^{-s(U_0+x)}] = \lambda p(0) \int_x^\infty \varphi(s,\xi)\mathrm{d}\xi, \quad s, x \in \mathbb{R}_+. \tag{2.28}$$

Choosing $s = 0$, for the workload distribution it follows

$$P(V_0 \le x) = 1 - \lambda p(0) \int_x^\infty \varphi(0,\xi)\mathrm{d}\xi, \quad x \in \mathbb{R}_+, \tag{2.29}$$

and choosing $x = 0$ in (2.29), for the probability that the system is empty we obtain

$$p(0) = \left(1 + \lambda \int_{\mathbb{R}_+} \varphi(0,\xi)\mathrm{d}\xi\right)^{-1}. \tag{2.30}$$

From (2.29) and (2.30) for the workload distribution we find

$$P(V_0 \le x) = p(0)\left(1 + \lambda \int_0^x \varphi(0,\xi)\mathrm{d}\xi\right), \quad x \in \mathbb{R}_+. \tag{2.31}$$

Note that (2.30), (2.25), and (2.27) imply

$$p(0) = \frac{1}{1+\lambda\Phi_1(0,0)}. \tag{2.32}$$

Further, from (2.31), (2.25), (2.27), and (2.32) for the LST $V^*(\theta) := E[e^{-\theta V_0}]$ of the workload V_0 we obtain

$$V^*(\theta) = \frac{1+\lambda\Phi_1(0,\theta)}{1+\lambda\Phi_1(0,0)}, \quad \theta \in \mathbb{C}_+. \tag{2.33}$$

The Poisson arrival process implies that (V_t, U_t), $t \in \mathbb{R}$, is a regenerative process with respect to the embedded time instants where a busy period starts. The duration of each cycle consists of a busy period and a subsequent idle period, which is exponentially distributed with parameter λ. Note that $V_t > 0$ if and only if there is a busy period running at t. Further, the intensity of the time instants where a busy period starts is given by $\lambda p(0)$. Thus the cycle formula for regenerative processes, cf. e.g. [Asm], provides

$$E[\mathbb{I}\{V_0 > 0\}e^{-sU_0}] = \lambda p(0) E\left[\int_0^D \mathbb{I}\{V_t^0 > 0\}e^{-sU_t^0}\mathrm{d}t\right], \tag{2.34}$$

where (V_t^0, U_t^0) has the distribution $P((V_t, U_t) \in (\cdot) \,|\, V_{0-} = 0, V_0 > 0)$ and D is the duration of the cycle which starts at time $t = 0$. For the LST

$Z^*(s) := E[e^{-sZ}]$ of the busy period Z starting at $t = 0$ from (2.34) it follows

$$E[\mathbb{I}\{V_0 > 0\}e^{-sU_0}] = \lambda p(0) E\left[\int_0^Z e^{-st}\mathrm{d}t\right] = \lambda p(0)\,\frac{1 - Z^*(s)}{s}\,. \tag{2.35}$$

Because of (2.28) for $x = 0$, from (2.35) we find

$$Z^*(s) = 1 - s\int_{\mathbb{R}_+} \varphi(s,\xi)\mathrm{d}\xi, \quad s \in \mathbb{R}_+, \tag{2.36}$$

which implies

$$Z^*(s) = 1 - \frac{s\Phi_1(s,0)}{1 + \lambda\Phi_2(s,0)}\,, \quad s \in \mathbb{R}_+, \tag{2.37}$$

because of (2.25). In view of (2.26), from (2.37) in particular we obtain

$$EZ = \Phi_1(0,0), \tag{2.38}$$

$$E[Z^2] = 2\partial_+(\lambda\Phi_1(0,0)\Phi_2(s,0) - \Phi_1(s,0))\Big|_{s=0}, \tag{2.39}$$

where ∂_+ denotes the right derivative with respect to s. Note that EZ is given by $p(0)$ and vice versa, cf. (2.32) and (2.38), due to the cycle formula, i.e.

$$EZ = \frac{1 - p(0)}{\lambda p(0)}\,. \tag{2.40}$$

2.2 Workload distribution in $M/GI/1 + GI$

The workload distribution for Kovalenko's impatience mechanism is given by (2.31) and (2.30). From (2.8) and (2.9) it follows $\varphi_2(0,x) = 0$, $x \in \mathbb{R}_+$, which implies $\varphi(0,x) = \varphi_1(0,x)$, $x \in \mathbb{R}_+$, in view of (2.7), cf. (2.27). Because of (2.9) and (2.13), thus $\varphi(0,x)$, $x \in \mathbb{R}_+$, is determined by the Volterra integral equation

$$\varphi(0,x) = \lambda\int_0^x K(x,\xi)\varphi(0,\xi)\mathrm{d}\xi + K(x,0), \quad x \in \mathbb{R}_+, \tag{2.41}$$

cf. [Kov, p. 206], [Dal, p. 197].

Since by (2.31) the density $v(x)$ of the work load distribution is related to $\varphi(0,x)$ via $v(x) = \lambda p(0)\varphi(0,x)$, $x \in (0,\infty)$, eq. (2.41) is equivalent to the corresponding equation for $v(x)$, cf. [Kov, p. 206]. For the $M/GI/1 + GI^w$ system its solution $\varphi(0,x) = \varphi_1(0,x)$ via the Neumann series (2.8), (2.9) has been given in [BBH, eq. (4.13)]. The density $v(x)$ and/or the LST $V^*(\theta)$ have been derived explicitly for several special cases: for $M/H_2/1+D^w$ in [XJA], for $M/GI/1+D^w$, $M/E_k/1+D^w$, $M/GI/1+M^w$ in [Dal], for $M/GI/1+E_k^w$ in [BBH], and for $M/GI/1+M^w$, $M/M/1+D^w$ in [St1], [St2].

For fixed $\alpha \in \mathbb{R}_+$, let $v(x,\alpha)$, $x \in (0,\infty)$, be the density of the workload distribution in $M/GI/1+GI$ with the mixed scheme (1.2), i.e. for $\bar{G}_w(x) := (\bar{C}(x+w)/\bar{C}(w))^\alpha$. Remember that $v(x,0)$ and $v(x,1)$ are the densities of the workload distributions in $M/GI/1{+}GI^w$ and $M/GI/1{+}GI^s$, respectively.

Theorem 2.2 *Let $\bar{G}_w(x)$ be given by* (1.2). *Then for $\alpha \in (0,\infty)$ it holds*

$$v(x,\alpha) = \frac{\bar{C}(x)^\alpha v(x,0)}{1-\int_{0+}^{\infty}(1-\bar{C}(\xi)^\alpha)v(\xi,0)\mathrm{d}\xi}\,, \quad x \in (0,\infty). \tag{2.42}$$

Proof. Let $\varphi(0,x,\alpha)$, $x \in \mathbb{R}_+$, be the solution of (2.41) for the $M/GI/1{+}GI$ system with the mixed scheme (1.2) for some $\alpha \in \mathbb{R}_+$, i.e. for

$$K(x,\xi) := \bar{C}(x)^\alpha \bar{C}(\xi)^{1-\alpha}\bar{B}(x-\xi), \quad 0 \le \xi \le x,$$

cf. (2.3) and (1.2). By putting in it follows that $\bar{C}(x)^\alpha\varphi(0,x,0)$, $x \in \mathbb{R}_+$, is a solution of (2.41) for the given value of α if $\varphi(0,x,0)$, $x \in \mathbb{R}_+$, is a solution of (2.41) for $\alpha = 0$. Thus it holds

$$\varphi(0,x,\alpha) = \bar{C}(x)^\alpha\varphi(0,x,0), \quad x \in \mathbb{R}_+. \tag{2.43}$$

From (2.29) we find

$$v(x,\alpha) = \lambda p(0,\alpha)\varphi(0,x,\alpha), \quad x \in (0,\infty),$$

where $p(0,\alpha)$ denotes the probability that the $M/GI/1+GI$ system with the mixed scheme (1.2) is empty. Because of (2.43), thus we obtain

$$v(x,\alpha) = \frac{p(0,\alpha)}{p(0,0)}\,\bar{C}(x)^\alpha v(x,0), \quad x \in (0,\infty). \tag{2.44}$$

Integrating over $x \in (0,\infty)$ and taking into account

$$p(0,0) + \int_{0+}^{\infty} v(\xi,0)\mathrm{d}\xi = 1 = p(0,\alpha) + \int_{0+}^{\infty} v(\xi,\alpha)\mathrm{d}\xi,$$

(2.42) follows from (2.44) after some algebra. □

Remark 2.2 *The results of the paper are easily generalized to the case where the Poisson arrival process is state-dependent having another intensity* λ_0 *if the system is empty. In particular, the parameter of the exponentially distributed empty period changes from* λ *to* λ_0 *while the distribution of the busy period does not depend on* λ_0.

3 Special cases

3.1 The $M/GI/1+M$ system

Consider the $M/GI/1+M$ system, i.e. $\bar{C}(u) := e^{-\gamma u}$, $u \in \mathbb{R}_+$, for some $\gamma \in (0,\infty)$, where $\bar{G}_w(x)$ is given by the mixed scheme (1.1) or (1.2) for some fixed $\alpha \in \mathbb{R}_+$, i.e. $\bar{G}_w(x) = e^{-\alpha\gamma x}$, $x \in \mathbb{R}_+$. In this model the maximal waiting time until beginning of service is exponentially distributed with parameter γ, and the maximal service time is exponentially distributed with parameter $\alpha\gamma$. Therefore (2.3) reads

$$K(x,\xi) = e^{-\gamma\xi}e^{-\alpha\gamma(x-\xi)}\bar{B}(x-\xi), \quad 0 \le \xi \le x.$$

Thus the kernel $K(x,\xi)$, $0 \le \xi \le x$, equals the kernel for the $M/GI/1+M^w$ system with the modified service time distribution

$$P(S>y) := e^{-\alpha\gamma y}\bar{B}(y), \quad y \in \mathbb{R}_+, \tag{3.1}$$

and the unchanged impatience distribution $\bar{C}(u) := e^{-\gamma u}$, $u \in \mathbb{R}_+$. Consequently, the distribution of the workload V_0 and the distribution of the busy period Z in the $M/GI/1+M$ system with the with the mixed scheme (1.1) or (1.2) are given by the corresponding distributions in the $M/GI/1+M^w$ system with the modified service time distribution (3.1). Moreover, the $M/GI/1+M$ system with the mixed scheme (1.1) or (1.2) is a special case of the $M/GI/1+PH$ system with the mixed scheme (1.1) analyzed in the following section, cf. (3.16) below.

The explicit formula for the LST $V^*(\theta)$ of the work load distribution for the $M/GI/1+M^w$ system given in [Dal, p. 203], cf. also [St1, p. 175], follows from (2.33) and (3.16) below, too. The LST $Z^*(s)$ of the busy period in the $M/GI/1+M^w$ system is a special result in [Sub]. More precisely, choosing in [Sub] the balking parameter $\beta = 1$ in (19) and $i = 0$, $z = 1$ in (21), one finds in a straight forward manner an explicit expression for $Z^*(s)$, which coincides with the expression resulting from our formulas (2.37) and (3.16) below.

3.2 The $M/GI/1+PH$ system

Let $C(u)$, $u \in \mathbb{R}_+$, be given by

$$\bar{C}(u) = \sum_{\kappa=1}^{k} p_\kappa e^{-\gamma_\kappa u}, \quad u \in \mathbb{R}_+, \tag{3.2}$$

for some $k \in \mathbb{N}$, where $\gamma_1, \dots, \gamma_k \in \{z \in \mathbb{C} : \Re z > 0\}$ and $p_1, \dots, p_k \in \mathbb{C}$ such that $p_1 + \ldots + p_k = 1$ and the r.h.s. of (3.2) is real-valued and monotonically decreasing. Further, for fixed $\alpha \in \mathbb{R}_+$ we use Kovalenko's mixed scheme (1.1), i.e. $\bar{G}_w(x) = \bar{C}(\alpha x + w)/\bar{C}(w)$, thus covering the $M/GI/1+PH^w$ system and the $M/GI/1+PH^s$ system as special cases for $\alpha = 0$ and $\alpha = 1$, respectively, cf. Section 1.

Remark 3.1 *The distributions given by* (3.2) *cover those phase-type distributions with no point mass at zero where the matrix of the transitions among the transient states has only simple eigenvalues, cf. [O'C]. Note that the distributions given by* (3.2) *are dense in the field of all distributions. The case of a point mass at zero can be treated by thinning the Poisson arrival process state-dependent, cf. Remark 2.2, the case of multiple eigenvalues can be treated as limiting case, cf. Section 3.3 below.*

Let $s \in \mathbb{R}_+$ be fixed in the following. In the case considered here (2.3) reads

$$K(x,\xi) = \sum_{\kappa=1}^{k} p_\kappa e^{-\gamma_\kappa x} \bar{B}(x-\xi) e^{(1-\alpha)\gamma_\kappa (x-\xi)}, \quad 0 \le \xi \le x, \tag{3.3}$$

and (2.10) reads

$$K_1(s,x,\xi) = \sum_{\kappa=1}^{k} p_\kappa e^{-\gamma_\kappa x} \Bigg(\bar{B}(x-\xi) e^{((1-\alpha)\gamma_\kappa - s)(x-\xi)} + s \int_0^{x-\xi} \bar{B}(\eta) e^{((1-\alpha)\gamma_\kappa - s)\eta} \mathrm{d}\eta \Bigg), \quad 0 \le \xi \le x. \tag{3.4}$$

Further, from (2.9) we find

$$a_1(s,x) = \bar{B}(x) \sum_{\kappa=1}^{k} p_\kappa e^{-(\alpha\gamma_\kappa + s)x},$$

$$a_2(s,x) = s \sum_{\kappa=1}^{k} p_\kappa e^{-\gamma_\kappa x} \int_0^x \bar{B}(\eta) e^{((1-\alpha)\gamma_\kappa - s)\eta} \mathrm{d}\eta, \quad x \in \mathbb{R}_+, \tag{3.5}$$

and thus for the Laplace transform of $a_j(s,\cdot)$, cf. (2.22), it follows

$$A_{0,1}(s,\theta) = m_S \sum_{\kappa=1}^{k} p_\kappa B_R^*(s+\theta+\alpha\gamma_\kappa),$$

$$A_{0,2}(s,\theta) = m_S \sum_{\kappa=1}^{k} p_\kappa \frac{s}{\theta+\gamma_\kappa} B_R^*(s+\theta+\alpha\gamma_\kappa), \quad \theta \in \mathbb{C}_+, \tag{3.6}$$

where $m_S := ES$ is the mean required service time and $B_R^*(\cdot)$ is the LST of the stationary residual service time distribution $B_R(x) = \int_0^x \bar{B}(\xi)\mathrm{d}\xi/m_S$, $x \in \mathbb{R}_+$. From (2.23) for $n = 1$ and $j = 1, 2$ we obtain

$$\begin{aligned} A_{1,j}(s,\theta) &= \int_{\mathbb{R}_+} e^{-\theta x} \int_0^x K_1(s,x,\eta) a_j(s,\eta)\mathrm{d}\eta\mathrm{d}x \\ &= \int_{\mathbb{R}_+} e^{-\theta\eta} \int_{\mathbb{R}_+} e^{-\theta y} K_1(s,y+\eta,\eta)\mathrm{d}y\, a_j(s,\eta)\mathrm{d}\eta, \quad \theta \in \mathbb{C}_+, \end{aligned} \tag{3.7}$$

where we applied Fubini's theorem and the substitution $y = x - \eta$ for the last equality. Further, from (3.4) after some algebra we find

$$\begin{aligned} &\int_{\mathbb{R}_+} e^{-\theta y} K_1(s,y+\eta,\eta)\mathrm{d}y \\ &\quad = \sum_{\kappa=1}^{k} p_\kappa e^{-\gamma_\kappa \eta} \int_{\mathbb{R}_+} e^{-(\theta+\gamma_\kappa)y} \Big(\bar{B}(y) e^{((1-\alpha)\gamma_\kappa - s)y} \\ &\qquad\qquad + s \int_0^y \bar{B}(\xi) e^{((1-\alpha)\gamma_\kappa - s)\xi} \mathrm{d}\xi \Big) \mathrm{d}y \\ &\quad = m_S \sum_{\kappa=1}^{k} p_\kappa e^{-\gamma_\kappa \eta} \frac{s+\theta+\gamma_\kappa}{\theta+\gamma_\kappa} B_R^*(s+\theta+\alpha\gamma_\kappa). \end{aligned} \tag{3.8}$$

In view of (2.22), thus from (3.7) for $j = 1, 2$ it follows

$$\begin{aligned} A_{1,j}(s,\theta) &= m_S \sum_{\kappa=1}^{k} p_\kappa \frac{s+\theta+\gamma_\kappa}{\theta+\gamma_\kappa} B_R^*(s+\theta+\alpha\gamma_\kappa) \int_{\mathbb{R}_+} e^{-(\theta+\gamma_\kappa)\eta} a_j(s,\eta)\mathrm{d}\eta \\ &= m_S \sum_{\kappa=1}^{k} p_\kappa \frac{s+\theta+\gamma_\kappa}{\theta+\gamma_\kappa} B_R^*(s+\theta+\alpha\gamma_\kappa) A_{0,j}(s,\theta+\gamma_\kappa), \quad \theta \in \mathbb{C}_+. \end{aligned} \tag{3.9}$$

From (2.23) and (2.11) for $n = 2, 3, \dots$ and $j = 1, 2$ we obtain

$$A_{n,j}(s,\theta) = \int_{\mathbb{R}_+} e^{-\theta x} \int_0^x \int_\xi^x K_1(s,x,\eta) K_{n-1}(s,\eta,\xi) \mathrm{d}\eta \, a_j(s,\xi) \mathrm{d}\xi \mathrm{d}x$$
$$= \int_{\mathbb{R}_+} e^{-\theta\eta} \int_{\mathbb{R}_+} e^{-\theta y} K_1(s,y{+}\eta,\eta) \mathrm{d}y \int_0^\eta K_{n-1}(s,\eta,\xi) a_j(s,\xi) \mathrm{d}\xi \mathrm{d}\eta,$$
$$\theta \in \mathbb{C}_+, \quad (3.10)$$

where we applied Fubini's theorem and the substitution $y = x - \eta$ for the last equality again. In view of (3.8) and (2.23), from (3.10) for $n = 2, 3, \dots$ and $j = 1, 2$ it follows

$$A_{n,j}(s,\theta) = m_S \sum_{\kappa=1}^{k} p_\kappa \, \frac{s{+}\theta{+}\gamma_\kappa}{\theta{+}\gamma_\kappa} \, B_R^*(s{+}\theta{+}\alpha\gamma_\kappa) \int_{\mathbb{R}_+} e^{-(\theta+\gamma_\kappa)\eta}$$
$$\int_0^\eta K_{n-1}(s,\eta,\xi) a_j(s,\xi) \mathrm{d}\xi \mathrm{d}\eta$$
$$= m_S \sum_{\kappa=1}^{k} p_\kappa \, \frac{s{+}\theta{+}\gamma_\kappa}{\theta{+}\gamma_\kappa} \, B_R^*(s{+}\theta{+}\alpha\gamma_\kappa) A_{n-1,j}(s,\theta{+}\gamma_\kappa), \quad \theta \in \mathbb{C}_+. \quad (3.11)$$

Summarizing, from (3.9) and (3.11) for fixed $j = 1, 2$ we find the recursion

$$A_{n+1,j}(s,\theta) = m_S \sum_{\kappa=1}^{k} p_\kappa \, \frac{s{+}\theta{+}\gamma_\kappa}{\theta{+}\gamma_\kappa} \, B_R^*(s{+}\theta{+}\alpha\gamma_\kappa) A_{n,j}(s,\theta{+}\gamma_\kappa),$$
$$\theta \in \mathbb{C}_+, \quad (3.12)$$

for $n \in \mathbb{Z}_+$, where $A_{0,j}(s,\theta)$ is given by (3.6).

Let $L := \{1, \dots, k\}$, and for $\ell \in L^n$ let $\ell = (\ell_1, \dots, \ell_n)$. Because of (3.6), for $n = 0$ and $j = 1, 2$ it holds the representation

$$A_{n,j}(s,\theta) = s^{j-1} m_S^{n+1} \sum_{\ell \in L^{n+1}} \frac{\prod_{m=1}^{n} (s{+}\theta{+}\gamma_{\ell_1}{+}\dots{+}\gamma_{\ell_m})}{\prod_{m=1}^{n+j-1} (\theta{+}\gamma_{\ell_1}{+}\dots{+}\gamma_{\ell_m})}$$
$$\times \prod_{m=1}^{n+1} p_{\ell_m} B_R^*(s{+}\theta{+}\gamma_{\ell_1}{+}\dots{+}\gamma_{\ell_{m-1}}{+}\alpha\gamma_{\ell_m}), \quad \theta \in \mathbb{C}_+. \quad (3.13)$$

Assume now that (3.13) holds for some $n \in \mathbb{Z}_+$. Then from (3.12) and (3.13) it follows

$$A_{n+1,j}(s,\theta) = m_S \sum_{\ell_0 \in L} p_{\ell_0} \, \frac{s{+}\theta{+}\gamma_{\ell_0}}{\theta{+}\gamma_{\ell_0}} \, B_R^*(s{+}\theta{+}\alpha\gamma_{\ell_0})$$

$$\times\ s^{j-1} m_S^{n+1} \sum_{\ell \in L^{n+1}} \frac{\prod_{m=1}^{n}(s+\theta+\gamma_{\ell_0}+\gamma_{\ell_1}+\ldots+\gamma_{\ell_m})}{\prod_{m=1}^{n+j-1}(\theta+\gamma_{\ell_0}+\gamma_{\ell_1}+\ldots+\gamma_{\ell_m})}$$

$$\times \prod_{m=1}^{n+1} p_{\ell_m} B_R^*(s+\theta+\gamma_{\ell_0}+\gamma_{\ell_1}+\ldots+\gamma_{\ell_{m-1}}+\alpha\gamma_{\ell_m})$$

$$= s^{j-1} m_S^{n+2} \sum_{\ell \in L^{n+2}} \frac{\prod_{m=1}^{n+1}(s+\theta+\gamma_{\ell_1}+\ldots+\gamma_{\ell_m})}{\prod_{m=1}^{n+j}(\theta+\gamma_{\ell_1}+\ldots+\gamma_{\ell_m})}$$

$$\times \prod_{m=1}^{n+2} p_{\ell_m} B_R^*(s+\theta+\gamma_{\ell_1}+\ldots+\gamma_{\ell_{m-1}}+\alpha\gamma_{\ell_m}), \quad \theta \in \mathbb{C}_+,$$

where $(\ell_0, \ldots, \ell_{n+1})$ is replaced by $(\ell_1, \ldots, \ell_{n+2})$ for the last equality. Therefore, induction on n provides that the explicit representation (3.13) holds for all $n \in \mathbb{Z}_+$. In view of (2.21), thus for $j = 1, 2$ we find the representation

$$\Phi_j(s,\theta) = \frac{s^{j-1}}{\lambda} \sum_{n=1}^{\infty} \varrho^n \sum_{\ell \in L^n} \frac{\prod_{m=1}^{n-1}(s+\theta+\gamma_{\ell_1}+\ldots+\gamma_{\ell_m})}{\prod_{m=1}^{n+j-2}(\theta+\gamma_{\ell_1}+\ldots+\gamma_{\ell_m})}$$

$$\times \prod_{m=1}^{n} p_{\ell_m} B_R^*(s+\theta+\gamma_{\ell_1}+\ldots+\gamma_{\ell_{m-1}}+\alpha\gamma_{\ell_m}), \quad s \in \mathbb{R}_+,\ \theta \in \mathbb{C}_+, \tag{3.14}$$

where $\varrho := \lambda m_S$ is the offered load. Note that in case of $s = 0$ and $j = 1$ the representation (3.14) simplifies to

$$\Phi_1(0,\theta) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \varrho^n \sum_{\ell \in L^n} \prod_{m=1}^{n} p_{\ell_m} B_R^*(\theta+\gamma_{\ell_1}+\ldots+\gamma_{\ell_{m-1}}+\alpha\gamma_{\ell_m}), \quad \theta \in \mathbb{C}_+. \tag{3.15}$$

Further, in case of $k = 1$, i.e. in case of exponentially distributed impatience $\bar{C}(u) = e^{-\gamma u}$, $u \in \mathbb{R}_+$, for $j = 1, 2$ the representation (3.14) simplifies to

$$\Phi_j(s,\theta) = \frac{s^{j-1}}{\lambda} \sum_{n=1}^{\infty} \varrho^n \frac{\prod_{m=1}^{n-1}(s+\theta+m\gamma)}{\prod_{m=1}^{n+j-2}(\theta+m\gamma)} \prod_{m=1}^{n} B_R^*(s+\theta+(m-1+\alpha)\gamma), \quad s \in \mathbb{R}_+,\ \theta \in \mathbb{C}_+. \tag{3.16}$$

Now, the LST $V^*(\theta)$ of the workload V_0, the probability $p(0)$ that the system is empty, and the first moment EZ of the busy period Z are given explicitly by (3.15) and (2.33), (2.32), (2.38), respectively. The LST $Z^*(s)$ and the second moment $E[Z^2]$ of the busy period Z are given explicitly by (3.14) for $\theta = 0$ and (2.37), (2.39), respectively.

3.3 The $M/GI/1+PH$ system in case of multiple eigenvalues

Let $C(u)$, $u \in \mathbb{R}_+$, be given by

$$\bar{C}(u) = \sum_{\nu=1}^{l} b_\nu(u) e^{-\beta_\nu u}, \quad u \in \mathbb{R}_+, \tag{3.17}$$

for some $l \in \mathbb{N}$, where $\beta_1, \ldots, \beta_l \in \{z \in \mathbb{C} : \Re z > 0\}$ and $b_\nu(\cdot)$ is a polynomial with complex coefficients of some degree d_ν such that $b_1(0) + \ldots + b_l(0) = 1$ and the r.h.s. of (3.17) is real-valued and monotonically decreasing. We assume here that $d := \max(d_1, \ldots, d_l) > 0$, cf. Section 3.2 for the case of $d = 0$. Further, let $\bar{G}_w(x)$ be given by Kovalenko's mixed scheme (1.1) for some fixed $\alpha \in \mathbb{R}_+$ again, i.e. $\bar{G}_w(x) = \bar{C}(\alpha x + w)/\bar{C}(w)$.

For fixed $g \in (0,1)$ and sufficiently small $h \in (0,1)$ we choose the phase-type distribution $C(u,g,h)$, $u \in \mathbb{R}_+$, given by

$$\bar{C}(u,g,h) := g e^{-\beta_0 u} + (1-g) \sum_{\nu=1}^{l} b_\nu \Big(\frac{1-e^{-hu}}{h} \Big) e^{-\beta_\nu u}, \quad u \in \mathbb{R}_+, \tag{3.18}$$

where $\beta_0 := \min(\Re\beta_1, \ldots, \Re\beta_l)/2$. Note that the function

$$f(u) := \sum_{\nu=1}^{l} b_\nu(u) e^{-\beta_\nu u} - \overline{\sum_{\nu=1}^{l} b_\nu(\overline{u}) e^{-\beta_\nu \overline{u}}}, \quad u \in \mathbb{C},$$

is identically zero as the r.h.s. of (3.17) is real-valued for $u \in \mathbb{R}_+$ and due to the principle of permanence. Applying the fact that any system of exponential functions $\{e^{\gamma_1 u}, \ldots, e^{\gamma_n u}\}$, where $\gamma_\nu \in \mathbb{C}$ such that $\gamma_\nu \neq \gamma_\mu$ for $\nu \neq \mu$, is linearly independent over the polynomials with complex coefficients due to the growth of the complex exponential function, to f, it follows that

$$\sum_{\nu=1}^{l} \mathbb{I}\{\beta_\nu = \gamma\} b_\nu(u) - \sum_{\nu=1}^{l} \mathbb{I}\{\beta_\nu = \overline{\gamma}\} \overline{b_\nu(\overline{u})} \equiv 0, \quad u \in \mathbb{C},\ \gamma \in \mathbb{C},$$

which implies that the r.h.s. of (3.18) is real-valued. Obviously, there exists $u_0 \in (0,\infty)$ such that $\frac{\partial}{\partial u} \bar{C}(u,g,h) < 0$ for $u \in (u_0, \infty)$, $h \in (0,1)$. Further, it holds

$$\lim_{h \downarrow 0} \frac{\partial}{\partial u} \bar{C}(u,g,h) = -g\beta_0 e^{-\beta_0 u} + (1-g) \frac{\partial}{\partial u} \bar{C}(u) \leq -g\beta_0 e^{-\beta_0 u_0}$$

uniformly for $u \in [0, u_0]$. Therefore, there exists $h_g \in (0,1)$ such that $\frac{\partial}{\partial u} \bar{C}(u,g,h) < 0$ for $u \in [0,u_0]$, $h \in (0,h_g)$. Thus the r.h.s. of (3.18) is monotonically decreasing with respect to u for $h \in (0, h_g)$. Moreover, there exists $M > 0$ such that for $g \in (0,1)$ and $h \in (0, h_g)$ it holds

$$\bar{C}(u,g,h) \leq M e^{-\beta_0 u}, \quad u \in \mathbb{R}_+. \tag{3.19}$$

The binomial theorem provides that

$$b_\nu\Big(\frac{1-e^{-hu}}{h}\Big) = \sum_{\mu=0}^{d_\nu} p_{\nu,\mu}(h) e^{-\mu h u}, \quad u \in \mathbb{R}_+, \tag{3.20}$$

where $h^d p_{\nu,\mu}(h)$ is a polynomial of degree less or equal to d. Let

$$(p_1(h), \ldots, p_k(h)) := (p_{1,0}(h), \ldots, p_{1,d_1}(h), p_{2,0}(h), \ldots, p_{l,d_l}(h)), \tag{3.21}$$

$$(\gamma_1(h), \ldots, \gamma_k(h)) := (\beta_1, \beta_1{+}h, \ldots, \beta_1{+}d_1 h, \beta_2, \ldots, \beta_l{+}d_l h), \tag{3.22}$$

where $k := (d_1+1) + \ldots + (d_l+1)$. Further, let $\gamma_0(h) := \beta_0$, $p_0(g,h) := g$, and $p_\kappa(g,h) := (1-g)p_\kappa(h)$, $\kappa = 1, \ldots, k$. Then (3.18) reads

$$\bar{C}(u,g,h) = \sum_{\kappa=0}^{k} p_\kappa(g,h) e^{-\gamma_\kappa(h) u}, \quad u \in \mathbb{R}_+. \tag{3.23}$$

For notational convenience we use the additional arguments g, h for the quantities connected with the impatience distribution (3.18) in the following. Let $s \in \mathbb{R}_+$, $\theta \in \mathbb{C}_+$ be fixed and let $g \in (0,1)$, $h \in (0, h_g)$. From (2.21), (3.13), and (3.23) for $j = 1, 2$ we obtain

$$\Phi_j(s,\theta,g,h) = \sum_{n=0}^{\infty} A_{n,j}(s,\theta,g,h)\lambda^n, \tag{3.24}$$

where for $n \in \mathbb{Z}_+$

$$\begin{aligned} A_{n,j}(s,\theta,g,h) = s^{j-1} m_S^{n+1} &\sum_{\ell \in L_0^{n+1}} \frac{\prod_{m=1}^{n}(s{+}\theta{+}\gamma_{\ell_1}(h){+}\ldots{+}\gamma_{\ell_m}(h))}{\prod_{m=1}^{n+j-1}(\theta{+}\gamma_{\ell_1}(h){+}\ldots{+}\gamma_{\ell_m}(h))} \\ &\times \prod_{m=1}^{n+1} p_{\ell_m}(g,h) B_R^*(s{+}\theta{+}\gamma_{\ell_1}(h){+}\ldots{+}\gamma_{\ell_{m-1}}(h){+}\alpha\gamma_{\ell_m}(h)), \end{aligned} \tag{3.25}$$

$L_0 := \{0, 1, \ldots, k\}$, and $\ell = (\ell_1, \ldots, \ell_n)$ for $\ell \in L_0^n$.

Let $n \in \mathbb{Z}_+$ and $j \in \{1,2\}$ be fixed. Note that $h^{d(n+1)} A_{n,j}(s,\theta,g,h)$ is a holomorphic function with respect to h for $\Re h > -2\beta_0/d$ because of (3.25). Further, from (2.24) and (3.19) it follows

$$|A_{n,j}(s,\theta,g,h)| \le \frac{(2M/\beta_0+2ES)^{n+1}}{(n+1)!}, \quad h \in (0,h_g). \tag{3.26}$$

Therefore, $A_{n,j}(s,\theta,g,h)$ has a removable singularity at $h=0$, and it holds

$$\lim_{h\downarrow 0} A_{n,j}(s,\theta,g,h) = \frac{1}{(d(n+1))!} \frac{\partial^{d(n+1)}}{\partial h^{d(n+1)}} h^{d(n+1)} A_{n,j}(s,\theta,g,h)\Big|_{h=0}. \tag{3.27}$$

Since the r.h.s. of (3.27) is a polynomial with respect to g, further we find

$$\lim_{g\downarrow 0}\lim_{h\downarrow 0} A_{n,j}(s,\theta,g,h) = \frac{1}{(d(n+1))!} \frac{\partial^{d(n+1)}}{\partial h^{d(n+1)}} h^{d(n+1)} A_{n,j}(s,\theta,0,h)\Big|_{h=0}. \tag{3.28}$$

As $\lim_{h\downarrow 0} \bar{C}(u,g,h) = \bar{C}(u,g,0) := ge^{-\beta_0 u} + (1-g)\bar{C}(u)$ locally uniformly for $u \in \mathbb{R}_+$, because of (2.3) and (2.9), for fixed $s \in \mathbb{R}_+$ and $j=1,2$ it holds $\lim_{h\downarrow 0} a_j(s,x,g,h) = a_j(s,x,g,0)$ locally uniformly for $x \in \mathbb{R}_+$, and because of (2.3), (2.10), and (2.11), for fixed $s \in \mathbb{R}_+$ and fixed $n = 1,2,\ldots$ it holds $\lim_{h\downarrow 0} K_n(s,x,\xi,g,h) = K_n(s,x,\xi,g,0)$ locally uniformly for $0 \le \xi \le x$. From (2.22), (2.23), (2.16), (2.19), and (3.19) thus we find that for $s \in \mathbb{R}_+$, $\theta \in \mathbb{C}_+$, and $g \in (0,1)$ it holds $\lim_{h\downarrow 0} A_{n,j}(s,\theta,g,h) = A_{n,j}(s,\theta,g,0)$. As also $\lim_{g\downarrow 0} \bar{C}(u,g,0) = \bar{C}(u,0,0) := \bar{C}(u)$ locally uniformly for $u \in \mathbb{R}_+$, analogously we find $\lim_{g\downarrow 0} A_{n,j}(s,\theta,g,0) = A_{n,j}(s,\theta,0,0) = A_{n,j}(s,\theta)$.

In view of (2.21), therefore for the impatience distribution (3.17) we obtain

$$\Phi_j(s,\theta) = \sum_{n=0}^{\infty} A_{n,j}(s,\theta)\lambda^n = \sum_{n=0}^{\infty} \Big(\lim_{g\downarrow 0}\lim_{h\downarrow 0} A_{n,j}(s,\theta,g,h)\Big)\lambda^n. \tag{3.29}$$

Because of $p_0(0,h) = 0$ and $p_\kappa(0,h) = p_\kappa(h)$, $\kappa = 1,\ldots,k$, from (3.29), (3.28), and (3.25) for $j=1,2$ we find the representation

$$\Phi_j(s,\theta) = \frac{s^{j-1}}{\lambda} \sum_{n=1}^{\infty} \frac{\varrho^n}{(dn)!} \times \sum_{\ell \in L^n} \frac{\partial^{dn}}{\partial h^{dn}} \frac{\prod_{m=1}^{n-1}(s+\theta+\gamma_{\ell_1}(h)+\ldots+\gamma_{\ell_m}(h))}{\prod_{m=1}^{n+j-2}(\theta+\gamma_{\ell_1}(h)+\ldots+\gamma_{\ell_m}(h))}$$

$$\times \prod_{m=1}^{n} h^d p_{\ell_m}(h) B_R^*(s+\theta+\gamma_{\ell_1}(h)+\ldots+\gamma_{\ell_{m-1}}(h)+\alpha\gamma_{\ell_m}(h))\Big|_{h=0},$$

$$s \in \mathbb{R}_+,\ \theta \in \mathbb{C}_+, \quad (3.30)$$

where $p_\kappa(h)$ and $\gamma_\kappa(h)$ are given by (3.17) and (3.20)–(3.22), cf. (3.14) for the case of $d = 0$. Note that $h^d p_\kappa(h)$ and $\gamma_\kappa(h)$ are polynomials of degree less or equal to d and to 1, respectively. In case of $s = 0$ and $j = 1$ the representation (3.30) simplifies to

$$\Phi_1(0,\theta) = \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{\varrho^n}{(dn)!} \sum_{\ell \in L^n} \frac{\partial^{dn}}{\partial h^{dn}} \prod_{m=1}^{n} h^d p_{\ell_m}(h)$$

$$\times B_R^*(\theta+\gamma_{\ell_1}(h)+\ldots+\gamma_{\ell_{m-1}}(h)+\alpha\gamma_{\ell_m}(h))\Big|_{h=0}, \quad \theta \in \mathbb{C}_+. \quad (3.31)$$

In case of multiple eigenvalues of the matrix of the transitions among the transient states, thus the LST $V^*(\theta)$ of the workload V_0, the probability $p(0)$ that the system is empty, and the first moment EZ of the busy period Z are given explicitly by (3.31) and (2.33), (2.32), (2.38), respectively, and the LST $Z^*(s)$ and the second moment $E[Z^2]$ of the busy period Z are given explicitly by (3.30) for $\theta = 0$ and (2.37), (2.39), respectively.

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