Hilbert Bases and the Facets of Special Knapsack Polytopes

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Abstract

Let a set N of items, a capacity $F \in \mathbb{N}$ and weights $a_i \in \mathbb{N}$, $i \in N$ be given. The 0/1 knapsack polytope is the convex hull of all 0/1 vectors that satisfy the inequality

$$\sum_{i \in N} a_i x_i \leq F.$$

In this paper we present a linear description of the 0/1 knapsack polytope for the special case where $a_i \in \{\mu, \lambda\}$ for all items $i \in N$ and $1 \leq \mu < \lambda \leq b$ are two natural numbers. The inequalities needed for this description involve elements of the Hilbert basis of a certain cone. The principle of generating inequalities based on elements of a Hilbert basis suggests further extensions.

Keywords: complete description, facets, Hilbert basis, knapsack polytope, knapsack problem, separation

1 Introduction and Notation

Let a set N of items, a capacity $F \in \mathbb{N}$ and weights $a_i \in \mathbb{N}$, $i \in N$ be given. The problem considered in this paper is the special case of the 0/1 knapsack problem, $\sum_{i \in N} a_i x_i \leq F$, $x_i \in \{0, 1\}, i \in N$ where $\mu < \lambda$ are given natural numbers, $N = N_1 \cup N_2$ is the set of items and N_1 contains all items of weight μ, N_2 contains all items of weight λ.

Whereas in case $N_1 = \emptyset$ or $N_2 = \emptyset$, the set of solutions to this problem defines a matroid, this is not in general true if both N_1 and N_2 are nonempty and $\mu \neq \lambda$. Nevertheless, maximizing a linear function $\sum_{i\in N} c_i x_i$ over the set $\{x \in \{0,1\}^{N_1\cup N_2} \mid \sum_{i\in N_1} \mu x_i + \sum_{i\in N_2} \lambda x_i \leq F\}$ can be performed in time that is polynomial in $|N_1| + |N_2|$ as can be easily seen by the following arguments.
Without loss of generality we assume that $N_1 = \{1, \ldots, n_1\}$, $N_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$ and that $c_1 \geq c_2 \geq \ldots \geq c_{n_1} \geq 0$, $c_{n_1+1} \geq c_{n_1+2} \geq \ldots \geq c_{n_1+n_2} \geq 0$. For every $t \in \{n_1+1, \ldots, n_1+n_2\}$ determine $s(t) := \min\{|N_1|, \lfloor \frac{F-(t-n_1)\lambda}{\mu} \rfloor$. An optimal solution to the problem $\max\{\sum_{i\in N} c_i x_i \mid x_i \in \{0,1\}, i \in N, \sum_{i\in N_1} \mu_i x_i + \sum_{i\in N_2} \lambda x_i \leq F\}$ is the vector $\sum_{w=1}^{s(t^\star)} e_w + \sum_{w=n_1+1}^{t^\star} e_w$ where $\sum_{w=1}^{s(t^\star)} c_w + \sum_{w=n_1+1}^{t^\star} c_w$ attains the maximum value of $\sum_{w=1}^{s(t)} c_w + \sum_{w=n_1+1}^{t} c_w$.
The fact that this special case of the 0/1 knapsack problem can be solved in polynomial time indicates that one can derive an explicit description of the associated polytope $P(\mu, \lambda, F) := \text{conv}\{x \in \{0,1\}^N \mid \sum_{i\in N_1} \mu x_i + \sum_{i\in N_2} \lambda x_i \leq F\}$ by means of inequalities. Indeed, this is true, as we show in this paper.

There is already an important literature on special cases of the 0/1 knapsack polytope for which a linear description is known.
Wolsey [11] showed that under certain restrictive assumptions the class of minimal cover inequalities describe the convex hull of the 0/1 solutions to the inequality $\sum_{i\in N} a_i x_i \leq F$. A subset $S \subseteq N$ is called a cover if $a(S) > F$. The cover is called minimal if $a(S \setminus \{i\}) \leq F$ for all $i \in S$. Padberg [7] introduced the notion of $(1,k)$-configurations, a generalization of minimal covers. A set $N' \cup \{z\}$ is called a $(1,k)$-*configuration* if $\sum_{i\in N'} a_i \leq F$, but $K \cup \{z\}$ is a minimal cover for all $K \subset N'$ with $|K| = k$. In [7] was proved that if $N = N' \cup \{z\}$ is a $(1,k)$-configuration, then the convex hull of the associated knapsack polyhedron is given by the inequalities $\sum_{i\in T} x_i + (|T| - l + 1)x_z \leq |T|$ where $T \subseteq N'$, $T \cup \{z\}$ is a $(1,l)$-configuration together with the inequalities $x_i \geq 0$, $x_i \leq 1$. Recently, Laurent and Sassano [5] showed that $|N|$ minimal cover inequalities suffice to describe the knapsack polytope provided that $a = (a_1, \ldots, a_n)$ is a weakly superincreasing sequence, i.e., $\sum_{j\geq q} a_j \leq a_{q-1}$ for all $q = 2, \ldots, n$ where $N = \{1, \ldots, n\}$. Finally, a complete description of the 0/1 knapsack polytope is known for the two cases

$$a_j = 1 \text{ or } a_j \in [\lfloor \tfrac{F}{3} \rfloor + 1, \ldots, \lfloor \tfrac{F}{2} \rfloor] \text{ for all } j \in N;$$

$$a_j = 1 \text{ or } a_j \in [\lfloor \tfrac{F}{2} \rfloor + 1, \ldots, F \text{ for all } j \in N.$$

In both cases, the facets of the corresponding polytopes are no longer necessarily minimal cover- or $(1,k)$-configuration inequalities, but are derived by means of a " weight-reduction" principle (see [10]).

One reason why many researchers are interested in new polyhedral results for knapsack problems is that such results often apply to more general cases. In fact, Crowder, Johnson and Padberg [3] have first shown that general 0/1 integer programs can be solved quite efficiently via branch and cut algorithms. The cutting plane phase of their code is essentially based on valid inequalities for the 0/1 knapsack polytopes associated with the rows of the given problem. Other applications include for instance the node capacitated graph partitioning problem [4]. Here the nodes of a graph must be partitioned into no more than k "clusters" such that the sum of the weights of the nodes within one cluster does not exceed a given capacity and the total sum of edges between nodes of different elements of the partition is minimized. For the corresponding polytope, valid inequalities can be derived that transform a knapsack inequality associated with the nodes and the capacity into a "cut-inequality" associated with the edges of the graph. Here new polyhedral results for the knapsack polytope directly apply to a better understanding of the more complex polytope.

This paper is organized as follows. In the remainder of this section we give as an example the description of the polytope associated with the 0/1 knapsack inequality $2x_1+2x_2+2x_3+2x_4+2x_5+3x_6+3x_7+3x_8+3x_9 \leq 8$ and introduce some notation. Section 2 deals with the two dimensional vectors $(-x, y)$ where $y = 1, \ldots, \mu$, $x = \lfloor \frac{y\lambda+r}{\mu} \rfloor$, $0 \leq r < \mu$ and μ, λ are two natural numbers. In particular, we present a recursive procedure for determing the Hilbert basis for the cone generated by those vectors.
Having established a procedure for computing this Hilbert basis, we show in Section 3 how the elements of the Hilbert basis can be transformed into valid inequalities for $\mathcal{P}(\mu, \lambda, F)$. In Section 4 we outline the proof that the inequalities of Section 3 (together with lower and upper bounds on the variables and one minimal cover inequality) describe $\mathcal{P}(\mu, \lambda, F)$. In Section 5 we finally discuss possible extensions.

Throughout the paper we use the following notation.

Let a set N of items, a capacity $F \in \mathbb{N}$ and weights $a_i \in \mathbb{N}$, $i \in N$ be given. The 0/1 knapsack polytope denoted by P is the convex hull of all 0/1 vectors that satisfy the *knapsack inequality* $\sum_{i \in N} a_i x_i \leq F$. The number F is called *knapsack capacity*.

Let two positive integer numbers $\mu < \lambda$ be given, the greatest common divisor between these numbers is denoted by $gcd(\mu, \lambda)$. For $t \in \{0, \mu\}$ and $r \in \{0, \mu-1\}$, we denote by $n_r(t)$ the integer division $\lfloor \frac{t\lambda+r}{\mu} \rfloor$. If $r = 0$, we also use the symbol $n(t)$ instead of $n_0(t)$. The two dimensional vector $(-n_r(t), t)$ is called *exchange vector*, because $n_r(t)$ is the maximum number of elements having weight μ that can be exchanged against t elements of weight λ plus a given value of r. For

$r \in \{0, \mu - 1\}$, the symbol $V(r)$ is used to denote the (unique) exchange vector $(-n(t), t)$ with $t\lambda - n(t)\mu = r$. Let N be a set of items that can be partitioned into two sets N_1 and N_2 such that $a_i = \mu$ for all $i \in N_1$ and $a_i = \lambda$ for all $i \in N_2$ where $1 \leq \mu < \lambda \leq F$ are integers and $gcd(\mu, \lambda) = 1$. The convex hull of all 0/1 vectors that satisfy the constraint $\sum_{i \in N_1} \mu x_i + \sum_{i \in N_2} \lambda x_i \leq F$ is denoted by $P(\mu, \lambda, F)$.

We say F_c a face of some polytope $\mathcal{P}$ induced by the inequality $c^T x \leq \gamma$, if $F_c = \{x \in \mathcal{P} \mid c^T x = \gamma\}$. Every $x \in F_c$ is also called a *root* of $c^T x \leq \gamma$. The inequalities $x_i \leq 1$, $i \in N$ and $x_i \geq 0$, $i \in N$ are called *trivial*. For real numbers τ_j, $j = 1, \ldots, n$ we define $\sum_{j=v}^{w} \tau_j := 0$ if $v > w$ and, for $I \subseteq \{1, \ldots, n\}$ we use the notation $\tau(I) := \sum_{i \in I} \tau_i$ with $\tau(\emptyset) = 0$. By e_u we denote the unit vector in $\mathbb{R}^d$ having a one in position u and a zero everywhere else.

Finally, let $W = \{w_1, \ldots, w_k\}$ be a finite set of integer vectors. A subset $W' \subseteq W$ is called *integer generating set* if every $w \in W$ is a nonnegative integer combination of the elements in W'. By $C(W)$ we denote the cone generated by the elements of W. A set $\mathcal{H}$ of integer vectors is called *integral Hilbert basis* if every integral vector $z \in C(\mathcal{H})$ is a nonnegative integer combination of the elements in $\mathcal{H}$ (see [8], section 16.4). The following well known result can be found in [8], Theorem 16.4: Each rational polyhedral cone is generated by an integral Hilbert basis. If C is pointed there is a unique minimal integral Hilbert basis generating C.

Example 1.1. Consider the knapsack polytope defined as the convex hull of all 0/1 vectors that satisfy the inequality

$$2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + 3x_6 + 3x_7 + 3x_8 + 3x_9 \leq 8.$$

A complete inequality description (checked by a program developed in [1]) is given by the trivial inequalities $x_i \geq 0,\ x_i \leq 1,\ i = 1, \ldots, 8$ and the following system of inequalities:

$$\begin{array}{rcccccccccl}
(1) & & & & & & +x_6 & +x_7 & +x_8 & +x_9 & \leq 2 \\
(2) & & +x_2 & & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(3) & & +x_2 & +x_3 & & +x_5 & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(4) & +x_1 & & & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(5) & +x_1 & & +x_3 & & +x_5 & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(6) & +x_1 & +x_2 & & & +x_5 & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(7) & +x_1 & +x_2 & +x_3 & & & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(8) & +x_1 & +x_2 & & +x_4 & & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(9) & +x_1 & & +x_3 & +x_4 & & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(10) & & +x_2 & +x_3 & +x_4 & & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(11) & & & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & +x_9 & \leq 3 \\
(12) & +x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & +x_8 & +2x_9 & \leq 4 \\
(13) & +x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & +2x_8 & +x_9 & \leq 4
\end{array}$$

(14)	$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+x_6$	$+2x_7$	$+x_8$	$+x_9$	≤ 4
(15)	$+x_1$	$+x_2$	$+x_3$	$+x_4$	$+x_5$	$+2x_6$	$+x_7$	$+x_8$	$+x_9$	≤ 4
(16)	$+x_1$	$+x_2$	$+x_3$	$+2x_4$	$+2x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(17)	$+x_1$	$+x_2$	$+2x_3$	$+x_4$	$+2x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(18)	$+x_1$	$+2x_2$	$+x_3$	$+x_4$	$+2x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(19)	$+2x_1$	$+x_2$	$+x_3$	$+x_4$	$+2x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(20)	$+x_1$	$+x_2$	$+2x_3$	$+2x_4$	$+x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(21)	$+x_1$	$+2x_2$	$+x_3$	$+2x_4$	$+x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(22)	$+2x_1$	$+x_2$	$+x_3$	$+2x_4$	$+x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(23)	$+x_1$	$+2x_2$	$+2x_3$	$+x_4$	$+x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(24)	$+2x_1$	$+x_2$	$+2x_3$	$+x_4$	$+x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(25)	$+2x_1$	$+2x_2$	$+x_3$	$+x_4$	$+x_5$	$+2x_6$	$+2x_7$	$+2x_8$	$+2x_9$	≤ 6
(26)	$+2x_1$	$+2x_2$	$+2x_3$	$+2x_4$	$+2x_5$	$+3x_6$	$+3x_7$	$+3x_8$	$+3x_9$	≤ 8

2 Exchange Vectors

Let a knapsack capacity F and positive integer numbers μ and λ such that $1 \leq \mu < \lambda \leq b$ and $gcd(\mu, \lambda) = 1$ be given. In this section we analyze the exchange vectors $(-n_r(t), t)$ and the corresponding residua $R((-n_r(t), t)) := \lambda t - n_r(t)\mu$, where $r \in \{0, \ldots, \mu-1\}$ and $t \in \{0, \ldots, \mu\}$. More precisely, let $t_{max}(r) > 0$ be the minimum positive number such that $R((-n_r(t_{max}(r)), t_{max}(r))) \leq 0$. Then, we determine an integral Hilbert basis for the cone generated by the set $\{(-n_r(t), t) \mid t = 0, \ldots, t_{max}(r)\}$. Having determined the Hilbert basis elements, we define in Section 3 classes of valid inequalities for $P(\mu, \lambda, F)$ that involve these elements. Throughout this section we always assume that two natural numbers μ and λ, $1 \leq \mu < \lambda \leq F$, $gcd(\mu, \lambda) = 1$ and an integer $r \in \{0, \ldots, \mu - 1\}$ are given.

We start with two easy and well known observations concerning the numbers $n_0(t)$, $t = 1, \ldots, t_{max}(0) = \mu$.

Observation 2.1. $R((-n(t_1), t_1)) + R((-n(t_2), t_2)) = R((-n(t_1 + t_2), t_1 + t_2))$ if and only if $n(t_1) + n(t_2) = n(t_1 + t_2)$.

Observation 2.2. Let $a, s, t < \mu$ be natural numbers. Then the relations $s = n(t)$ and $a = R((-n(t), t)$ hold if and only if $0 < t\lambda - s\mu = a < \mu$. If $t\lambda - s\mu < 0$, then $s > n(t)$ holds.

Before making precise the relation between the integral Hilbert basis of a certain cone and the vectors $(-n(t), t)$, $t = 1, \ldots, \mu$, let us give an example.

Example 2.3. Setting $\mu := 34$ and $\lambda := 47$, the subsequent table shows the values $n(t) \times \mu \quad || \quad t \times \lambda \quad || \quad R((-n(t), t))$ for $t = 1, \ldots, \mu$.

1	×	34	$\parallel$	1	×	47	$\parallel$	13
2	×	34	$\parallel$	2	×	47	$\parallel$	26
4	×	34	$\parallel$	3	×	47	$\parallel$	5
5	×	34	$\parallel$	4	×	47	$\parallel$	18
6	×	34	$\parallel$	5	×	47	$\parallel$	31
8	×	34	$\parallel$	6	×	47	$\parallel$	10
9	×	34	$\parallel$	7	×	47	$\parallel$	23
11	×	34	$\parallel$	8	×	47	$\parallel$	2
12	×	34	$\parallel$	9	×	47	$\parallel$	15
13	×	34	$\parallel$	10	×	47	$\parallel$	28
15	×	34	$\parallel$	11	×	47	$\parallel$	7
16	×	34	$\parallel$	12	×	47	$\parallel$	20
17	×	34	$\parallel$	13	×	47	$\parallel$	33
19	×	34	$\parallel$	14	×	47	$\parallel$	12
20	×	34	$\parallel$	15	×	47	$\parallel$	25
22	×	34	$\parallel$	16	×	47	$\parallel$	4
23	×	34	$\parallel$	17	×	47	$\parallel$	17
24	×	34	$\parallel$	18	×	47	$\parallel$	30
26	×	34	$\parallel$	19	×	47	$\parallel$	9
27	×	34	$\parallel$	20	×	47	$\parallel$	22
29	×	34	$\parallel$	21	×	47	$\parallel$	1
30	×	34	$\parallel$	22	×	47	$\parallel$	14
31	×	34	$\parallel$	23	×	47	$\parallel$	27
33	×	34	$\parallel$	24	×	47	$\parallel$	6
34	×	34	$\parallel$	25	×	47	$\parallel$	19
35	×	34	$\parallel$	26	×	47	$\parallel$	32
37	×	34	$\parallel$	27	×	47	$\parallel$	11
38	×	34	$\parallel$	28	×	47	$\parallel$	24
40	×	34	$\parallel$	29	×	47	$\parallel$	3
41	×	34	$\parallel$	30	×	47	$\parallel$	16
42	×	34	$\parallel$	31	×	47	$\parallel$	29
44	×	34	$\parallel$	32	×	47	$\parallel$	8
45	×	34	$\parallel$	33	×	47	$\parallel$	21
47	×	34	$\parallel$	34	×	47	$\parallel$	0

In this example the vectors $(-1,1)$, $(-4,3)$, $(-11,8)$, $(-29,21)$ and $(-47,34)$ are an integer generating set for the vectors $(-n(t),t)$, $t=1,\ldots,\mu$ and $R((-1,1)) > R((-4,3)) > R((-11,8)) > R((-29,21)) > R((-47,34))$. Moreover,

$$R((-4,3)) < R((-n(t),t)) \text{ for all } 1 \le t < 3,$$

$$R((-11,8)) < R((-n(t),t)) \text{ for all } 1 \le t < 8,$$

$$R((-29,21)) < R((-n(t),t)) \text{ for all } 1 \le t < 21,$$

$$R((-47,34)) < R((-n(t),t)) \text{ for all } 1 \le t < 34.$$

These properties of the integer generating set for the vectors $(-n(t),t)$, $t = 1,\ldots,\mu$ are indeed not random, but hold in general as we now show.

2.4 Recursive construction of an integer generating set for the exchange vectors.

Set $R_0 := 0$, $h_0 := (0,0)$, $l_0 := (0,0)$ and $\sigma_0 := 0$.

Set $R_1 := \lambda - n(1)\mu$, $h_1 := (-n(1),1)$, $l_1 := h_1$ and $\sigma_1 := 1$.

For $i \geq 2$ perform the following steps until $R_{i-1} = 0$.

- Compute $\sigma_i := \max\{0 \leq \sigma \mid \sigma R_{i-1} + R(l_{i-1}) \leq \mu\}$.
- Set $R_i := (\sigma_i + 1)R_{i-1} + R(l_{i-1}) - \mu$.
- Set $h_i := (\sigma_i + 1)h_{i-1} + l_{i-1} - (1,0)$.
- Set $l_i := \sigma_i h_{i-1} + l_{i-1}$.
- Set $i := i + 1$.

Let τ denote the index with $R_\tau = 0$. In the following we show that $h_1, \ldots, h_\tau$ is an integer generating set for the vectors $(-n(t),t)$, $t = 1, \ldots, \mu$ and that $h_i = (-h_i^1, h_i^2)$ satisfies $R(h_i) \leq R((-n(t),t))$ for all $1 \leq t \leq h_i^2$ and $l_i = (-l_i^1, l_i^2)$ satisfies $R(l_i) \geq R((-n(t),t))$ for all $1 \leq t \leq h_i^2 - 1$.

First note that by induction on i one can easily convince oneself that the vectors h_i and l_i, $i = 1, \ldots, \tau$, are exchange vectors and that $R_i = R(h_i)$ holds. Secondly, $R_i := (\sigma_i + 1)R_{i-1} + R(l_{i-1}) - \mu = R_{i-1} + R(l_{i-1}) - \mu < R_{i-1}$ for $i = 1, \ldots, \tau$. Hence, $R_1 > R_2 > \ldots > R_\tau = 0$. Similarly, $R(l_0) \leq R(l_1) \leq \ldots \leq R(l_\tau)$, because $R(h_{i-1}) \geq 0$ and $\sigma_i \geq 0$. We now prove that $h_1, \ldots, h_\tau$ is an integer generating set for the exchange vectors. To establish this we need the following lemma.

Lemma 2.5. For $i = 2, \ldots, \tau$ the following properties are satisfied.

(a) $R(l_i) - R((-n(t),t)) \geq R_{i-1}$ for all $1 \leq t < l_i^2$.

(b) $R(l_i) \geq R((-n(t),t))$ for all $1 \leq t < h_i^2$.

(c) For every $h_i^2 + 1 \leq t \leq h_{i+1}^2$ there exist nonnegative integers ϵ_u, $u = 1, \ldots, i$ such that $(-n(t),t) = \sum_{u=1}^{i} \epsilon_u h_u$.

Proof. For $i = 2$ the properties are easily verified. Suppose, $i \geq 3$ and (a), (b), (c) is true for all $j \leq i - 1$. We now show that they are true for i as well. Recall that $l_i = \sigma_i h_{i-1} + l_{i-1}$ and $h_i = (\sigma_i + 1)h_{i-1} + l_{i-1} - (1,0)$.

(a) Let $(-n(t), t)$ be an exchange vector with $t < l_i^2$. We write t as $t = \sigma h_{i-1}^2 + t'$ where $t' = t - \sigma h_{i-1}^2 < h_{i-1}^2$. Clearly, $0 \leq \sigma \leq \sigma_i$ and $0 \leq t' < l_{i-1}^2$ if $\sigma = \sigma_i$. Moreover, $\lambda t - \mu(\sigma h_{i-1}^1 + n(t')) = \lambda(\sigma h_{i-1}^2 + t') - \mu(\sigma h_{i-1}^1 + n(t')) = \sigma R(h_{i-1}) + R((-n(t'), t'))$. Since $\sigma_i R_{i-1} + R(l_{i-1}) < \mu$ and since by assumption of the induction $R(l_{i-1}) \geq R((-n(v), v))$ for all $1 \leq v < h_{i-1}^2$, we obtain $0 < \sigma R(h_{i-1}) + R((-n(t'), t')) < \mu$. By Observation $O2$ this yields $n(t) = \sigma h_{i-1}^1 + n(t')$. Furthermore, if $\sigma < \sigma_i$, then $R(l_i) - R((-n(t), t)) \geq R_{i-1}$. Otherwise, $\sigma = \sigma_i$ and $t' < l_{i-1}^2$. By assumption of the induction we know $R(l_{i-1}) - R((-n(t'), t')) > R_{i-2} > R_{i-1}$. Thus, $R(l_i) - R((-n(t), t)) = \sigma_i R(h_{i-1}) + R(l_{i-1}) - \sigma_i R(h_{i-1}) - R((-n(t), t)) > R_{i-2} > R_{i-1}$. This completes the proof of (a).

(b) Let $(-n(t), t)$ be an exchange vector with $t < h_i^2$. We write t as $t = \sigma h_{i-1}^2 + t'$ where $t' = t - \sigma h_{i-1}^2 < h_{i-1}^2$. Then, $0 \leq \sigma \leq \sigma_i + 1$ and $0 \leq t' < l_{i-1}^2$ if $\sigma = \sigma_i + 1$. We first consider the case where $\sigma \leq \sigma_i$. By assumption of the induction we have $R((-n(t'), t')) \leq R(l_{i-1})$, because $t' < h_{i-1}^2$. Hence, $0 < \lambda t - \mu(\sigma h_{i-1}^1 + n(t')) = \lambda(\sigma h_{i-1}^2 + t') - \mu(\sigma h_{i-1}^1 + n(t')) = \sigma R(h_{i-1}) + R((-n(t'), t')) \leq \sigma_i R_{i-1} + R(l_{i-1}) < \mu$. By Observation 02 this yields $n(t) = \sigma h_{i-1}^1 + n(t')$. Moreover, $R(l_i) - R((-n(t), t)) = R(l_i) - \sigma R(h_{i-1}) - R((-n(t'), t')) \geq R(l_i) - \sigma_i R(h_{i-1}) - R(l_{i-1}) = 0$.

In case $\sigma = \sigma_i + 1$ we know $t' < l_{i-1}^2$. By (a), $R(l_{i-1}) - R((-n(t'), t')) > R_{i-1}$,and hence, $0 < \lambda t - \mu(\sigma h_{i-1}^1 + n(t')) = \sigma R(h_{i-1}) + R((-n(t'), t')) \leq (\sigma_i + 1)R_{i-1} + R(l_{i-1}) - R_{i-1} = \sigma_i R_{i-1} + R(l_{i-1}) = R(l_i) < \mu$. By Observation 02, $n(t) = \sigma h_{i-1}^1 + n(t')$ and $R(l_i) - R((-n(t), t)) \geq 0$ follows. Thus, statement (b) is verified.

(c) Let $(-n(t), t)$ be an exchange vector with $t < h_{i+1}^2$. We write t as $t = \sigma h_i^2 + t'$ where $t' = t - \sigma h_i^2 < h_i^2$. By definition, $\sigma \leq \sigma_{i+1} + 1$ and if $\sigma = \sigma_{i+1} + 1$, then $t' < l_i^2$. If $\sigma \leq \sigma_{i+1}$, then by similar arguments as in case (b) we obtain $n(t) = \sigma h_i^1 + n(t')$. By assumption of the induction there exist nonnegative integers ϵ_u, $u = 1, \ldots, i-1$ such that $(-n(t'), t') = \sum_{u=1}^{i-1} \epsilon_u h_u$. Hence, $(-n(t), t) = \sum_{u=1}^{i} \epsilon_u h_u$ where $\epsilon_i = \sigma$.

If $\sigma = \sigma_{i+1} + 1$, then $t' < l_i^2$ and by (a), $R((-n(t'), t') \leq R(l_i) - R_{i-1}$. Hence, $0 < \lambda t - \mu(\sigma h_i^1 + n(t')) = \sigma R_i + R((-n(t'), t')) \leq (\sigma_{i+1} + 1)R_i + R(l_i) - R_{i-1} < \sigma_i R_i + R(l_i) = R(l_{i+1}) < \mu$. Therefore, Observation 02 yields $n(t) = \sigma h_i^1 + n(t')$. By assumption of the induction there exist nonnegative integers ϵ_u, $u = 1, \ldots, i-1$ such that $(-n(t'), t') = \sum_{u=1}^{i-1} \epsilon_u h_u$. Hence, $(-n(t), t) = \sum_{u=1}^{i} \epsilon_u h_u$ where $\epsilon_i = \sigma$. This completes the proof. ■

As a corollary of Lemma 2.5 we immediately obtain $R_i = \min\{R((-n(t), t)) \mid t = 1, \ldots, h_{i+1}^2 - 1\}$, because $R(h_i) < R(h_{i-1}) < \ldots < R(h_1)$. In addition, the set $\mathcal{H} := \{h_1, \ldots, h_\tau\}$ is an integer generating set for the set of exchange vectors. We now briefly show that $\mathcal{H}$ is not only an integer generating set, but even an integral Hilbert basis for the cone $C(\mathcal{H})$.

Theorem 2.7. The set $\mathcal{H} = \{h_1, \ldots, h_\tau\}$ defined via (2.4) is an integral Hilbert basis for the cone $C(\mathcal{H})$.

Proof. It is easy to see that the two extreme rays of the cone are the lines passing through the points $(0,0)$, $(-\lambda, \mu)$ and $(0,0)$, $(-n(1), 1)$, respectively. Suppose, there exist points in $C(\mathcal{H})$ that are not nonnegative integer combinations of the exchange vectors $(-n(t), t)$, $t = 1, \ldots, \mu$. Let $(-x, y)$ be such a point with y minimal. We know $(-x, y) = \alpha(-\lambda, \mu) + \beta(-n(1), 1) = (-(\alpha\lambda + \beta n(1)), \alpha\mu + \beta)$ where $\alpha \geq 0$, $\beta \geq 0$ and x and y are integers. Clearly, $0 < \alpha < 1$ and $0 < \beta < 1$ holds. Hence, $\alpha\mu + \beta \leq \mu$ and consequently, $(-n(y), y)$ is an exchange vector. Moreover, $R((-x, y)) = \lambda y - \mu x$ is integer and since $\lambda y - \mu x = \beta(\lambda - n(1)\mu)$, we obtain $0 < R((-x, y)) < \mu$. By Observation $O2$, $x = n(y)$ holds, a contradiction. Since $\mathcal{H}$ is an integer generating set for the exchange vectors the statement follows. ■

By now we have analyzed the exchange vectors $(-n_r(t), t)$ and their residua for the special case that $r = 0$. To end this section we deal with the case $r > 0$. First note that $n_r(t) = n_0(t + v) - n_0(v)$ where $(-n(v), v) = V(r)$. We now show that $t_{max}(r) = \min\{t > 0 \mid R((-n_0(t), t)) \geq \mu - r\}$ and that $(-n_r(t), t) = (-n(t), t)$ for $t = 1, \ldots, t_{max}(r) - 1$. Both relations are quite obvious for the following reasons.

$n_r(t) = \lfloor \frac{r+t\lambda}{\mu} \rfloor = \lfloor \frac{r+R((-n(t),t))+n(t)\mu}{\mu} \rfloor = \lfloor \frac{r+R((-n(t),t))}{\mu} \rfloor + n(t)$. Hence, $n_r(t) = n(t)$ and $R((-n_r(t), t)) := \lambda t - \mu n_r(t) = R((-n_0(t), t))$ if and only if $R((-n(t), t))$ $< \mu - r$. Consequently, $t_{max}(r) = \min\{t > 0 \mid R((-n(t), t) \geq \mu - r\}$. Taking our discussions for the case $r = 0$ into account it follows that $\mathcal{H}_r := \{h_i \mid i = 1, \ldots \tau, h_i^2 < t_{max}(r)\} \cup \{(-n(t_{max}(r)) - 1, t_{max}(r))\}$ is an integer generating set for the exchange vectors $(-n_r(t), t)$, $t = 1, \ldots t_{max}(r)$. In addition, $\mathcal{H}_r$ is a Hilbert basis for the cone generated by these exchange vectors.

3 The Facets of $P(\mu, \lambda, F)$

In this section we establish a link between the elements of the Hilbert basis $\mathcal{H}_r$ and the facets of $P(\mu, \lambda, F)$. Throughout this section we assume that natural numbers μ, λ, F with $gcd(\mu, \lambda) = 1$ and nonempty subsets N_1, N_2 are given. Before explaining the relation between the elements of $\mathcal{H}_r$ and the facets of $P(\mu,\lambda,F)$ in more detail let us introduce the notion of "λ-maximum" with respect to an inequality.

Definition Let $\sum_{i\in N_1\cup N_2} d_i x_i \le \delta$ be an inequality. A natural number t is called λ-maximum with respect to $d^T x \le \delta$ if for all vectors $x \in P(\mu,\lambda,F)$ with $|\{i \in N_2 \mid x_i = 1\}| > t$, $d^T x < \delta$ holds and if there exists a vector $x^0 \in P(\mu,\lambda,F)$ satisfying $|\{i \in N_2 \mid x_i^0 = 1\}| = t$ and $d^T x^0 = \delta$.

For every number $t \in \{1,\dots,|N_2|\}$ such that $|N_1| > \lfloor \frac{F-t\lambda}{\mu} \rfloor$ we now generate a series of inequalities where t is the λ-maximum. We proceed as follows.

Choose $I_1 \subseteq N_1$, $s := |I_1| = \lfloor \frac{F-t\lambda}{\mu} \rfloor$ and $I_2 \subseteq N_2$, $t = |I_2|$ and set $r := F - t\lambda - s\mu$. Due to the choice of s and t we have $0 \le r < \mu$. Furthermore, let $h_i = (-h_i^1, h_i^2)$ be some element of $\mathcal{H}_r = \{h_1,\dots,h_\tau\}$ satisfying

- $h_i^1 \le |N_1 \setminus I_1|$,
- $h_i^2 \le |I_2|$

and consider the inequality

$$\sum_{i\in N_1} h_i^2 x_i + \sum_{i\in N_2} h_i^1 x_i \le s h_i^2 + t h_i^1. \tag{3.1}$$

Of course, for every subset $V_1 \subseteq N_1$, $V_2 \subseteq N_2$, $|V_1| = s$, $|V_2| = t$ the vector $\sum_{v\in V_1\cup V_2} e_v$ satisfies the inequality at equation. Moreover, choosing $V_1 \subseteq N_1$, $|V_1| = s + h_i^1$ and $V_2 \subseteq N_2$, $|V_2| = t - h_i^2$ and setting $x_v = 1$ if $v \in V_1 \cup V_2$, $x_v = 0$ otherwise, yields a root of the above inequality. Unfortunately, the inequality is not always valid for $P(\mu,\lambda,F)$. More precisely, we will show later the following.

Under the assumption that t is the λ-maximum with respect to $\sum_{i\in N_1} h_i^2 x_i + \sum_{i\in N_2} h_i^1 x_i \le s h_i^2 + t h_i^1$, it is valid for $P(\mu,\lambda,F)$ if and only if one of the following conditions is satisfied

$$i = \tau,$$

$$h_{i+1}^1 > |N_1 \setminus I_1|,$$

$$h_{i+1}^2 > |I_2|.$$

If none of the three conditions holds, the above inequality must be modified to be valid for $P(\mu, \lambda, F)$ in a way we outline now. In this case, there are four possibilities to determine an inequality. We first demonstrate some of these possibilities on an example.

Example 3.2. For the polytope defined as the convex hull of all 0/1 vectors that satisfy the inequality in 0/1 variables

$$\sum_{i=1}^{8} 5x_i + \sum_{i=9}^{14} 7x_i \leq 35,$$

the inequalities

(i) $6x_1 + 6x_2 + 6x_3 + 6x_4 + 6x_5 + 5x_6 + 5x_7 + 5x_8 + 8x_9 + 8x_{10} + 8x_{11} + 8x_{12} + 8x_{13} + 8x_{14} \leq 40$,

(ii) $3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 + 3x_6 + 3x_7 + 3x_8 + 5x_9 + 4x_{10} + 4x_{11} + 4x_{12} + 4x_{13} + 4x_{14} \leq 21$,

(iii) $4x_1 + 4x_2 + 4x_3 + 4x_4 + 4x_5 + 4x_6 + 4x_7 + 4x_8 + 6x_9 + 6x_{10} + 6x_{11} + 5x_{12} + 5x_{13} + 5x_{14} \leq 28$.

define facets (checked by the program developed in [1]). The Hilbert basis $\mathcal{H}_0$ consists of the three vectors $(-1, 1)$, $(-4, 3)$ and $(-7, 5)$. It is easily checked that for the inequalities (i), (ii), (iii) the vector $e_9 + e_{10} + e_{11} + e_{12} + e_{13}$ is a root. Moreover, choosing the Hilbert basis element $h_i = (-4, 3)$ we obtain in all three cases that the vector $\sum_{v \in V_1 \cup V_2} e_v$ defined via $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{9, 10\}$ is a tight point. However, the inequality $\sum_{i \in N_1} 3x_i + \sum_{i \in N_2} 4x_i \leq 20$ is not valid for the corresponding polytope, because for $h_{i+1} = (-7, 5)$ we have $3 \times 7 + 0 \times 5 > 20$ and $h^1_{i+1} \leq |N_1 \setminus I_1|$, $h^2_{i+1} \leq |N_2|$. To obtain valid inequalities we have the choice either to reduce some of the coefficients in N_1 from κh^2_i to a smaller value or to increase coefficients in N_2 from κh^1_i to a bigger value (κ is some natural number) such that the vector $\sum_{v \in V_1 \cup V_2} e_v$ with $V_1 = \{1, \ldots, 7\}$, $V_2 = \emptyset$ is tight for the corresponding inequalities.

Let us now formalize these possibilities. We assume that $I_1 \subseteq N_1$, $s := |I_1| = \lfloor \frac{F - t\lambda}{\mu} \rfloor$ and $I_2 \subseteq N_2$, $t = |I_2|$ is given and that $r := F - t\lambda - s\mu$ denotes the residuum. Furthermore, $h_i = (-h^1_i, h^2_i)$ is some element of $\mathcal{H}_r = \{h_1, \ldots, h_\tau\}$ satisfying

- $h^1_i \leq |N_1 \setminus I_1|$,

- $h_i^2 \leq |I_2|$.

Finally we assume that $i < \tau$, $h_{i+1}^1 \leq |N_1 \setminus I_1|$ and $h_{i+1}^2 \leq |I_2|$, i.e., the inequality of type (3.1) is not valid for $\mathcal{P}(\mu, \lambda, F)$.

(3.3.) We choose an integer $h_i^2 \leq j < h_{i+1}^2$ and a subset $J_2 \subseteq I_2$, $|J_2| = j$. With this subset J_2 we associate an inequality such that the coefficients of the items in N_1 are all equal to some value a, where the coefficients of the items in $I_2 \setminus J_2$ are all equal to a value b' and where the coefficients of the items in $(N_2 \setminus I_2) \cup J_2$ are all equal to a value $b < b'$. The appropriate choice of the numbers a, b, b' is determined by the two equations

$$h_i^1 a = h_i^2 b,$$

$$h_{i+1}^1 a = jb + (h_{i+1}^2 - j)b',$$

which yields $a = \kappa h_i^2$, $b = \kappa h_i^1$, $\kappa = h_{i+1}^2 - j$ and $b' = h_{i+1}^1 h_i^2 - j h_i^1$. Choosing a, b, b' as indicated here we will show later that the inequality

$$\sum_{i \in N_1} a x_i + \sum_{i \in I_2 \setminus J_2} b' x_i + \sum_{i \in (N_2 \setminus I_2) \cup J_2} b x_i \leq sa + jb + (t - j)b'$$

is valid for $P(\mu, \lambda, F)$. It is not difficult to see that for every subset $V_1 \subseteq N_1$, $I_2 \setminus J_2 \subseteq V_2 \subseteq N_2$, $|V_1| = s$, $|V_2| = t$ the vector $\sum_{v \in V_1} e_v + \sum_{v \in V_2} e_v$ satisfies the inequality at equation. Moreover, choosing $V_1 \subseteq N_1$, $|V_1| = s + h_i^1$ and $I_2 \setminus J_2 \subseteq V_2 \subseteq N_2$, $|V_2| = t - h_i^2$ and setting $x_v = 1$ if $v \in V_1 \cup V_2$, $x_v = 0$ otherwise, yields a root of the above inequality. Finally, we have that every vector $\sum_{v \in V_1 \cup V_2} e_v$ satisfies the above inequality at equation for which $V_1 \subseteq N_1$, $|V_1| = s + h_{i+1}^1$ and $V_2 \subseteq I_2 \setminus J_2$, $|V_2| = t - h_{i+1}^2$.

(3.4.) We choose an integer $h_i^1 \leq j < h_{i+1}^1$ and a subset $J_1 \subseteq N_1 \setminus I_1$, $|J_1| = j$. With J_1 we associate an inequality such that the coefficients of the items in N_2 are all equal to some value d, where the coefficients of the items in $I_1 \cup J_1$ are all equal to a value c and where the coefficients of the items in $N_1 \setminus (I_1 \cup J_1)$ are all equal to a value $c' < c$. The numbers c, c', d are a solution to the system

$$h_i^1 c = h_i^2 d,$$

$$jc + (h_{i+1}^1 - j)c' = h_{i+1}^2 d,$$

which yields $c = \kappa h_i^2$, $d = \kappa h_i^1$, $\kappa = h_{i+1}^1 - j$ and $c' = h_{i+1}^2 h_i^1 - j h_i^2$. Now consider the inequality

$$\sum_{i \in I_1 \cup J_1} c x_i + \sum_{i \in N_1 \setminus (I_1 \cup J_1)} c' x_i + \sum_{i \in N_2} d x_i \leq sc + td.$$

The roots of this inequality are the vectors $\sum_{v\in V_1\cup V_2} e_v$ such that

$$V_1 \subseteq I_1\cup J_1,\ V_2\subseteq N_2,\ |V_1| = s,\ |V_2| = t \text{ or}$$
$$V_1 \subseteq I_1\cup J_1,\ |V_1| = s + h_i^1 \text{ and } V_2\subseteq N_2,\ |V_2| = t - h_i^2 \text{ or}$$
$$I_1\cup J_1 \subseteq V_1 \subseteq N_1,\ |V_1| = s + h_{i+1}^1 \text{ and } V_2\subseteq N_2,\ |V_2| = t - h_{i+1}^2.$$

The remaining cases are $1 \le j < h_i^2$ or $1 \le j < h_i^1$, respectively. In both cases an inequality similar to the one above can be derived.

(3.5.) We choose an integer $1 \le j < h_i^2$, a subset $J_2 \subseteq I_2$, $|J_2| = j$ and determine numbers α, β, β' via the following system

$$h_i^1\alpha = j\beta + (h_i^2 - j)\beta',$$
$$h_{i+1}^1\alpha = j\beta + (h_{i+1}^2 - j)\beta'.$$

Solving these equations yields $\alpha = \kappa(h_{i+1}^2 - h_i^2)$, $\beta' = \kappa(h_{i+1}^1 - h_i^1)$, $\kappa = j$ and $\beta = -(h_i^2 - j)(h_{i+1}^1 - h_i^1) + h_i^1(h_{i+1}^2 - h_i^2)$. Choosing α, β, β' as indicated here we present conditions such that the inequality

$$\sum_{i\in N_1} \alpha x_i + \sum_{i\in I_2\setminus J_2} \beta' x_i + \sum_{i\in (N_2\setminus I_2)\cup J_2} \beta x_i \le s\alpha + j\beta + (t-j)\beta'$$

is valid for $P(\mu, \lambda, F)$.

(3.6.) We choose an integer $1 \le j < h_i^1$, a subset $J_1 \subseteq N_1 \setminus I_1$, $|J_1| = j$ and define numbers γ, γ', δ via the equations

$$j\gamma + (h_i^1 - j)\gamma' = h_i^2\delta,$$
$$j\gamma + (h_{i+1}^1 - j)\gamma' = h_{i+1}^2\delta.$$

This yields $\gamma' = \kappa(h_{i+1}^2 - h_i^2)$, $\delta = \kappa(h_{i+1}^1 - h_i^1)$, $\kappa = j$ and $\gamma = h_i^2(h_{i+1}^1 - h_i^1) - (h_i^1 - j)(h_{i+1}^2 - h_i^2)$. Under certain conditions, the inequality

$$\sum_{i\in I_1\cup J_1} \gamma x_i + \sum_{i\in N_1\setminus(I_1\cup J_1)} \gamma' x_i + \sum_{i\in N_2} \delta x_i \le s\gamma + t\delta$$

is valid for $P(\mu, \lambda, F)$.

Having introduced the inequalities (3.1), (3.3), (3.4), (3.5) and (3.6) we now deal with the question when they are valid and facet defining for $P(\mu, \lambda, F)$. For this purpose we first present three easy, yet technical lemmas. The corresponding proofs are left to the Appendix.

Lemma 3.7. Let natural numbers $\mu < \lambda$ with $gcd(\mu, \lambda) = 1$ and $0 \leq r < \mu$ be given. For every $h_i \in \mathcal{H}_r$ and $h_j \in \mathcal{H}_r$, $i \neq j$ the following relations hold.

(1) $h_j^2 h_i^1 - h_j^1 h_i^2 > 0$, if $j < i$.

(2) $h_j^2 h_i^1 - h_j^1 h_i^2 < 0$, if $j > i$.

Lemma 3.8. Let natural numbers $\mu < \lambda$ with $gcd(\mu, \lambda) = 1$ and $0 \leq r < \mu$ be given and let $a_1 \geq a_2 \geq \ldots \geq a_{n_r(t_{max}(r))}$ and $b_1 \leq b_2 \leq \ldots \leq b_{t_{max}(r)}$ be two sequences of nonnegative integers such that

$$\sum_{v=1}^{h_i^1} a_v = \sum_{v=1}^{h_i^2} b_v \text{ and } \sum_{v=1}^{n(t)} a_v < \sum_{v=1}^{t} b_v, \text{ for all } t < h_i^2$$

where $h_i = (-h_i^1, h_i^2) \in \mathcal{H}_r = \{h_1, \ldots, h_\tau\}$ and $i < \tau$. Then the following statements are true.

(1) $\sum_{v=1}^{n_r(t)} a_v \leq \sum_{v=1}^{t} b_v$ for all $t < h_{i+1}^2$ and $\sum_{v=1}^{n_r(t)} a_v < \sum_{v=1}^{t} b_v$ for all $h_i^2 + 1 \leq t < h_{i+1}^2$, provided that $a_1 > a_{h_i^1}$ or $b_1 < b_{h_i^2}$.

(2) If $\sum_{v=1}^{h_{i+1}^1} a_v = \sum_{v=1}^{h_{i+1}^2} b_v$, it follows that $\sum_{v=1}^{n_r(t)} a_v \leq \sum_{v=1}^{t} b_v$ for all $t \leq h_\tau^2$ and $\sum_{v=1}^{n_r(t)} a_v < \sum_{v=1}^{t} b_v$ for all $h_{i+1}^2 + 1 \leq t \leq h_\tau^2$, provided that $a_{h_{i+1}^1} < a_{h_i^1+1}$ or $b_{h_{i+1}^2} > b_{h_i^2+1}$.

(3) If $\sum_{v=1}^{h_{i+1}^1} a_v < \sum_{v=1}^{h_{i+1}^2} b_v$, then $\sum_{v=1}^{n_r(t)} a_v < \sum_{v=1}^{t} b_v$ for all $h_{i+1}^2 < t \leq h_\tau^2$.

Lemma 3.9. Let natural numbers $\mu < \lambda$ with $gcd(\mu, \lambda) = 1$ and $0 \leq r < \mu$ be given and let $a_1 \geq a_2 \geq \ldots \geq a_{n_1}$ and $b_1 \leq b_2 \leq \ldots \leq b_{n_2}$, $n_r(n_2) = n_1$ be two sequences of nonnegative integers such that $\sum_{v=1}^{n(t)} a_v < \sum_{v=1}^{t} b_v$, for all $t < h_\tau^2$.

(1) If $\sum_{v=1}^{n_r(h_\tau^2)} a_v \leq \sum_{v=1}^{h_\tau^2} b_v$, then $\sum_{v=1}^{n_r(t)} a_v \leq \sum_{v=1}^{t} b_v$ for all $t \geq h_\tau^2$.

(2) If $\sum_{v=1}^{n_r(h_\tau^2)} a_v < \sum_{v=1}^{h_\tau^2} b_v$, then $\sum_{v=1}^{n_r(t)} a_v < \sum_{v=1}^{t} b_v$ for all $t \geq h_\tau^2$.

(3) If $\sum_{v=1}^{n_r(h_\tau^2)} a_v = \sum_{v=1}^{h_\tau^2} b_v$ and if $a_1 > a_{n_r(h_\tau^2)}$ or $b_1 < b_{h_\tau^2}$, then $\sum_{v=1}^{n_r(t)} a_v < \sum_{v=1}^{t} b_v$ for all $t > h_\tau^2$.

Proposition 3.10. Let $I_1 \subseteq N_1$, $s := |I_1| = \lfloor \frac{F-t\lambda}{\mu} \rfloor$ and $I_2 \subseteq N_2$, $t = |I_2|$ and set $r := F - t\lambda - s\mu$. Furthermore, let $h_i = (-h_i^1, h_i^2)$ be some element of $\mathcal{H}_r = \{h_1, \ldots, h_\tau\}$ satisfying $h_i^1 \leq |N_1 \setminus I_1|$ and $h_i^2 \leq |I_2|$.

Provided that t is the λ-maximum, the inequality (3.1),

$$\sum_{i \in N_1} h_i^2 x_i + \sum_{i \in N_2} h_i^1 x_i \leq s h_i^2 + t h_i^1$$

is valid for $P(\mu, \lambda, F)$ if and only if one of the following conditions is satisfied

(V1) $i = \tau$,

(V2) $h_{i+1}^1 > |N_1 \setminus I_1|$,

(V3) $h_{i+1}^2 > |I_2|$.

In addition, the inequality defines a facet of $P(\mu, \lambda, F)$ if and only if the following two conditions (O1) and (O2) are satisfied.

(O1) $s > 0$ or $s + h_i^1 < |N_1|$ or $|N_1| = 1$,

(O2) $t - h_i^2 > 0$ or $t < |N_2|$ or $|N_2| = 1$.

Proof. We start with proving the validity statements. First notice that the above conditions guarantee the existence of sets J_1 and J_2 with $I_1 \subseteq J_1 \subseteq N_1$, $|J_1| = s + h_i^1$ and $J_2 \subseteq I_2$, $|J_2| = t - h_i^2$.

Suppose, none of the conditions (V1), (V2), (V3) is satisfied. By Lemma 3.7 we have that $h_{i+1}^2 h_i^1 - h_{i+1}^1 h_i^2 < 0$. Choose sets V_1 and V_2 such that $V_1 \subseteq N_1$, $|V_1| = s + h_{i+1}^1$ and $V_2 \subseteq N_2$, $|V_2| = t - h_{i+1}^2$. Such sets exist since (V1), (V2), (V3) do not hold. The vector $\sum_{v \in V_1 \cup V_2} e_v$ is an element of $P(\mu, \lambda, F)$ and $\sum_{i \in N_1} h_i^2 x_i + \sum_{i \in N_2} h_i^1 x_i = (s + h_{i+1}^1) h_i^2 + (t - h_{i+1}^2) h_i^1 > s h_i^2 + t h_i^1$.

To prove the converse direction we assume that (V1) or (V2) or (V3) is satisfied. Let $x \in P(\mu, \lambda, F)$ and set $V_1 := \{v \in N_1 \mid x_v = 1\}$, $V_2 := \{v \in N_2 \mid x_v = 1\}$. We can assume that $|V_2| < t$ (otherwise, x clearly satisfies the inequality). Let $v_2 > 0$ such that $|V_2| = t - v_2$. Then, $|V_1| \leq s + n_r(v_2)$.

If (V1) is satisfied, then by Lemmas 3.7 and 3.9 $\sum_{v=s+1}^{s+n_r(v_2)} h_i^2 \leq \sum_{v=t-v_2+1}^{t} h_i^1$ and hence the inequality is valid.
If (V2) holds, then $|V_1| < s + h_{i+1}^1$. Define v_1' via $|V_1| = s + v_1'$ and let v_2' be the

smallest natural number such that $n_r(v_2') \geq v_1$. Clearly, $v_2 \geq v_2'$ and $h^2_{i+1} \geq v_2'$. By Lemma 2.5, $(-n_r(v_2'), v_2') = \sum_{u=1}^{i} \epsilon_u h_u$ and by Lemma 3.7, $h^2_j h^1_i - h^1_j h^2_i > 0$ for all $j = 1, \ldots, i-1$. Therefore, $|V_1| h^2_i + |V_2| h^1_i \leq (s + n_r(v_2')) h^2_i + (t - v_2') h^1_i = s h^2_i + t h^1_i - \sum_{u=1}^{i} \epsilon_u (h^2_u h^1_i - h^1_u h^2_i) \leq s h^2_i + t h^1_i$.
If (V3) is satisfied, we conclude that $v_2 < h^2_{i+1}$. By Lemma 2.5, $(-n_r(v_2), v_2) = \sum_{u=1}^{i} \epsilon_u h_u$. Since $h^2_j h^1_i - h^1_j h^2_i > 0$ for all $j = 1, \ldots, i-1$ (Lemma 3.7), we obtain $|V_1| h^2_i + |V_2| h^1_i \leq (s + n_r(v_2)) h^2_i + (t - v_2) h^1_i = s h^2_i + t h^1_i - \sum_{u=1}^{i} \epsilon_u (h^2_u h^1_i - h^1_u h^2_i) \leq s h^2_i + t h^1_i$. This proves that the inequality is valid.

We now turn to the second statement.
Suppose, (O1) or (O2) do not hold. W.l.o.g. we assume (O1) is not satisfied, i.e., $s = 0$ and $s + h^1_i = |N_1|$ and $|N_1| > 1$. By Lemma 3.7, $h^2_j h^1_i - h^1_j h^2_i > 0$ for all $j = 1, \ldots, i-1$. Hence, every vector x satisfying (3.1) at equality either satisfies $x_i = 0$ for all $i \in N_1$ or $x_i = 1$ for all $i \in N_1$. Since $|N_1| > 1$, x also satisfies the equation $x_1 - x_{|N_1|} = 0$ and consequently, the dimension of the face induced by inequality (3.1) is less or equal than $|N| - 2$. Analogous arguments apply if (O2) is not satisfied.
Conversely, let $c^T x \leq \gamma$ be a facet defining inequality of $P(\mu, \lambda, F)$ such that every root of (3.1) satisfies $c^T x = \gamma$. If $|N_1| > 1$, there exist $\emptyset \neq V_1 \subseteq N_1$, $V_1 \neq N_1$ with $|V_1| = s$ or $|V_1| = s + h^1_i$. Let $V_2 \subseteq N_2$ with $|V_2| = t$ if $|V_1| = s$ and $|V_2| = t - h^2_i$ if $|V_1| = s + h^1_i$. Then, $x := \sum_{v \in V_1 \cup V_2} e_v$ is a root of (3.1). Moreover, for every $u \in V_1$, $u' \in N_1 \setminus V_1$, the vector $x' := x - e_u + e_{u'}$ is a root of (3.1). Therefore, $c^T x = c^T x'$. This yields $c_u = c_{u'}$ for all $u, u' \in N_1$, since u and u' can be chosen arbitrarily. By analogous arguments we obtain $c_u = c_{u'}$ for all $u, u' \in N_2$. Finally, let $V_1 \subseteq N_1$, $|V_1| = s$, $V_2 \subseteq N_2$, $|V_2| = t$ and $V_1' \subseteq N_1$, $|V_1'| = s + h^1_i$, $V_2' \subseteq N_2$, $|V_2'| = t - h^2_i$. Since x and x' are roots of (3.1) we immediately obtain $h^1_i c_u = h^2_i c_v$ where $u \in N_1$, $v \in V_2$. This shows that $c^T x \leq \gamma$ and inequality (3.1) are equal up to multiplication by a scalar. ■

The next question to be raised is when the inequalities (3.3) – (3.6) are valid and facet defining for $P(\mu, \lambda, F)$. This question is answered by Proposition 3.11. Here we show that under mild assumptions such inequalities are valid. In addition, for each type of inequality (3.3) – (3.6) necessary and sufficient conditions are presented such that the corresponding inequality defines a facet of $P(\mu, \lambda, F)$. It turns out that the statements as well as the corresponding proofs are quite similar for the four types of inequalities and we decided to outline just one such proof in detail.

Proposition 3.11. Let $I_1 \subseteq N_1$, $s := |I_1| = \lfloor \frac{F - t\lambda}{\mu} \rfloor$ and $I_2 \subseteq N_2$, $t = |I_2|$ and set $r := F - t\lambda - s\mu$. Furthermore, let $h_i = (-h^1_i, h^2_i)$ be some element of $\mathcal{H}_r = \{h_1, \ldots, h_\tau\}$ such that $i < \tau$, $h^1_{i+1} \leq |N_1 \setminus I_1|$ and $h^2_{i+1} \leq |I_2|$ (these three conditions guarantee that the corresponding inequality of type (3.1) is not valid).

(1) Let $h^2_i \le j < h^2_{i+1}$ be some integer and let $J_2 \subseteq I_2$, $|J_2| = j$. Set $a = \kappa h^2_i$, $b = \kappa h^1_i$, $\kappa = h^2_{i+1} - j$ and $b' = h^1_{i+1} h^2_i - j h^1_i$. If t is the λ-maximum, the inequality (3.3),

$$\sum_{i \in N_1} a x_i + \sum_{i \in J_2 \cup (N_2 \setminus I_2)} b x_i + \sum_{i \in I_2 \setminus J_2} b' x_i \le sa + jb + (t-j)b'$$

is valid for $P(\mu, \lambda, F)$. In addition, it defines a facet of $P(\mu, \lambda, F)$ if and only if the following two conditions are satisfied

(F1) $t < |N_2|$ or $j > h^2_i$ or $h^2_i = 1$,

(F2) $h^2_{i+1} < t$ or $j = h^2_{i+1} - 1$.

(2) Let $h^1_i \le j < h^1_{i+1}$ be some integer and let $J_1 \subseteq N_1 \setminus I_1$, $|J_1| = j$. Set $c = \kappa h^2_i$, $d = \kappa h^1_i$, $\kappa = h^1_{i+1} - j$ and $c' = h^2_{i+1} h^1_i - j h^2_i$. If t is the λ-maximum, the inequality (3.4),

$$\sum_{i \in I_1 \cup J_1} c x_i + \sum_{i \in N_1 \setminus (I_1 \cup J_1)} c' x_i + \sum_{i \in N_2} d x_i \le sc + td$$

is valid for $P(\mu, \lambda, F)$. In addition, it defines a facet of $P(\mu, \lambda, F)$ if and only if the following two conditions are satisfied

(F3) $s > 0$ or $j > h^1_i$ or $h^1_i = 1$,

(F4) $h^1_{i+1} < |N_1 \setminus I_1|$ or $j = h^1_{i+1} - 1$.

(3) Let $1 \le j < h^2_i$ be some integer and let $J_2 \subseteq I_2$, $|J_2| = j$. Set $\alpha = \kappa(h^2_{i+1} - h^2_i)$, $\beta' = \kappa(h^1_{i+1} - h^1_i)$, $\kappa = j$ and $\beta = h^1_i(h^2_{i+1} - h^2_i) - (h^2_i - j)(h^1_{i+1} - h^1_i)$. We require that the numbers α, β, β' satisfy the properties

$v\beta - \lfloor \frac{v\lambda - r}{\mu} \rfloor \alpha < 0$ for all values of v with $1 \le v \le |N_2| - t$ and $\lfloor \frac{v\lambda - r}{\mu} \rfloor \le s$.

$-v\beta + n_r(v)\alpha < 0$ for all numbers v with $1 \le v \le j$.

$-j\beta - v\beta' + n_r(j+v)\alpha < 0$ for all numbers v with $1 \le v < h^1_i - j$.

Then, the inequality (3.5),

$$\sum_{i \in N_1} \alpha x_i + \sum_{i \in I_2 \setminus J_2} \beta' x_i + \sum_{i \in (N_2 \setminus I_2) \cup J_2} \beta x_i \le s\alpha + j\beta + (t-j)\beta'$$

is valid for $P(\mu, \lambda, F)$. In addition, it defines a facet of $P(\mu, \lambda, F)$ if and only $t < |N_2|$ or $j = 1$.

(4) Let $1 \le j < h^1_i$ be some integer and let $J_1 \subseteq N_1 \setminus I_1$, $|J_1| = j$. Set $\gamma = \kappa(h^2_{i+1} - h^2_i)$, $\delta = \kappa(h^1_{i+1} - h^1_i)$, $\kappa = j$ and $\gamma' = h^2_i(h^1_{i+1} - h^1_i) - (h^1_i - j)(h^2_{i+1} - h^2_i)$. We require that the numbers γ, δ, γ' satisfy the properties

$v\delta - \lfloor \frac{v\lambda - r}{\mu} \rfloor \gamma < 0$ for all values of v with $1 \leq v \leq |N_2| - t$ and $\lfloor \frac{v\lambda - r}{\mu} \rfloor \leq s$.

$-v\delta + n_r(v)\gamma < 0$ for all numbers $1 \leq v$ with $n_r(v) \leq j$.

$-v\delta + j\gamma + (n_r(v) - j)\gamma' < 0$ for all numbers $1 \leq v$ with $j < n_r(v) < h^1_i$.

Then, the inequality (3.6),

$$\sum_{i \in I_1 \cup J_1} \gamma x_i + \sum_{i \in N_1 \setminus (I_1 \cup J_1)} \gamma' x_i + \sum_{i \in N_2} \delta x_i \leq s\gamma + t\delta$$

is valid for $P(\mu, \lambda, F)$. In addition, it defines a facet of $P(\mu, \lambda, F)$ if and only if $s > 0$ or $j = 1$.

Proof of (1). For ease of notation we denote the corresponding inequality by $c^T x \leq \gamma$. First notice that by Lemma 3.7 we obtain $b' = h^1_{i+1} h^2_i - j h^1_i > h^2_{i+1} h^1_i - j h^1_i = b$. W.l.o.g. we assume that $I_2 = \{n_1 + 1, \ldots, n_1 + t\}$ and $J_2 = \{n_1 + t - j + 1, \ldots, n_1 + t\}$. Let $V_1 \subseteq N_1$ and $V_2 \subseteq N_2$ such that $x = \sum_{v \in V_1 \cup V_2} e_v$ is feasible. If $|V_2| \geq t$, $c^T x \leq \gamma$ obviously holds, because t is the λ-maximum. Hence $|V_2| = t - v$ with $v \geq 1$ and $|V_1| \leq s + \lfloor \frac{v\lambda + r}{\mu} \rfloor = s + n_r(v)$. Moreover, we can assume that $V_1 = \{1, \ldots, |V_1|\}$ and $V_2 = \{n_1 + 1, \ldots, n_1 + t - v\}$. We write $(-n_r(v), v) = \sum_{u \in \mathcal{H}_r} \epsilon_u h_u$. In case, $v < h^2_{i+1}$ we immediately obtain $c^T x \leq \gamma$, because $h^2_j h^1_i - h^1_j h^2_i > 0$ for all $j < i$ and $\epsilon_j = 0$ for all $j > i$ and $\epsilon_j \geq 0$ for all $j \leq i$. If $v = h^2_{i+1}$, the inequality is satisfied as well. It remains the case $v > h^2_{i+1}$. By definition, $c_{s+1}, \ldots, c_{s+n_r(v)}$ and $c_t, \ldots, c_{t-v+1}$ are sequences of numbers as investigated in Lemma 3.8 (2) and Lemma 3.9. Hence, $\sum_{w=s+1}^{s+n_r(v)} c_w \leq \sum_{w=t-v+1}^{t} c_w$ and it follows that x satisfies the above inequality.

We now turn to the facet statement. Let $c^T x \leq \gamma$ denote the above inequality. Suppose, condition (F1) is not satisfied. Then, $t = |N_2|$ and $j = h^2_i > 1$. Hence, $|J_1| \geq 2$ and every $x \in F_c$ satisfies the equation $x_u - x_w = 0$ where $u, w \in J_1$, $u \neq w$. Similarly, if condition (F2) is not satisfied, then, $h^2_{i+1} = t$ and $j < h^2_{i+1}$. Therefore, $|I_2 \setminus J_2| \geq 2$ and every $x \in F_c$ satisfies the equation $x_u - x_w = 0$ where $u, w \in I_2 \setminus J_2$, $u \neq w$.

This shows that the conditions (F1) and (F2) are necessary. To prove that both conditions are sufficient such that an inequality of type (3.3) defines a facet of $P(\mu, \lambda, F)$ is straight forward and we omit further details.

Proof of (2). This proof is very similar to the one outlined above. We just notice that $c = \kappa h^2_i$, $d = \kappa h^1_i$, $\kappa = h^1_{i+1} - j$ and $c' = h^2_{i+1} h^1_i - j h^2_i < h^1_{i+1} h^2_i - j h^2_i = \kappa h^2_i = c$.

Proof of (3). Let $c^T x \leq \gamma$ denote the inequality of type (3.5). Again, we first conclude that $\beta = h_i^1(h_{i+1}^2 - h_i^2) - (h_i^2 - j)(h_{i+1}^1 - h_i^1) = h_i^1 h_{i+1}^2 - h_i^1 h_i^2 - h_i^2 h_{i+1}^1 + h_i^2 h_i^1 + j(h_{i+1}^1 - h_i^1) = h_i^1 h_{i+1}^2 - h_{i+1}^1 h_i^2 + \beta' < \beta'$. Secondly, the three conditions guarantee that t is the λ-maximum of the inequality and that every vector $\sum_{v \in V_1 \cup V_2} e_v$ with $V_1 \subseteq N_1$, $V_2 \subseteq N_2$, $|V_2| = t - v$, $v < h_i^2$, $|V_1| = s + \lfloor \frac{v\lambda + r}{\mu} \rfloor$ $< s + h_i^1$ satisfies $c^T x < \gamma$. Taking these facts into account, one can proof statement (3) analogously to (1).

Proof of (4). This proof is similar to the ones above. ■

To end this section we finally introduce a class of inequalities that can be viewed as a degenerate case of the inequalities (3.1). We choose numbers s, t, r, l with the properties $1 \leq t \leq |N_2|$, $s = \lfloor \frac{F - t\lambda}{\mu} \rfloor$, $r = F - t\lambda - s\mu$, $1 \leq l < n_r(1)$ and $0 \leq s$, $s + l \leq |N_1|$. For every subset $I_1 \subseteq N_1$, $|I_1| = s + l$, the inequality

$$\text{(3.12)} \qquad \sum_{i \in I_1} x_i + \sum_{i \in N_2} l x_i \leq s + lt$$

is valid for $P(\mu, \lambda, F)$ if t is the λ-maximum. It is facet defining for $P(\mu, \lambda, F)$ if and only if $s > 0$ or $l = 1$. These two statements can be shown quite easily.

4 A Complete Description of $P(\mu, \lambda, F)$

Having introduced the five classes of inequalities in the previous section we now show that they are together with the trivial inequalities and the inequality $\sum_{i \in N_2} x_i \leq \lfloor \frac{F}{\lambda} \rfloor$ sufficient to decribe the polytope $P(\mu, \lambda, F)$.

Theorem 4.1. The trivial inequalities, the inequality $\sum_{i \in N_2} x_i \leq \lfloor \frac{F}{\lambda} \rfloor$, the inequalities (3.1), (3.3), (3.4), (3.5), (3.6) and (3.12) completely describe $P(\mu, \lambda, F)$.

Proof. Let $c^T x \leq \gamma$ define the nontrivial facet F_c of $P(\mu, \lambda, F)$. W.l.o.g. we assume that $N_1 = \{1, \ldots, n_1\}$, $N_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$ and $c_1 \geq \ldots \geq c_{n_1}$ and $c_{n_1+1} \geq \ldots \geq c_{n_1+n_2}$. Let t be the λ-maximum with respect to $c^T x \leq \gamma$. If $c_1 = 0$, then it is easy to see that $F_c \subseteq \{x \in P(\mu, \lambda, F) \mid \sum_{i \in N_2} x_i = \lfloor \frac{F}{\lambda} \rfloor\}$. Otherwise, $c_1 > 0$ and we can assume that the vector x^0 defined via $x_v^0 = 1$ for $v = 1, \ldots, \lfloor \frac{F - t\lambda}{\mu} \rfloor$, $x_v^0 = 1$ for $v = n_1 + 1, \ldots, n_1 + t$, $x_v^0 = 0$, else is a root of F_c (for if not, then $F_c \subseteq \{x \in P(\mu, \lambda, F) \mid x_1 = 1\}$). Set $s := \lfloor \frac{F - t\lambda}{\mu} \rfloor$ and $r := F - t\lambda - s\mu$. Since F_c is a nontrivial facet, not every root satisfies the equation $\sum_{i \in N_2} x_i = t$. Thus, there exist numbers $t' < t$ and $s' > s$ and a root $x \in F_c$ such that $x_v = 1$ if $v = 1, \ldots, s'$, $x_v = 1$ if $v = n_1 + 1, \ldots, n_1 + t'$, $x_v = 0$ else. Let t_1 denote the maximum number t' with the above properties, i.e., there exist numbers $t_1 < t$ and $s_1 > s$ and a root $x^1 \in F_c$ such that $x^1_v = 1$ if $v = 1, \ldots, s_1$, $x^1_v = 1$ if $v = n_2+1, \ldots, n_2+t_1$, $x^1_v = 0$, otherwise. Of course, $s_1 \leq \lfloor \frac{F-t_1\lambda}{\mu} \rfloor$. Notice that for every number $t' < t$, $\lfloor \frac{F-t'\lambda}{\mu} \rfloor = \lfloor \frac{F-t\lambda+(t-t')\lambda}{\mu} \rfloor = \lfloor \frac{r+s\mu+(t-t')\lambda}{\mu} \rfloor = s + \lfloor \frac{r+(t-t')\lambda}{\mu} \rfloor$ $= s + n_r(t-t')$.

We distinguish the two cases: (1) $s_1 = s + n_r(t-t_1)$ and (2) $s_1 < s + n_r(t-t_1)$.

(1) $s_1 = s+n_r(t-t_1)$. Since both x^0 and x^1 are roots of F_c and due to the choice of t_1 we derive the following relations:

(R1) $\sum_{v=s+1}^{s+n_r(t-t')} c_v < \sum_{v=n_1+t'+1}^{n_1+t} c_v$ for all $t > t' > t_1$.

(R2) $\sum_{v=s+1}^{s+n_r(t-t_1)} c_v = \sum_{v=n_1+t_1+1}^{n_1+t} c_v$.

Let $\mathcal{H}_r = \{h_1, \ldots, h_\tau\}$ denote the Hilbert basis as introduced in Section 2. From Lemma 2.5 we know that every vector $(-n_r(t-t'), t-t')$ with $t-t' \leq h^2_\tau$ can be written as $\sum_{u=1}^{\tau} \epsilon_u h_u$ where h_u are the elements in $\mathcal{H}_r$ and $\epsilon_u \geq 0$, ϵ_u integer $(u = 1, \ldots, \tau)$. Moreover, $c_{s+1} \geq c_{s+2} \geq \ldots \geq c_{n_1}$ and $c_{n_1+t} \leq c_{n_1+t-1} \leq \ldots \leq c_{n_1+1}$ are two sequences of numbers as investigated in Lemmas 3.8 and 3.9. In particular, we conclude from Lemma 3.9 that $t - t_1 \leq h^2_\tau$. Furthermore, Lemma 2.5, (R1) and (R2) imply that $t - t_1 = h^2_i$ for some $i \in \{1, \ldots, \tau\}$. Therefore by Lemma 3.8, $\sum_{v=s+1}^{s+n_r(t-t')} c_v \leq \sum_{v=n_1+t'+1}^{n_1+t} c_v$ for all $h^2_i < t-t' < h^2_{i+1}$, $t' \geq 0$, $n_r(t-t') \leq n_1 - s$ and $\sum_{v=s+1}^{s+n_r(t-t')} c_v < \sum_{v=n_1+t'+1}^{n_1+t} c_v$ for all $h^2_i < t-t' < h^2_{i+1}$, $t' \geq 0$, $n_r(t-t') \leq n_1 - s$, provided that $c_{s+1} > c_{s_1}$ or $c_t < c_{t_1+1}$.

(a) Suppose, $i = \tau$ or $h^1_{i+1} > n_1 - s$ or $h^2_{i+1} > t$.

By Lemma (3.8) and (3.9) we obtain that every $x \in F_c$ satisfies $|\{i \in N_1 \mid x_i = 1\}| = s + \sigma(s_1 - s)$ and $|\{i \in N_2 \mid x_i = 1\}| = t - \sigma(t - t_1)$ for some $\sigma \geq 0$, σ integer. Thus F_c is defined by the inequality $\sum_{i \in N_1} h^2_i x_i + \sum_{i \in N_2} h^1_i x_i \leq s h^2_i + t h^1_i$.

(b) Suppose, $i < \tau$ and $h^1_{i+1} \leq n_1 - s$ and $h^2_{i+1} \leq t$. We distinguish the two subcases:

(b1) $c_{s_1} = c_{s+1}$ and $c_{n_1+t} = c_{n_1+t_1+1}$. Hence, $c_u = c_v = \kappa h^2_i$ for all u, v, $s+1 \leq u \leq v \leq s_1$ and $c_u = c_v = \kappa h^1_i$ for all $n_1 + t_1 + 1 \leq u \leq v \leq n_1 + t$ where κ is is some positive integer. Moreover, $(-n_r(v), v) = \sum_{u=1}^{i} \epsilon_u h_u$ for $h^2_i + 1 \leq v \leq h^2_{i+1} - 1$ and nonnegative integers ϵ_u (Lemma 2.5) and consequently, $\sum_{w=s+1}^{s+n(v)} c_w \leq \sum_{w=n_1+t-v+1}^{n_1+t} c_w$. By Lemma 3.7 we know that $h^2_{i+1}\kappa h^1_i - h^1_{i+1}\kappa h^2_i < 0$. Since the inequality $c^T x \leq \gamma$ is valid there exists $h^2_{i+1} > j \geq h^2_i$ such that $c_{t-j} >$

κh_i^1 or there exists $h_{i+1}^1 > j \geq h_i^1$ such that $c_{j+1} < \kappa h_i^2$. Now it follows that $\sum_{v=s+1}^{s+h_{i+1}^1} c_v = \sum_{v=n_1+t-h_{i+1}^2+1}^{n_1+t} c_v$. For if not then $\sum_{v=s+1}^{s+h_{i+1}^1} c_v < \sum_{v=n_1+t-h_{i+1}^2+1}^{n_1+t} c_v$ and together with Lemma 3.8 (3) and Lemma (3.9) , every root $x \in F_c$ would satisfy the equation $\sum_{i\in N_1} h_i^2 x_i + \sum_{i \in N_2} h_i^1 x_i = sh_i^2 + th_i^1$, which contradicts the assumption that F_c is a facet of $P(\mu, \lambda, F)$. Since there exists $h_i^1 \leq j < h_{i+1}^1$ with $c_{j+1} < \kappa h_i^2 = c_{s_1}$ or there exists $h_{i+1}^2 > j \geq h_i^2$ with $c_{t-j} > \kappa h_i^1 = c_{n_1+t_1+1}$ we obtain $\sum_{u=s+1}^{s+n_r(v)} c_u < \sum_{u=n_1+t-v+1}^{n_1+t} c_u$ for all $v = h_{i+1}^2, \ldots, h_\tau^2$ (Lemma 3.8 (2)) and $\sum_{u=s+1}^{s+n_r(v)} c_u < \sum_{u=n_1+t-v+1}^{n_1+t} c_u$ for all $v > h_\tau^2$ (Lemma 3.9). This implies that $c^T x \leq \gamma$ is of the type (3.3) or (3.4).

(b2) $c_{s_1} < c_{s+1}$ or $c_{n_1+t} > c_{n_1+t_1+1}$. By Lemma 3.8 (1) we obtain $\sum_{u=s+1}^{s+n_r(v)} c_u < \sum_{u=n_1+t-v+1}^{n_1+t} c_u$ for all $h_i^2 < v < h_{i+1}^2$. Further, if $\sum_{u=s+1}^{s+h_{i+1}^1} c_u < \sum_{u=n_1+t-h_{i+1}^2+1}^{n_1+t} c_u$, then by Lemma 3.8 (3) and Lemma 3.9, $\sum_{u=s+1}^{s+n_r(v)} c_u < \sum_{u=n_1+t-v+1}^{n_1+t} c_u$ for all $v > h_{i+1}^2$. Hence, every $x \in F_c$ would satisfy the equation $\sum_{i\in N_1} h_i^2 x_i + \sum_{i\in N_2} h_i^1 x_i = sh_i^2 + th_i^1$ and consequently, $dim(F_c) \leq |N| - 2$, a contradiction. Therefore, $\sum_{u=s+1}^{s+h_{i+1}^1} c_u = \sum_{u=n_1+t-h_{i+1}^2+1}^{n_1+t} c_u$. Taking Lemmas 3.8 and 3.9 into account, it follows that $c^T x \leq \gamma$ is of the type (3.5) or (3.6). This completes the analysis of the case (1), i.e., $s_1 = s + n_r(t - t_1)$.

(2) $s_1 < s + n_r(t - t_1)$. Then, $t_1 = t - 1$ for the following reasons:

Suppose, $t_1 < t-1$ and let s' denote the maximum number such that $s + n_r(s') \leq s_1$. Then, $s' < t - t_1$ follows and $s_1 - (s + n_r(s')) < n_r(1) \leq n(1) + 1$. Since the inequality is valid and due to the choice of t_1 and s_1 we conclude that $\sum_{u=s+1}^{s+n_r(s')} c_u < \sum_{u=n_1+t-s'+1}^{n_1+t} c_u$. Moreover, $\sum_{w=s+n_r(s')+1}^{s_1} c_w \leq \sum_{w=s+1}^{s+n(1)} c_w \leq c_{n_1+t} \leq c_{n_1+t-s'}$. Thus, $\sum_{w=s+1}^{s_1} c_w = \sum_{w=s+1}^{s+n_r(s')} c_w + \sum_{s+n_r(s')+1}^{s_1} c_w < \sum_{w=n_1+t-s'+1}^{n_1+t} c_w + c_{n_1+t-s'} \leq \sum_{w=n_1+t-t_1+1}^{n_1+t} c_w$, a contradiction since $\sum_{w=s+1}^{s_1} c_w = \sum_{w=n_1+t-t_1+1}^{n_1+t} c_w$. Therefore, $t_1 = t - 1$ and $s_1 < s + n_r(1)$ and we obtain $(\star)$ $\sum_{w=s+1}^{s_1} c_w = c_{n_1+t}$. Since $s_1 < s + n_r(1)$, for every $s_1 + 1 \leq v \leq n_1$ the vector $\sum_{w=s+1}^{s_1} e_w + e_v + \sum_{w=n_1+1}^{n_1+t_1} e_w$ is feasible. Together with $(\star)$ we obtain $c_v = 0$. Set $\kappa = s_1 - s$. Then, every $x \in F_c$ satisfies the equation $\sum_{i=1}^{s_1} x_i + \sum_{i=n_1+1}^{n_1+n_2} \kappa x_i = s + \kappa t$ and consequently, $c^T x \leq \gamma$ is of the type (3.12). ■

Let us end this section with a brief remark on the separation problem for the inequalities of $P(\mu, \lambda, F)$. Let $y \in \mathbb{R}^{N_1 \cup N_2}$ be a fractional solution. W.l.o.g. we assume that $N_1 = \{1, \ldots, n_1\}$, $N_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$ and $y_1 \geq \ldots \geq y_{n_1}$, $y_{n_1+1} \geq \ldots \geq y_{n_1+n_2}$. Choose a number $t \in [1, \ldots, n_2]$ and check whether there exists a violated inequality of the above types. Setting $r = F - t\lambda - \mu \lfloor \frac{F - t\lambda}{\mu} \rfloor$, we can construct the Hilbert basis $\mathcal{H}_r$ in polynomial time. For every $h_i \in \mathcal{H}_r$ we can check in polynomial time whether the inequality of type (3.1) is valid and in

case it is, whether it is violated.
For the inequalities (3.3) – (3.6) we can proceed accordingly. Having established the sorting of the values y_i in N_1 and N_2, the number t and the element h_i, we just check for every $j \in [1, \dots, h^1_{i+1} - 1]$ and $j \in [1, \dots, h^2_{i+1} - 1]$ whether the corresponding inequality of type (3.3), (3.4), (3.5) or (3.6) is valid and if so, whether it is violated. Since all numbers j, t and $h^1_i, h^2_i, h^1_{i+1}, h^2_{i+1}$ are polynomial in $|N|$, a polynomial running time for solving the separation problem is obtained. Similar arguments apply for separating the inequalities of type (3.12).

5 Extensions

Having seen how Hilbert basis elements of a certain cone can be transformed into valid inequalities for a special knapsack polytope, the natural question is how to extend or apply this concept to more general cases. Within this section we briefly discuss some directions that might be of interest for further investigations.
Let a 0/1 integer programming problem of the form $Ax \leq F$ be given where $a_i \in \mathbb{N}^m$, $i \in N$ denote the columns of A. Suppose that N can be partitioned into two sets N_1 and N_2 say, and suppose there exist values $\mu, \lambda, F' \in \mathbb{N}$ such that $t_1\mu + t_2\lambda \leq F'$ if and only if for every subset $T_1 \subseteq N_1$, $|T_1| = t_1$ and $T_2 \subseteq N_2$, $|T_2| = t_2$, the relation $\sum_{i \in T_1} a_i + \sum_{i \in T_2} a_i \leq F$ holds. Then, the convex hull of all 0/1 vectors satisfying $Ax \leq F$ is equal to $\mathrm{conv}\{x \in \{0,1\}^{N_1 \cup N_2} \mid \sum_{i \in N_1} \mu x_i + \sum_{i \in N_2} \lambda x_i \leq F'\}$ and an inequality description is given by Theorem 4.1.

One way to apply this result to more general cases is to incorporate the lifting and complementing of variables (see [6] and [11] for details on this subject). Roughly speaking, the idea is the following. Let $Ax \leq F$ be a 0/1 integer program where $a_i \in \mathbb{R}^m$, $i \in N$ denote the columns. We choose $C \subseteq N$ and $S \subseteq N \setminus C$ and consider the problem $\sum_{i \in S} a_i x_i \leq F - \sum_{i \in C} a_i$. If the set S can be partitioned into S_1 and S_2 and if there exist numbers μ, λ, F' such that $t_1\mu + t_2\lambda \leq F'$ if and only if $\sum_{i \in T_1} a_i + \sum_{i \in T_2} a_i \leq F$ for all $T_1 \subseteq S_1$, $|T_1| = t_1$ and $T_2 \subseteq S_2$, $|T_2| = t_2$, the inequalities (3.1), (3.3) – (3.6) and (3.12) are valid for $\mathrm{conv}\{x \in \{0,1\}^S \mid \sum_{i \in S_1} a_i x_i \leq F - \sum_{i \in C} a_i\}$. Any of these inequalities can now be expanded by ordering the variables in $N \setminus S$ and subsequently compute appropriate coefficients for those variables not considered before. Of course, each of these computation steps needs not be polynomial in the encoding length of the problem $Ax \leq F$ and for practical purposes approximate coefficients are usually determined. However, can one go further? Is it possible, for instance, to choose a valid inequality for $\mathrm{conv}\{x \in \{0,1\}^{N_1 \cup N_2} \mid \sum_{i \in N_1} \mu_1 x_i + \sum_{i \in N_2} \mu_2 x_i \leq F\}$ and one for $\mathrm{conv}\{x \in \{0,1\}^{N_3 \cup N_4} \mid \sum_{i \in N_3} \mu_3 x_i + \sum_{i \in N_4} \mu_4 x_i \leq G\}$ and combine them appropriately to obtain a valid inequality for $\mathrm{conv}\{x \in \{0,1\}^{N_1 \cup N_2 \cup N_3 \cup N_4} \mid \sum_{i=1}^{4} \sum_{j \in N_i} \mu_i x_j \leq F + G\}$? If this is true, what is the precise relation to the Hilbert basis associated with the exchange vectors of the knapsack problem $\sum_{i=1}^{4} \sum_{j \in N_i} \mu_i x_j \leq F + G$? Finding an answer to this problem is not only of theoretical interest, but also has an algorithmic impact, because Hilbert bases can be computed by the Buchberger algorithm. The latter algorithm plays an important role for computational algebra in the setting of Gröbner bases (see [2], [9]).

Besides this, there are several generalizations of the problem

$$\sum_{i \in N_1} \mu x_i + \sum_{i \in N_2} \lambda x_i \leq F$$

that seem interesting. How does the inequality description look like if we replace $x_i \in \{0,1\}$ by $x_i \in \{0,1,\ldots,u_i\}$ where u_i is some natural number ($i \in N$)?
Is there a way to extend Theorem 4.1 to the polytope $\mathrm{conv}\{x \in \{0,1\}^{\cup_{i=1}^{3} N_i} \mid \sum_{i \in N_1} x_i + \sum_{i \in N_2} \mu x_i + \sum_{i \in N_3} \lambda x_i \leq F\}$ where N_1, N_2, N_3 is the set of items with weight 1, μ, λ, respectively? Finally, the mixed integer program $\mathrm{conv}\{x \in \{0,1\}^{N_1 \cup N_2}, y \in [0,1]^{N_1 \cup N_2} \mid \sum_{i \in N_1 \cup N_2} y_i \leq F, y_i \leq \mu x_i, i \in N_1;\ y_i \leq \lambda x_i, i \in N_2\}$ is a natural generalization of the knapsack problem $\sum_{i \in N_1} \mu x_i + \sum_{i \in N_2} \lambda x_i \leq F$ and it certainly would be interesting to understand its facial structure.

Acknowledgement. I wish to thank Günter Ziegler and Alexander Martin for several valuable discussions on this subject.

Appendix

Proof of Lemma 3.7.

(i) Let $j < i$ be given. From the generation of the integer Hilbert basis $\mathcal{H}_r$ we know that $R_j > R_i$. Hence, $h_j^2 \lambda - h_j^1 \mu > h_i^2 \lambda - h_i^1 \mu$. Equivalently, $h_j^1(\frac{h_j^2}{h_j^1}\lambda - \mu) > h_i^1(\frac{h_i^2}{h_i^1}\lambda - \mu)$. Since $h_j^1 < h_i^1$, we conclude that $\frac{h_j^2}{h_j^1}\lambda - \mu > \frac{h_i^2}{h_i^1}\lambda - \mu$ and consequently, $\frac{h_j^2}{h_j^1} > \frac{h_i^2}{h_i^1}$. Moreover, $\frac{h_j^2}{h_j^1} h_i^1 - h_i^2 > \frac{h_i^2}{h_i^1} h_i^1 - h_i^2 = 0$. Then, $h_j^2 h_i^1 - h_i^2 h_j^1 > 0$ follows. This proves statement (i).

(ii) Let $j > i$ be given. In this case we have that $R_j < R_i$ and $h_j^1 > h_i^1$. By the same computations as outlined above the statement follows. ■

Proof of Lemma 3.8.

(1) Let $\mathcal{H}_r = \{h_1, \ldots, h_\tau\}$ be the Hilbert basis as introduced in Section 2 and let $(-n_r(t), t)$ be given with $h_i^2 < t < h_{i+1}^2$. From Lemma 2.5 we know that $(-n_r(t), t) = \sum_{u=1}^{i} \epsilon_u h_u$ where $\epsilon_u \geq 0$, integer and $\epsilon_i > 0$. Since $a_1 \geq a_2 \geq \ldots \geq a_{n_r(t_{max}(r))}$ and $b_1 \leq b_2 \leq \ldots \leq b_{t_{max}(r)}$ we obtain $\sum_{v=1}^{n_r(t)} a_v \leq \sum_{u=1}^{i} \epsilon_u \sum_{w=1}^{h_u^1} a_w \leq \sum_{u=1}^{i} \epsilon_u \sum_{w=1}^{h_u^2} b_w \leq \sum_{v=1}^{t} b_v$. If $a_1 > a_{h_i^1}$, then $\sum_{v=1}^{n_r(t)} a_v < \sum_{u=1}^{i} \epsilon_u \sum_{w=1}^{h_u^1} a_w$ and if $b_1 < b_{h_i^2}$, then $\sum_{u=1}^{i} \epsilon_u \sum_{w=1}^{h_u^2} b_w < \sum_{v=1}^{t} b_v$. Hence, statement (1) is true.

(2) Let $(-n_r(t), t)$ be given with $t > h_{i+1}^2$. We write $t = l + \sigma(h_{i+1}^2 - h_i^2)$ where $h_i^2 \leq l < h_{i+1}^2$. Now we first show that $n_r(t) < n_r(l) + \sigma(h_{i+1}^1 - h_i^1)$ holds.

$$
\begin{aligned}
\lfloor \tfrac{r+t\lambda}{\mu} \rfloor = \lfloor \tfrac{r+l\lambda+\sigma(h_{i+1}^2-h_i^2)\lambda}{\mu} \rfloor \quad & = \\
\lfloor \tfrac{r+l\lambda}{\mu} + \tfrac{\sigma(h_{i+1}^2-h_i^2)\lambda}{\mu} \rfloor \quad & = \\
\lfloor \tfrac{R((-n_r(l),l))+\sigma(R_{i+1}-R_i)}{\mu} + n_r(l) + \sigma(h_{i+1}^1 - h_i^1) \rfloor \quad & = \\
n_r(l) + \sigma(h_{i+1}^1 - h_i^1) + \lfloor \tfrac{R((-n_r(l),l))+\sigma(R_{i+1}-R_i)}{\mu} \rfloor \quad & \leq \\
n_r(l) + \sigma(h_{i+1}^1 - h_i^1) \quad &
\end{aligned}
$$

Henceforth, $\sum_{v=1}^{n_r(t)} a_v \leq \sum_{v=1}^{n_r(l)+\sigma(h_{i+1}^1-h_i^1)} a_v =: A$ and

$$
\begin{aligned}
A &= \sum_{v=1}^{n_r(l)} a_v + \sum_{u=0}^{\sigma-1} \sum_{v=n_r(l)+u(h_{i+1}^1-h_i^1)+1}^{n_r(l)+(u+1)(h_{i+1}^1-h_i^1)} a_v \\
&\leq \sum_{v=1}^{n_r(l)} a_v + \sum_{u=0}^{\sigma-1} \sum_{v=h_i^1+u(h_{i+1}^1-h_i^1)+1}^{h_i^1+(u+1)(h_{i+1}^1-h_i^1)} a_v \\
&\leq \sum_{v=1}^{n_r(l)} a_v + \sigma \sum_{v=h_i^1+1}^{h_{i+1}^1} a_v \\
&= \sum_{v=1}^{n_r(l)} a_v + \sigma \sum_{v=h_i^2+1}^{h_{i+1}^2} b_v \\
&\leq \sum_{v=1}^{l} b_v + \sigma \sum_{v=h_i^2+1}^{h_{i+1}^2} b_v \\
&\leq \sum_{v=1}^{l} b_v + \sum_{u=0}^{\sigma-1} \sum_{v=l+u(h_{i+1}^2-h_i^2)+1}^{l+(u+1)(h_{i+1}^2-h_i^2)} b_v \\
&= \sum_{v=1}^{t} b_v.
\end{aligned}
$$

In fact, strict inequality is true if $a_{h_i^1+1} > a_{h_{i+1}^1}$ or $b_{h_i^2+1} < b_{h_{i+1}^2}$.

(3) This statement can be shown similarly to the proof of (ii). ■

Proof of Lemma 3.9.

This proof is analogous to the proof of Lemma 3.8. We briefly outline the steps. Let $(-n_r(t), t)$ be given with $t > h_\tau^2$. We write $t = \sigma h_\tau^2 + t'$ where $0 \leq t' < h_\tau^2$ and by similar arguments as used in the proof of Lemma 3.8 we obtain $n_r(t) \leq \sigma h_\tau^1 + n_r(t')$. Due to the ordering of the numbers a_i, $i = 1, \ldots, n_1$ and b_i, $i = 1, \ldots, n_2$ and since $\sum_{v=1}^{n_r(t')} a_v \leq \sum_{v=1}^{t'} b_v$, we easily obtain statements (1), (2) and (3). ■

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