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Symmetry Breaking Bifurcations of Chaotic Attractors

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Research supported in part by the Deutsche Forschungsgemeinschaft,
the Konrad-Zuse-Zentrum für Informationstechnik Berlin
and ONR Grant N00014-94-1-0317.

Preprint SC 94–27 (October 1994)

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Abstract

In an array of coupled oscillators *synchronous chaos* may occur in the sense that all the oscillators behave identically although the corresponding motion is chaotic. When a parameter is varied this fully symmetric dynamical state can lose its stability, and the main purpose of this paper is to investigate which type of dynamical behavior is expected to be observed once the loss of stability has occurred. The essential tool is a classification of Lyapunov exponents based on the symmetry of the underlying problem. This classification is crucial in the derivation of the analytical results but it also allows an efficient computation of the dominant Lyapunov exponent associated with each symmetry type. We show how these dominant exponents determine the stability of invariant sets possessing various instantaneous symmetries and this leads to the idea of *symmetry breaking bifurcations of chaotic attractors*. Finally the results and ideas are illustrated for several systems of coupled oscillators.

Keywords: Lyapunov exponents, coupled oscillators, symmetry, bifurcation

Contents

1 Introduction

The use of symmetry in the analysis of classical local bifurcation problems is now well established [15]. More recently, symmetry has been used to aid understanding of chaotic dynamical behavior, e.g. [7, 6, 8, 22, 3]. Particular attention has been paid to *symmetry increasing bifurcations*, at which symmetrically related attractors collide to give one attractor with more symmetry than the ones before the collision. In those cases, typically, the attractors before as well as after the bifurcation possess symmetry just on average in the sense that the attractors as sets are left invariant under certain symmetry transformations but most points inside the attractor have no symmetry properties. Such a symmetry of an attractor is inherited by the underlying invariant measure and this fact can be used to produce striking colored pictures (see [11]).

In this article we consider *symmetry breaking bifurcations* of chaotic attractors. For motivation consider a system of partial differential equations on the line — with periodic boundary conditions, say — that behaves chaotically although the corresponding motion is constant in space at each instant of time. In other words, we have a complicated dynamical motion which has both translational and reflectional symmetry in space at each instant of time. The results of this paper give insight — from a symmetry point of view — into the change in the dynamics which is expected to occur if the variation of a system parameter leads to the instability of such a fully symmetric dynamical state.

Another example is provided by the consideration of *synchronous chaos* in coupled oscillator systems (see [16], [27]). Synchronous chaos describes a situation for which all the oscillators are behaving identically and chaotically. In this context one of the main interests lies in the derivation of criteria which guarantee the stability of the synchronous chaotic state (see [16]). Our considerations give rise to such criteria as well as to results concerning the symmetry type of the invariant sets which exist once synchronous chaos has become unstable.

Abstractly speaking, we are concerned with the following scenario: before the bifurcation the dynamical behavior is restricted to a fixed point space of the underlying group action and we show that a loss of stability may lead to the existence of invariant sets which have strictly less symmetry than the attracting set before criticality. Hence, in contrast to symmetry increasing bifurcations, the attractors possess nontrivial instantaneous symmetries — at least before the bifurcation — and a bifurcation leads to a decrease in these symmetry properties. Simple Z_2 symmetry breaking bifurcations of chaotic attractors due to a crisis have been observed before [28]. However, our approach is strongly related to the one in [1] in the sense that we also compute certain Lyapunov exponents to obtain stability results. In fact, we will show that Lyapunov exponents can be classified in terms of symmetry properties, and, moreover, it will be possible to derive restrictions on the symmetries of the invariant sets existing after the bifurcation by taking the different symmetry types of Lyapunov exponents into account.

Let us be more precise. In [1] it has been shown that for the motion of a two-dimensional dynamical system inside an invariant subspace the associated two Lyapunov exponents — one belonging to the motion inside the subspace, the other one belonging to a normal direction — can be computed by two limiting processes where for each one just a single entry of the underlying Jacobian is involved. Using the symmetry we will show that in general a related decomposition of the Jacobian can always be obtained as soon as the invariant set under consideration is lying inside a nontrivial fixed point space of the group action. In fact, the Jacobian matrix for the system decomposes into diagonal blocks associated with the isotypic components of the underlying space, and hence the Lyapunov exponents can be labelled according to those different blocks. For classical steady state or Hopf bifurcation problems it is a well known fact that this procedure is very efficient in the detection and computation of bifurcation points since it significantly reduces the numerical effort for the computation of corresponding eigenvalues (e.g. [30], [29], [13], [4]). Here we will make use of this reduction to compute the dominant Lyapunov exponent associated with each symmetry type in an efficient way.

Simultaneously the classification of Lyapunov exponents leads to some results determining the structure of the invariant sets which exist once a Lyapunov exponent associated with a certain isotypic component becomes positive. In particular, we will find restrictions on the symmetry types of those sets by means of the type of the isotypic component which is involved (cf. Theorem 3.5). Moreover, we also show that further bifurcations can sometimes be detected based only on the computation of the Lyapunov exponents for the (unstable) fully symmetric flow. Since in simulations this motion is much cheaper to compute than the complete dynamical behavior, this result is of particular importance for a numerical stability analysis.

A more detailed outline of the paper is as follows. In Sec. 2 we begin by recalling the notion of Lyapunov exponents and we introduce the reader to the basic group representation theory that is needed. Then by a combination of these two notions we obtain the desired blockdiagonal structure of the Jacobian which allows a classification as well as an efficient computation of Lyapunov exponents. Restrictions on the symmetry types of invariant sets existing after symmetry breaking bifurcations are stated in Sec. 3. In Sec. 4 we concentrate on the specific case of coupled oscillators, and we will show analytically that for a certain type of coupling there is a strong relationship between the magnitudes of Lyapunov exponents belonging to the different symmetry types. Finally, in Sec. 5, all the ideas and results are illustrated for several systems of coupled oscillators. In particular, we consider coupled Lorenz equations and coupled Duffing oscillators.

2 Lyapunov exponents and symmetry

2.1 Some basic notation

We denote by $\varphi_t : \mathrm{R}^n \to \mathrm{R}^n$ the flow of an ordinary differential equation

$$\dot{x} = f(x),$$

where $f : \mathrm{R}^n \to \mathrm{R}^n$ may additionally depend on parameters. Our main purpose is to analyse symmetry related changes in complicated dynamical behavior. We treat these dynamical states as invariant sets in phase space and we will frequently make use of the following ideas: A subset $A \subset \mathrm{R}^n$ is

1. an *invariant* set if
$$\varphi_t(A) = A \quad \text{for all } t \geq 0;$$
2. an *attracting set* with *fundamental neighborhood* U if A is an invariant set and for every open set $V \supset A$ we have $\varphi_t(U) \subset V$ for sufficiently large t. In particular, if A is closed,
$$A = \bigcap_{t \geq 0} \varphi_t(U);$$
3. an *attractor* if it is a compact attracting ω-limit set $\omega(x)$ for some $x \in \mathrm{R}^n$.

Recall that an ω-limit set $\omega(x)$ is defined as

$$\omega(x) = \{y \in \mathrm{R}^n : \text{there exists } t_j \to \infty \text{ such that } \varphi_{t_j}(x) \to y.\}$$

Obviously asymptotically stable steady states and stable periodic orbits are attractors. Moreover there is the following more general example (see [26]):

Example 2.1 If $U \subset \mathrm{R}^n$ is open and the closure of $\varphi_t(U)$ is compact and contained in U for all sufficiently large t, then the set $\cap_{t \geq 0} \varphi_t(U)$ is a compact attracting set with fundamental neighborhood U.

2.2 Lyapunov exponents

In this subsection we recall the definition of Lyapunov exponents and some of their basic properties. We will present a measure theoretic approach and follow essentially part of the exposition in [10].

An *invariant measure* μ associated with the underlying dynamical system is a measure for which

$$\mu(\varphi_t(B)) = \mu(B) \quad \text{for all measurable } B \subset \mathrm{R}^n \text{ and } t \geq 0.$$

To avoid technicalities we do not specify the underlying measure space. However, we implicitly assume that we are working on a compact space with the usual Borel σ-algebra.

For a given invariant set A there always exists an invariant measure with $\mu(A) = 1$ (cf. [21]). Additionally this measure may be chosen to be *ergodic*, i.e.,

$$\mu(B) \in \{0, 1\}$$

for each invariant set $B \subset \mathrm{R}^n$.

Let $y(t)$ be the solution of the underlying dynamical system with $y(0) = x$ and let $\Phi_x(t)$ be the solution of the variational equation

$$\dot{\Phi} = Df(y(t))\Phi$$

with initial condition $\Phi(0) = I$.

The following result was first proved in [23].

Theorem 2.2 *Let μ be an ergodic measure with compact support. Then, for μ-almost all x, the following limits exist:*

$$\lim_{t\to\infty} \left(\Phi_x(t)^t \Phi_x(t)\right)^{\frac{1}{2t}} = \Lambda_x,$$

$$\lim_{t\to\infty} \frac{1}{t} \ln \|\Phi_x(t)v\| = \lambda^{(i)} \quad \text{if } v \in E_x^{(i)} - E_x^{(i+1)},$$

where $\lambda^{(1)} > \lambda^{(2)} > \cdots$ are the logarithms of the eigenvalues of Λ_x, and $E_x^{(i)}$ is the sum of the eigenspaces corresponding to the eigenvalues which are less than or equal to

The numbers $\lambda^{(i)}$ are called *Lyapunov exponents*. Notice that these numbers depend on the underlying ergodic measure μ.

A nonlinear analogue of the subspaces $E_x^{(i)}$ is provided by the concept of *invariant manifolds*. Let $\lambda < 0$, $\epsilon > 0$ and define

$$V_x^s(\lambda, \epsilon) = \{y \in \mathrm{R}^n : d(\varphi_t(x), \varphi_t(y)) \le \epsilon e^{\lambda t} \quad \text{for all } t \ge 0\},$$

where $d(u, v)$ denotes the distance of u and v. If for two Lyapunov exponents $\lambda^{(i-1)} < \lambda < \lambda^{(i)}$, then the set $V_x^s(\lambda, \epsilon)$ is in fact, for μ-almost all x and small ϵ, a piece of a differentiable manifold, called the *local stable manifold* at x. This manifold is tangent at x to the linear subspace $E_x^{(i)}$ and has the same dimension.

The *global stable manifold* at x is defined by

$$V_x^{(i)s} = \bigcup_{t>0} \varphi_{-t}(V_x^s(\lambda, \epsilon)),$$

with negative λ between $\lambda^{(i-1)}$ and $\lambda^{(i)}$. We also define the *stable manifold* at x by

$$V_x^s = \left\{ y \in \mathrm{R}^n : \lim_{t\to\infty} \frac{1}{t} \ln d(\varphi_t(x), \varphi_t(y)) < 0 \right\}.$$

One can show that this is the largest of the stable manifolds, i.e. it is equal to $V_x^{(i)s}$ where $\lambda^{(i)}$ is the largest negative Lyapunov exponent.

The *unstable manifolds* $V_x^u(\lambda, \epsilon)$, $V_x^{(i)u}$ and V_x^u are defined analogously (one essentially has to replace t by $-t$ in the definitions).

Theorem 2.3 ([10]) *If A is a closed attracting set and $x \in A$ then $V_x^u \subset A$.*

We state the proof of this theorem since it is both simple and illustrative.

Proof: Let U be a fundamental neighborhood of A and $y \in V_x^u$. Then $\varphi_{-\tau}(y) \in U$ for sufficiently large τ. To see this observe that $\varphi_{-\tau}(y)$ is close to $\varphi_{-\tau}(x) \in A$. Therefore

$$y \in \cap_{\tau > T} \varphi_\tau(U) = \cap_{\tau \geq 0} \varphi_\tau(U) = A.$$

∎

We end this section with the following basic but crucial observation which will be used frequently in Sec. 3.

Proposition 2.4 *Let $V \subset \mathrm{R}^n$ be a subspace and let $A \subset V$ be a closed invariant set inside V. Suppose that $B \subset A$ is an invariant set such that for a corresponding ergodic measure μ_B one of the Lyapunov exponents $\lambda^{(i)}$ is greater than zero and the related unstable subspace is not in V. Then the invariant set A cannot be attracting.*

Proof: Observe that μ_B is also an ergodic measure for A. Since A is closed and since the unstable subspace is contained in the tangent space of $V_x^{(i)u}$ at x the result now immediately follows from Theorem 2.3. ∎

2.3 Symmetry

Our aim is to show how symmetry can be used to classify different types of Lyapunov exponents and how this may simplify their computation. For this we have to introduce some basic facts from group representation theory. To make the reading of this subsection more comprehensible we start with a simple example which illustrates the general ideas and the results below.

Suppose that we want to compute eigenvalues of a matrix M which commutes with an involution R,

$$RM = MR \quad \text{and} \quad R^2 = I.$$

We show that this problem can be decomposed into two subproblems of lower dimension. The crucial observation is that the subspaces consisting of symmetric and antisymmetric vectors, i.e. the spaces

$$\begin{aligned} X_1 &= \{v \in \mathrm{R}^n : Rv = v\}, \\ X_2 &= \{v \in \mathrm{R}^n : Rv = -v\}, \end{aligned}$$

are left invariant under M. It can easily be verified that

$$\mathrm{R}^n = X_1 \oplus X_2,$$

and therefore M has a blockdiagonal structure according to this decomposition. As a consequence, by a change of coordinates one can decompose the eigenvalue problem into two different subproblems, namely the computation of eigenvalues of M restricted to X_1 and X_2 respectively.

Example 2.5 We consider the matrix

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 3 & 0 \\ 1 & 0 & 1 & 2 \\ 3 & 0 & 3 & 2 \end{pmatrix},$$

which commutes with

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Obviously $R^2 = I$ and the subspaces X_1 and X_2 of symmetric and antisymmetric vectors can be written as

$$\begin{aligned} X_1 &= \mathrm{R}\{(1,0,1,0)^t, (0,1,0,1)^t\}, \\ X_2 &= \mathrm{R}\{(1,0,-1,0)^t, (0,1,0,-1)^t\}. \end{aligned}$$

In those coordinates M takes the form

$$M = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 6 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

and the eigenvalues are 0, 2 and $2(1 \pm \sqrt{3})$.

After this introductory example we now consider the general case in which a compact Lie group Γ (e.g. a finite group) of transformations in $\mathbf{O}(n)$ is acting on R^n. We assume that a matrix $M \in \mathrm{R}^{n,n}$ commutes with Γ,

$$\gamma M = M\gamma \quad \text{for all } \gamma \in \Gamma.$$

In Example 2.5 $\Gamma = \{I, R\} \cong \mathrm{Z}_2$, where Z_2 is the group generated by one reflection.

The *fixed point space* $\mathrm{Fix}(\Sigma)$ of a subgroup Σ of Γ is the subspace of R^n in which each element is fixed under all the transformations in Σ,

$$\mathrm{Fix}(\Sigma) = \{v \in \mathrm{R}^n : \sigma v = v \quad \text{for all } \sigma \in \Sigma\}.$$

We will also make use of some elementary representation theory which we now briefly describe. For a more detailed introduction the reader is referred to the book [15].

A subspace $V \subset \mathrm{R}^n$ is called Γ-*invariant* if

$$\gamma v \in V \quad \text{for all } v \in V,\ \gamma \in \Gamma.$$

Of particular importance are the “smallest” nontrivial Γ-invariant subspaces: The invariant subspace V is Γ-*irreducible* if it contains no proper nontrivial Γ-invariant subspace. One can identify — and we will do this frequently — the irreducible subspace with the underlying abstract *irreducible representation*, which is represented by the set of transformations realising the action of Γ on the irreducible subspace V.

It is known (see e.g. [19]) that for an irreducible subspace V the set

$$\mathcal{L}_\Gamma(V) = \{L : V \to V \text{ linear} : \gamma L = L\gamma \text{ for all } \gamma \in \Gamma\}$$

is isomorphic to R, C or H, where H denotes the quaternions. This result enables a classification of irreducible subspaces and according to this we have irreducible subspaces (or representations) of *real*, *complex* or *quaternionic* type.

We may decompose R^n as a direct sum of irreducible subspaces

$$\mathrm{R}^n = V_1 \oplus \cdots \oplus V_k,$$

and group together those V_i on which Γ acts isomorphically. Then we obtain the *isotypic decomposition*

$$\mathrm{R}^n = X_1 \oplus \cdots \oplus X_l,$$

where each *isotypic component* X_j is the sum of isomorphic irreducible subspaces.

Theorem 2.6 *The isotypic decomposition is unique, and, moreover, each isotypic component is left invariant by matrices which commute with the action of* Γ.

Example 2.7 In the introductory example X_1 and X_2 are precisely the isotypic components corresponding to the two nonisomorphic one-dimensional irreducible representations of Z_2.

Since each isotypic component is an invariant subspace for M there is a blockdiagonalisation of M according to the decomposition into isotypic components. Using this we classify the eigenvalues of M in the following way: We say that an eigenvalue λ of M corresponds to the irreducible subspace (or representation) V if λ is an eigenvalue of a block which belongs to the corresponding isotypic component.

We finish our digression to group representation theory with the following observation which is of particular interest in numerical implementations. In the case where irreducible representations of dimension greater or equal to two are present in an isotypic component there is an even finer decomposition of R^n into M-invariant subspaces (cf. eg [17], [29]). To see this write one isotypic component X as $X = V \oplus \cdots \oplus V$ where the V's are isomorphic irreducible subspaces of R^n. Denote by M_X the restriction of M to X. Then, according to this decomposition of X, M_X can be written as

$$M_X = (M_{ij})_{1 \leq i,j \leq l}, \quad M_{ij} : V \to V.$$

It follows from a simple computation that each M_{ij} has to commute with the action of Γ, i.e. $M_{ij} \in \mathcal{L}_\Gamma(V)$. Hence, each M_{ij} can be identified with a number in R, C or H depending on the type of the irreducible subspace V. Therefore M_X defines nothing else than an $m \times m$ matrix over R, C or H respectively.

We summarise this in the following theorem.

Theorem 2.8 ([17],[29]) *Let M be a matrix that commutes with the action of a compact Lie group Γ. Let X be the isotypic component corresponding to the irreducible representation V.*

Then X decomposes into subspaces $W_i \subset X$ with the following properties:

1. *The dimension of W_i is*
$$\dim W_i = d \cdot \frac{\dim X}{\dim V},$$
where $d = 1, 2, 4$ if V is of real, complex or quaternionic type respectively.

2. *The W_i are invariant under M and all these spaces are transformed in the same way.*

Remark 2.9 The projections onto the isotypic components and their invariant subspaces can easily be computed. They can also be found in eg. [17], [29], [4].

To illustrate the foregoing notions and results we end this subsection with the following example, which we will meet again in the examples of Sec. 5.

Example 2.10 We consider the action of the finite group $\Gamma = \mathbf{D}_4$ on R^8. Here $\mathbf{D}_4$ denotes the dihedral group of order four which is the symmetry group of the square. This group consists of 8 elements, which in this case are generated by the transformations

$$R = \begin{pmatrix} 0 & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix},$$

where I is the 2×2 identity matrix. Geometrically, R and S can be interpreted as the counterclockwise rotation by 90° and a reflection respectively, both leaving a square invariant.

It is easy to verify that commuting matrices have the form

$$\begin{pmatrix} A & B & C & B \\ B & A & B & C \\ C & B & A & B \\ B & C & B & A \end{pmatrix}, \tag{2.1}$$

where A, B and C are arbitrary real 2×2 matrices. It is easy to check that the subspaces

$$\mathrm{R}^8, \quad \mathrm{R}\{(1,0,1,0,1,0,1,0)^t, (1,0,-1,0,1,0,-1,0)^t\}$$

are Γ-invariant but not Γ-irreducible, whereas the following three subspaces are additionally Γ-irreducible:

$$\mathrm{R}\{(1,0,1,0,1,0,1,0)^t\}, \quad \mathrm{R}\{(1,0,-1,0,1,0,-1,0)^t\},$$

$$\mathrm{R}\{(0,0,1,0,0,0,-1,0)^t, (1,0,0,0,-1,0,0,0)^t\}.$$

In fact, up to isomorphism, these are the only irreducible representations which are present and the corresponding three isotypic components are

$$\begin{aligned} X_1 &= \{(z,z,z,z)^t \in \mathrm{R}^8 : z \in \mathrm{R}^2\}, \\ X_2 &= \{(z,-z,z,-z)^t \in \mathrm{R}^8 : z \in \mathrm{R}^2\}, \\ X_3 &= \{(z_1,z_2,-z_1,-z_2)^t \in \mathrm{R}^8 : z_j \in \mathrm{R}^2, j = 1,2\}. \end{aligned}$$

Hence

$$\mathrm{R}^8 = X_1 \oplus X_2 \oplus X_3$$

and each X_i is an invariant subspace for all the matrices commuting with Γ.

Finally, since all the irreducible representations of $\mathbf{D}_4$ are of real type, we know by Theorem 2.8 that there exist two two-dimensional invariant subspaces in X_3. Those can be written as

$$\begin{aligned} X_3^1 &= \{(z, 0, -z, 0)^t \in \mathrm{R}^8 : z \in \mathrm{R}^2\}, \\ X_3^2 &= \{(0, z, 0, -z)^t \in \mathrm{R}^8 : z \in \mathrm{R}^2\} \end{aligned}$$

and with respect to the decomposition

$$\mathrm{R}^8 = X_1 \oplus X_2 \oplus X_3^1 \oplus X_3^2$$

the matrix in (2.1) takes the form

$$\begin{pmatrix} A + C + 2B & 0 & 0 & 0 \\ 0 & A + C - 2B & 0 & 0 \\ 0 & 0 & A - C & 0 \\ 0 & 0 & 0 & A - C \end{pmatrix}.$$

This establishes the final blockdiagonalisation.

2.4 Lyapunov exponents and symmetry

In this subsection we combine the notions of Lyapunov exponents and symmetry. We assume that the underlying dynamical system,

$$\dot{x} = f(x),$$

is Γ-*equivariant*, i.e. the mapping f satisfies

$$\gamma f(x) = f(\gamma x) \quad \text{for all } \gamma \in \Gamma, \tag{2.2}$$

where Γ is a compact Lie group. Let us illustrate the notion of Γ-equivariance by two examples.

Example 2.11 (a) In the *Lorenz system*

$$\begin{aligned} \dot{x} &= \sigma(y - x), \\ \dot{y} &= \rho x - y - xz, \\ \dot{z} &= -\beta z + xy, \end{aligned}$$

where σ, ρ and β are real parameters, the first two components of the right hand side are odd in x and y whereas the third component is even in x and y.

Hence (2.2) is satisfied for $\Gamma = \{I, \kappa\} \cong \mathrm{Z}_2$, where I is the 3×3-identity matrix and

$$\kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) We consider a system of p identical *coupled oscillators* of the form

$$\dot{z}_j = g(z_{j-1}, z_j, z_{j+1}), \quad (j = 1, \ldots, p),$$

where $z_j \in \mathrm{R}^m$, and $g(u, v, w) = g(w, v, u)$. (Here $z_0 = z_p$ and $z_{p+1} = z_1$.) Systems of this type naturally arise via a spatial discretisation of a partial differential equation on the real line with periodic boundary conditions.

In this case the symmetry group Γ is isomorphic to the dihedral group $\mathbf{D}_p$, the symmetry group of the regular p-gon, and the reflections and rotations of Γ are represented by block-permutation matrices. An example of such a group action has already been given in Example 2.10. In that example we had $p = 4$ and $m = 2$.

Due to the equivariance of the underlying dynamical system the dynamical behavior of the system may also possess symmetry properties. This has been known for a long time for steady states or periodic solutions, and, more recently, the concept has been extended to cover additionally more complicated dynamical behavior (see e.g. [7], [8], [6] or [9]). Mathematically this is phrased in terms of symmetries of invariant sets. The symmetry $S(A)$ of an invariant set A is given by the subgroup of Γ which fixes A setwise,

$$S(A) = \{\gamma \in \Gamma : \gamma A = A\}.$$

Observe that an invariant set may have the symmetry $S(A) = \Gamma$ although every single point inside A has less symmetry. In this sense $S(A)$ just describes the symmetry on average of the underlying dynamical behavior. Examples of this situation are given in Sec. 5.

For our purposes we additionally have to introduce the subgroup of $S(A)$ which fixes each single point inside A (cf. [22]),

$$S_{fix}(A) = \{\gamma \in \Gamma : \gamma x = x \quad \text{for all } x \in A\}.$$

Note that

$$A \subset \mathrm{Fix}(\Sigma) \text{ if and only if } \Sigma \subset S_{fix}(A).$$

Lemma 2.12 *Suppose that A_Γ is an invariant set with $S_{fix}(A_\Gamma) = \Gamma$. For $x \in A_\Gamma$ let $y(t)$ be the trajectory with $y(0) = x$. Then the solution $\Phi_x(t) \in \mathrm{R}^{n,n}$ of the corresponding variational equation*

$$\dot{\Phi} = Df(y(t))\Phi, \quad \Phi(0) = I, \tag{2.3}$$

commutes with the action of Γ,

$$\gamma\Phi_x(t) = \Phi_x(t)\gamma \quad \textit{for all } \gamma \in \Gamma \textit{ and } t \geq 0.$$

Proof: First observe that for every point $x \in \mathrm{Fix}(\Gamma)$ the Jacobian $Df(x)$ commutes with the action of Γ, i.e.

$$\gamma Df(x) = Df(x)\gamma \quad \text{for all } \gamma \in \Gamma.$$

This is immediate by the chain rule. In particular, since A_Γ is an invariant set inside $\mathrm{Fix}(\Gamma)$,

$$\gamma Df(y(t)) = Df(y(t))\gamma \quad \text{for all } \gamma \in \Gamma \text{ and } t \geq 0.$$

It follows that whenever Φ is a solution of (2.3) then $\gamma\Phi$ and $\Phi\gamma$ are solutions of the variational equation with initial condition $\gamma\Phi(0) = \Phi(0)\gamma = \gamma$. Since solutions of such initial value problems are unique we obtain the desired result. □

By this lemma we may apply the results of the previous subsection to see that there is a time independent change of coordinates such that for all $t \geq 0$ the matrices $\Phi(t)$ can be transformed into a blockdiagonal structure related to the different irreducible representations of Γ (see Theorem 2.6). Combining this with the facts that the elements of Γ are orthogonal matrices and that for any power of a positive definite matrix the eigenspaces are the same we arrive at the following result.

Proposition 2.13 *The matrix*

$$\Lambda_x = \lim_{t\to\infty} (\Phi_x(t)^t \Phi_x(t))^{\frac{1}{2t}}$$

commutes with the action of Γ.

As a consequence of Proposition 2.13 the matrix Λ_x can also be transformed into a blockdiagonal structure and its eigenvalues can be classified by the corresponding irreducible representations (see Theorem 2.6). By Theorem 2.2 the logarithms of these eigenvalues are the Lyapunov exponents and in this way we obtain the desired classification of Lyapunov exponents by means of symmetry.

Remark 2.14 Since there exist irreducible representations of dimension greater than one it follows that in systems with symmetry generically multiple Lyapunov exponents may occur (cf. Theorem 2.8).

In the examples of Sec. 5, we are mainly concerned with finite groups. However, we prove here one result for continuous groups. It is well known that problems which are equivariant with respect to continuous groups have singular Jacobian matrices when evaluated at steady state solutions. It has also recently been proved that maps with O(2) symmetry always have one zero Lyapunov exponent [5]. We now generalise this result for general symmetry groups.

Theorem 2.15 *Suppose that f satisfies the equivariance condition (2.2) for some (continuous) Lie group Γ with $\dim\Gamma = d$. Let Σ be an isotropy subgroup of Γ of dimension d_Σ and let μ_Σ be an ergodic measure with compact support on* $\mathrm{Fix}(\Sigma)$*. Suppose that the corresponding invariant set A in* $\mathrm{Fix}(\Sigma)$ *has maximal symmetry Σ. Then $f_\Sigma = f|\mathrm{Fix}(\Sigma)$ has $d - d_\Sigma$ zero Lyapunov exponents.*

Proof: Let $\mathcal{L}$ be the Lie algebra of Γ and let $\mathcal{L}_\Sigma$ be the subalgebra associated with the subgroup Σ. By differentiating with respect to the one-parameter subgroups of Γ, it is easily verified that for any x

$$gf(x) = Df(x)gx, \quad \text{for all } g \in \mathcal{L}.$$

Let $y(t)$ be a trajectory with $y(0) = x \in \mathrm{Fix}(\Sigma)$. Then

$$\begin{aligned} \dot{\overline{gy}} &= g\dot{y} \\ &= gf(y(t)) \\ &= Df(y(t))gy(t). \end{aligned}$$

Thus, $gy(t)$ is a solution of the variational equation and is therefore given by $gy(t) = \Phi_x(t)v$ for some v. Setting $t = 0$ gives $gy(0) = gx = \Phi_x(0)v = v$. Thus,

$$gy(t) = \Phi_x(t)gx.$$

If $g \in \mathcal{L}_\Sigma$, then $gx = 0$ and so the derived solution of the variational equation is the trivial one. Similarly, for each $g \in \mathcal{L} - \mathcal{L}_\Sigma$, $gy(t)$ is a nontrivial solution of the variational equation and so choosing $v = gx$ in Theorem 2.2 gives a Lyapunov exponent for μ_Σ almost all x which we denote λ_g, given by

$$\begin{aligned} \lambda_g &= \lim_{t\to\infty} \frac{1}{t} \ln ||\Phi_x(t)gx|| \\ &= \lim_{t\to\infty} \frac{1}{t} \ln ||gy(t)|| \\ &\leq \lim_{t\to\infty} \frac{1}{t} \ln(||g||\,||y(t)||) \\ &\leq \lim_{t\to\infty} \frac{1}{t} \ln(C_1 C_2) \end{aligned}$$

where $||g|| \leq C_1$ since g is a bounded linear operator and $||y(t)|| \leq C_2$ since A is compact and the trajectory is therefore bounded. Thus, $\lambda_g \leq 0$.

Suppose now that $\lambda_g < 0$. We aim to find a contradiction. From the above, we have $\lambda_g = \lim_{t\to\infty} \frac{1}{t} \ln ||gy(t)||$ and so, for any $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all $t > N$

$$\left| \frac{1}{t} \ln ||gy(t)|| - \lambda_g \right| < \epsilon.$$

This implies that

$$\frac{1}{t} \ln ||gy(t)|| < \lambda_g + \epsilon$$

and since $\lambda_g < 0$, we can choose $\epsilon > 0$ such that $\lambda_g + \epsilon < 0$. Then

$$||gy(t)|| < e^{t(\lambda_g+\epsilon)} \quad \to \quad 0 \quad \text{as} \quad t \to \infty.$$

Thus, $\lim_{t\to\infty} gy(t) = 0$.

Since $g \notin \mathcal{L}_\Sigma$ then $gx \neq 0$ and so $\lim_{t\to\infty} gy(t) \neq 0$, by the assumption that the maximal symmetry of A in $\mathrm{Fix}(\Sigma)$ is Σ. (We note that if $x \in \mathrm{Fix}(\Sigma)$ but A has symmetry Δ for some subgroup Δ of Γ such that $\Sigma \subset \Delta$ and $\dim \Delta > \dim \Sigma = d_\Sigma$, then $\lim_{t\to\infty} gy(t)$ will be zero for some $g \in \mathcal{L} - \mathcal{L}_\Sigma$.) This gives a contradiction and so we conclude that $\lambda_g \geq 0$.

Combining these results gives $\lambda_g = 0$. Thus, by taking $d - d_\Sigma$ linearly independent elements of $\mathcal{L} - \mathcal{L}_\Sigma$, we obtain $d - d_\Sigma$ linearly independent solutions of the variational equation which give rise to $d - d_\Sigma$ zero Lyapunov exponents. $\square$

Remark 2.16 This result applies to noncompact as well as compact groups. A special case of this result is autonomous dynamical systems which have the continuous, one dimensional group of time translations. Thus, any trajectory which does not settle down to a steady state solution will give rise to a zero Lyapunov exponent, as is well known for autonomous systems.

2.5 Efficient computation of Lyapunov exponents

The dominant Lyapunov exponent associated with each isotypic component is the most important quantity to compute since it indicates whether the invariant set A_Γ is stable with respect to perturbations in the direction of the particular isotypic component. Using the block decomposition of Theorem 2.8, these are now easily computed.

Let W_k be one copy of the invariant subspaces in Theorem 2.8 corresponding to the isotypic component X_k, $k = 1, \ldots, \ell$. Suppose that $y(t) \in \mathrm{Fix}(\Gamma)$ is a fully symmetric solution of the underlying dynamical system. Then at each time the

Jacobian $Df(y(t))$ leaves the space W_k invariant and we denote its restriction by $D_k(y(t)) : W_k \to W_k$. It follows that the dominant Lyapunov exponent associated with the isotypic component X_k is given by

$$\lim_{t\to\infty} \frac{1}{t} \ln \|\Phi_k(t)\| \tag{2.4}$$

where $\Phi_k(t)$ is the solution of the reduced variational equation

$$\dot{\Phi}_k = D_k(y(t))\Phi_k$$

with initial condition $\Phi_k(0) = I$. Obviously, the computation of the smaller Lyapunov exponents can also be done by use of this reduction process.

There are several important numerical advantages in using this approach. We mention three:

(i) This method is very efficient computationally since only the flow in $\mathrm{Fix}(\Gamma)$ is required and only the reduced matrices $D_k(y(t))$ are used and so no effort is wasted on the parts of $Df(y(t))$ which contain redundant information.

(ii) As we have already previously pointed out, symmetry may lead to the existence of multiple Lyapunov exponents. In fact, by Theorem 2.8 the Lyapunov exponent computed in (2.4) has multiplicity r_k/d_k, where r_k is the dimension of the corresponding irreducible representation and $d_k = 1, 2, 4$ depending on its type. Obviously, if the symmetry is not taken into account *a priori*, such a nontrivial multiplicity of Lyapunov exponents may lead to numerical problems in their computation, but the method described above avoids these difficulties.

(iii) Using the symmetry we have decomposed the problem of the computation of all the Lyapunov exponents into ℓ subproblems which do not depend on each other. Therefore this approach allows the use of parallel computers for this problem in a very effective way.

3 Bifurcation of attractors

In this section we state our main results concerning the bifurcation of attractors. In the first subsection we consider the case of bifurcations from a Γ-symmetric attractor. As the main result (Theorem 3.5) we derive an upper and a lower bound on the instantaneous symmetries of bifurcating attracting sets. Here we make use of the above classification of Lyapunov exponents with respect to irreducible representations.

In the second subsection we show that in certain cases even further bifurcations can be detected by the computation of Lyapunov exponents for the fully symmetric motion which had *previously* lost its stability.

3.1 Primary bifurcations

For the sake of simplicity we just consider primary bifurcations from Γ-symmetric attractors. However, bifurcations from Σ-symmetric attractors can be treated in the same way for any subgroup Σ of Γ.

Observe that by (2.2) for each subgroup $\Sigma \subset \Gamma$ the fixed point space $\mathrm{Fix}(\Sigma)$ is invariant under the flow of the dynamical system. We say that an invariant set $A_\Sigma \subset \mathrm{Fix}(\Sigma)$ is *attracting inside* $\mathrm{Fix}(\Sigma)$ if A_Σ is an attracting set for the dynamical system which is the restriction of the original system to that fixed point space.

We assume that there is a closed invariant set $A_\Gamma \subset \mathrm{Fix}(\Gamma)$ which is attracting inside $\mathrm{Fix}(\Gamma)$. Let μ be an ergodic measure such that $\mathrm{supp}(\mu) \cap A_\Gamma \neq \emptyset$ and denote the corresponding dominant Lyapunov exponents associated with each isotypic component by

λ_Γ	corresponding to the trivial representation of Γ, and
λ_V	corresponding to a (nontrivial) irreducible representation V of Γ.

We denote by Σ_V the kernel of the representation V, i.e.

$$\Sigma_V = \{\gamma \in \Gamma : \gamma|V = I\}.$$

Hence, by definition, Σ_V is the minimal symmetry type of all the elements in V. As before we identify the space V with the underlying irreducible representation.

In all that follows we make the following general assumption: There is a ball B in phase space such that on the boundary ∂B the flow is pointing inside B.

Since Lyapunov exponents do not in general depend continuously on system parameters, it does not seem to be appropriate to define a bifurcation point by a value of a parameter at which one Lyapunov exponent becomes zero. Also it is not guaranteed that A_Γ is attracting if all the Lyapunov exponents are negative with the possible exception of λ_Γ (cf. [1] and Theorem 3.8). This is the reason why all of the following results are formulated "beyond criticality", where one of the significant Lyapunov exponents is already greater than zero. Thus, for the remainder of this subsection, we assume that there is a nontrivial irreducible representation W with

$$\lambda_W > 0.$$

Proposition 3.1 *For each proper subgroup Σ of Γ for which*

$$\mathrm{Fix}(\Sigma) \cap W \neq \{0\}$$

there exists a closed attracting set A_Σ inside $\mathrm{Fix}(\Sigma)$ with the following properties:

(a) $A_\Gamma \subset A_\Sigma$,

(b) $S_{fix}(A_\Sigma)$ *is a proper subgroup of* Γ.

Proof: At first we restrict the flow to $\mathrm{Fix}(\Sigma)$ and consider the dynamical system on that space. By the existence of the invariant ball B there is a closed attracting set A_Σ inside $\mathrm{Fix}(\Sigma) \cap B$. To see this define $A_\Sigma = \cap_{t \geq 0} \varphi_t(U)$ where U is the interior of $\mathrm{Fix}(\Sigma) \cap B$ (cf. [10]). By construction, $A_\Gamma \subset A_\Sigma$.

For contradiction suppose that A_Σ is contained in $\mathrm{Fix}(\Gamma)$. Since W is a nontrivial irreducible representation of Γ we have that $\mathrm{Fix}(\Gamma) \cap W = \{0\}$. Moreover, $A_\Gamma \subset A_\Sigma$ and $\mathrm{supp}(\mu) \cap A_\Gamma \neq \emptyset$, and hence we may apply Proposition 2.4 by letting $A = A_\Sigma$, $B = A_\Gamma$ and $V = \mathrm{Fix}(\Gamma)$ to see that A_Σ cannot be attracting inside $\mathrm{Fix}(\Sigma)$. This produces the desired contradiction and completes the proof of the proposition. ∎

Remark 3.2 (a) Observe that this last result can also be applied to the case of classical local symmetry breaking bifurcations. In that case the Lyapunov exponents are the eigenvalues of the steady states under consideration and Proposition 3.1 guarantees the existence of invariant sets after the bifurcation which are not fully symmetric even when all the branches are bifurcating subcritically.

(b) If we additionally assume that the set $\mathrm{Fix}(\Sigma) \cap B$ is attracting, then the resulting invariant set A_Σ is not just attracting inside $\mathrm{Fix}(\Sigma)$ but is an attracting set for the full system.

(c) Note that when $S(A_\Sigma) \neq \Gamma$ then there exist at least two distinct invariant sets A_Σ, B_Σ which are *conjugate* to each other, i.e., there is a γ in Γ such that

$$B_\Sigma = \gamma A_\Sigma \quad \text{and} \quad B_\Sigma \neq A_\Sigma.$$

Definition 3.3 The invariant set A is an *attractor inside* $\mathrm{Fix}(\Sigma)$ if it is a compact attracting ω-limit set inside $\mathrm{Fix}(\Sigma)$.

We recall the following result which — in a slightly different setting — has first been stated in [7].

Proposition 3.4 *Let* A *be an attractor and* $\gamma \in \Gamma$. *Then*

$$A \cap \gamma A \neq \emptyset \text{ if and only if } \gamma A = A.$$

Proof: One can verbally transfer the proof of Proposition 4.8 in [22]. ∎

In the case where the invariant set A_Σ in Proposition 3.1 is an attractor inside $\mathrm{Fix}(\Sigma_W)$ we can use Proposition 3.4 to characterise its symmetry on average precisely and, moreover, we can derive an upper bound on the pointwise symmetry of A_Σ. This is the content of the following theorem.

Theorem 3.5 *Suppose that A_Σ is a closed attracting set in* $\mathrm{Fix}(\Sigma_W)$*. Then*

$$\Sigma \subset S_{fix}(A_\Sigma) \subset \Sigma_W.$$

If additionally A_Σ is an attractor in $\mathrm{Fix}(\Sigma_W)$ *then*

$$S(A_\Sigma) = \Gamma.$$

Proof: For contradiction suppose that there is a $\tilde\sigma \in \tilde\Sigma = S_{fix}(A_\Sigma)$ which is not in Σ_W. Then the minimality of Σ_W implies that there are elements in W which are not fixed by $\tilde\sigma$ and it follows

$$\dim(\mathrm{Fix}(\tilde\Sigma) \cap W) < \dim(\mathrm{Fix}(\Sigma_W) \cap W) = \dim W.$$

Since $A_\Gamma \subset A_\Sigma$ and since A_Σ is a closed attracting set in $\mathrm{Fix}(\Sigma_W)$ we may proceed as in the proof of Proposition 3.1 and use Proposition 2.4 to conclude that A_Σ cannot be attracting in $\mathrm{Fix}(\Sigma_W)$ if it is contained in $\mathrm{Fix}(\tilde\Sigma)$. This yields the desired contradiction. Hence $S_{fix}(A_\Sigma) \subset \Sigma_W$ as claimed.

Now suppose that additionally A_Σ is an attractor in $\mathrm{Fix}(\Sigma_W)$. By Proposition 3.1 $A_\Sigma \cap \gamma A_\Sigma \supset A_\Gamma \neq \emptyset$ for all $\gamma \in \Gamma$ and therefore it follows from Proposition 3.4 that $\gamma A_\Sigma = A_\Sigma$. Hence $S(A_\Sigma) = \Gamma$. □

Remark 3.6 It follows from Theorem 3.5 that for high dimensional irreducible representations W the set A_Σ will have very little pointwise symmetry. For instance, for the two-dimensional representations of $\mathbf{D}_p$ the kernel Σ_W consists of only the identity. Hence, in that case each closed attracting set A_Σ has trivial instantaneous symmetry.

The previous theorem has the following immediate but interesting consequence.

Corollary 3.7 *There exists a closed attracting set A_Σ inside* $\mathrm{Fix}(\Sigma_W)$ *such that*

$$S_{fix}(A_\Sigma) = \Sigma_W.$$

Proof: In Proposition 3.1 we may choose $\Sigma = \Sigma_W$. With this choice Theorem 3.5 implies that

$$\Sigma_W \subset S_{fix}(A_\Sigma) \subset \Sigma_W$$

giving the desired result. □

We end this subsection by describing those fixed point spaces relative to which A_Γ still remains "stable" after the bifurcation. For this we recall a specific version of a result from [1]. For an introduction of the notions *SBR-measure* or *nonuniformly hyperbolic* invariant sets occurring in the following theorem the reader is refered to [31].

Theorem 3.8 *Let V be a flow-invariant subspace of* R^n *of dimension $m < n$ and let A be a nonuniformly hyperbolic invariant set inside V. Suppose that there exists a corresponding SBR-measure for which the $n - m$ normal Lyapunov exponents are negative. Then there is a set of positive Lebesgue measure in phase space which is forward asymptotic to A.*

Remark 3.9 Observe that although all the normal Lyapunov exponents are negative it is in general not guaranteed that all points in a full neighborhood of A in R^n are attracted to A. (For an explanation of this fact see [2].) However, for the case in which the SBR-measure is additionally absolutely continuous with respect to the Lebesgue measure on V one can show that inside a neighborhood of A the quotient of the Lebesgue measure of points which are not approaching A and the Lebesgue measure of points which are approaching A is going to zero if the neighborhood is shrinking to A. For an example in which a corresponding *riddled basin* occurs see again [1].

Corollary 3.10 *Let A_Γ be a nonuniformly hyperbolic set and let μ be an SBR-measure. Suppose that for an irreducible representation W*

$$\lambda_W > 0$$

and that all the other Lyapunov exponents are negative with the possible exception of λ_Γ. Then for each subgroup Δ of Γ with the property that

$$\mathrm{Fix}(\Delta) \cap W = \{0\}$$

there is a set of positive Lebesgue measure on $\mathrm{Fix}(\Delta)$ *which is forward asymptotic to A_Γ.*

Proof: The Lyapunov exponents of A_Γ in $\mathrm{Fix}(\Delta)$ are all negative, with the possible exception of λ_Γ, and so the result follows immediately by Theorem 3.8. $\square$

3.2 Further bifurcations

Let A_Σ be a closed invariant set with $S_{fix}(A_\Sigma) = \Sigma$ and suppose that

$$A_\Gamma = A_\Sigma \cap \mathrm{Fix}(\Gamma) \neq \emptyset.$$

As above we denote the dominant Lyapunov exponents associated with different isotypic components for the motion A_Γ by λ_Γ and λ_V where the V's are the nontrivial irreducible representations of Γ.

In the previous subsection Proposition 2.4 turned out to be very useful in the derivation of bounds on the symmetry types of the "bifurcating" invariant sets A_Σ. Here we will see that it also allows us to determine further bifurcations from A_Σ by the computation of the Lyapunov exponents of A_Γ. Obviously this is of particular numerical importance.

Proposition 3.11 *Suppose that $\lambda_W > 0$, where the irreducible subspace W is not contained in* $\mathrm{Fix}(\Sigma)$. *Let $\Delta \subset \Gamma$ be a subgroup such that*

$$(\mathrm{Fix}(\Delta) \cap W) - (\mathrm{Fix}(\Sigma) \cap W) \neq \emptyset.$$

Then the invariant set $A_\Sigma \cap \mathrm{Fix}(\Delta)$ cannot be attracting in $\mathrm{Fix}(\Delta)$.

Proof: Restrict the underlying dynamical system to $\mathrm{Fix}(\Delta)$ and let $V = \mathrm{Fix}(\Sigma) \cap \mathrm{Fix}(\Delta)$, $A = A_\Sigma \cap \mathrm{Fix}(\Delta)$ and $B = A_\Gamma$. Observe that $\mathrm{Fix}(\Gamma) \subset V$ and that $A_\Sigma \cap \mathrm{Fix}(\Delta) \neq \emptyset$ since $A_\Gamma \neq \emptyset$ is contained in $\mathrm{Fix}(\Delta)$. Moreover, the condition $(\mathrm{Fix}(\Delta) \cap W) - (\mathrm{Fix}(\Sigma) \cap W) \neq \emptyset$ guarantees that $\mathrm{Fix}(\Delta) \cap W$ is not contained in V. Hence the result follows by Proposition 2.4. $\square$

Remark 3.12 If $\Delta = \{Id\}$ is the trivial subgroup of Γ then $W \subset \mathrm{Fix}(\Delta)$ is automatically satisfied and in this case we may conclude that A_Σ cannot be attracting. That is, the set A_Σ has lost its stability.

We will illustrate the usefulness of this result in the numerical examples of Sec. 5.

4 Lyapunov exponents for coupled oscillators

In this section, we consider systems of coupled oscillators and show that in some cases the Lyapunov exponents of diagonal blocks associated with non-trivial irreducible representations are related to those of the block associated with the trivial irreducible representation (i.e. on the fixed point space) in a predetermined way. Our considerations are similar to the exposition in [15], Chapter XVIII, §4. However, eventually here it turns out to be more convenient to analyse general linear coupling arrangements rather than just the structure of couplings which are related to the underlying symmetry of the system.

4.1 Structure of the Jacobian

Consider the system

$$\dot{x} = f(x), \qquad f : \mathbf{R}^m \to \mathbf{R}^m \tag{4.1}$$

which can be either autonomous or non-autonomous. We denote the Jacobian matrix of f along a trajectory $x(t)$ by

$$A(t) = Df(x(t)).$$

Coupling p such systems together using any *linear* coupling gives rise to the system

$$\dot{x}_i = f(x_i) + \sum_{j=1}^{p} B_{i,j} x_j, \qquad i = 1, \ldots, p \tag{4.2}$$

where $B_{i,j}$ are constant $m \times m$ matrices. We write this system as

$$\underline{\dot{x}} = \underline{F}(\underline{x}) + \mathcal{B}\underline{x}$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \in \mathbf{R}^{mp}, \qquad \underline{F}(\underline{x}) = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_p) \end{bmatrix}, \quad \underline{F} \;:\; \mathbf{R}^{mp} \to \mathbf{R}^{mp}$$

and $\mathcal{B}$ is an $mp \times mp$ block matrix consisting of the $B_{i,j}$ blocks. We assume that the coupling matrix $\mathcal{B}$ can also be expressed as

$$\mathcal{B} = C \otimes D$$

where the $m \times m$ matrix $D \neq 0$ describes the coupling between any two oscillators and the $p \times p$ matrix C describes the connections between the different oscillators and the strength of the couplings. Thus, $B_{i,j} = c_{i,j} D$. We assume that the couplings are not directed and this implies that the coupling matrix C is symmetric.

We ignore any symmetries associated with the individual oscillators and note that the coupling will lead to some symmetry Γ of this system related to permutations of the oscillators and so Γ is a subgroup of the permutation group S_p. The matrix representation of this group will have the form

$$P(\gamma) \otimes I_m, \qquad \gamma \in \Gamma \subset S_p \tag{4.3}$$

where I_m is the identity on $\mathbf{R}^m$ and P is the natural representation of the subgroup Γ of S_p on $\mathbf{R}^p$. Due to the structure of $\underline{F}(\underline{x})$, it is equivariant with respect to the whole group S_p and so the group Γ is determined by the symmetries of the coupling. In particular, the coupling terms are equivariant under the action of the particular permutation $\gamma \in S_p$ if

$$CP(\gamma) = P(\gamma)C.$$

We require that there must be a solution where all the individual oscillators behave identically and so we assume that

$$\sum_{j=1}^{p} B_{i,j} = 0 \quad \text{for all} \quad i = 1, \ldots, p$$

or equivalently,

$$\sum_{j=1}^{p} c_{i,j} = 0 \quad \text{for all} \quad i = 1, \ldots, p. \tag{4.4}$$

We consider only Lyapunov exponents corresponding to motions for which the oscillators all behave identically, that is when the solution has the full instantaneous Γ symmetry. The Jacobian matrix $\mathcal{J}$ of the coupled system (4.2) at such a solution can be written as a $p \times p$ block matrix given by

$$\mathcal{J} = I_p \otimes A(t) + C \otimes D. \tag{4.5}$$

We now consider the block diagonalisation of $\mathcal{J}$ which is guaranteed by Theorem 2.8 and show that it is closely related to the corresponding block diagonalisation of the coupling matrix C. We write this block diagonalisation as

$$\mathcal{D}^s = \text{diag}(\mathcal{D}^s_i).$$

Using the concept of *symmetry adapted bases* (e.g. [29]) one can easily write down a change of co-ordinates involving a nonsingular matrix Q such that

$$\mathcal{D}^s = Q^{-1}\mathcal{J}Q. \tag{4.6}$$

Since the symmetry Γ acts only by permutation of the oscillators and acts as the identity on each individual oscillator (see (4.3)), the matrix Q can be written as

$$Q = \tilde{Q} \otimes I_m. \tag{4.7}$$

We now derive the form of $\mathcal{D}^s$.

Proposition 4.1 *The block diagonalisation of $\mathcal{J}$ is given by*

$$\mathcal{D}^s = I_p \otimes A(t) + \tilde{Q}^{-1}C\tilde{Q} \otimes D. \tag{4.8}$$

Proof: Using the definitions of $\mathcal{J}$ in (4.5) and Q in (4.7) gives

$$\begin{aligned}
\mathcal{D}^s &= Q^{-1}\mathcal{J}Q \\
&= (\tilde{Q}^{-1} \otimes I_m)(I_p \otimes A(t) + C \otimes D)(\tilde{Q} \otimes I_m) \\
&= I_p \otimes A(t) + \tilde{Q}^{-1}C\tilde{Q} \otimes D
\end{aligned}$$

as required. ∎

This result shows that the block diagonalisation of the coupling matrix C using the matrix $\tilde{Q}$ determines the block diagonalisation $\mathcal{D}^s$ of $\mathcal{J}$. This is not surprising since it is the coupling matrix C which determines the symmetry of the system, as described above.

The block diagonalisation $\mathcal{D}^s$ given in (4.8) is achieved using the symmetry. However, using the fact that C is a symmetric matrix, a further decomposition can be obtained in some cases. This is derived using the fact that the symmetric matrix C is similar to a diagonal matrix with its (real) eigenvalues ν_i on the diagonal. Now the decomposition (4.8) is valid for any nonsingular matrix $\tilde{Q}$. Thus, we now choose $Q = \hat{Q} \otimes I_m$ where $\hat{Q}$ is the matrix of eigenvectors of C. In this case, we obtain a block diagonalisation of $\mathcal{J}$ given by

$$\mathcal{D}^e = I_p \otimes A(t) + \hat{Q}^{-1} C \hat{Q} \otimes D$$

which consists of p diagonal blocks, each of size $m \times m$, given by

$$\mathcal{D}_i^e = A(t) + \nu_i D.$$

Note that the condition (4.4) implies that C has a zero eigenvalue and so one block, which we take to be the first, is simply

$$\mathcal{D}_1^e = A(t).$$

Since the eigenvector of C associated with the zero eigenvalue is given by $[1, 1, \ldots, 1]^T$, the block $\mathcal{D}_1^e$ must be a sub-block of $\mathcal{D}_1^s$ associated with the trivial irreducible representation, corresponding to motion in which all the oscillators are in phase.

Clearly, if the block diagonalisation of C using $\tilde{Q}$ leads to a diagonal matrix, then $\mathcal{D}^s = \mathcal{D}^e$ and so no further simplification is obtained in this way.

This eigenvalue decomposition of the Jacobian has also been considered in [12] for systems of coupled oscillators, but the symmetry in the system was not considered. Clearly, this decomposition can also be used for standard bifurcation analysis of coupled oscillators.

Finally, we note that it is only the linearisation $\mathcal{J}$ which decomposes in this way. When dealing with the full nonlinear system, the only invariant spaces are the usual fixed point spaces.

4.2 Lyapunov exponents

From the analysis of the previous section, it is clear that the most efficient method of computing the Lyapunov exponents and determining the bifurcations which occur, is by using the eigenvalue decomposition $\mathcal{D}^e$ into p blocks of size $m \times m$. It is easily verified that in the new coordinates, the solution to the variational equation must have a similar block structure to $\mathcal{D}^e$ and so the variational equation decomposes into the p independent equations

$$\dot{\Phi}_i = (A(t) + \nu_i D)\Phi_i, \quad \Phi_i(0) = I, \qquad i = 1, \ldots, p.$$

For these subproblems, the Lyapunov exponents associated with each system are given by

$$\lambda_j^{(i)} = \lim_{t\to\infty} \frac{1}{t}||\Phi_j(t)v_j||, \quad v_j \in E_j^{(i)} - E_j^{(i+1)}, \qquad j = 1, \ldots, p$$

where the notation is a natural extension of that used in Theorem 2.2. Clearly the eigenspaces for the Lyapunov exponents as subspaces of the full space $\mathbf{R}^{mp}$ can be constructed from their individual eigenspaces and will depend on the relative magnitudes of the Lyapunov exponents of different blocks. We now show that in some cases, there is a close relationship between the dominant Lyapunov exponents of the different subsystems.

Proposition 4.2 *Suppose that the coupling matrix D is symmetric and commutes with $A(t)$. Let $\lambda^{(j)}$ be the largest Lyapunov exponent associated with the matrix $D_j^e = A(t) + \nu_j D$. Then*

$$\lambda^{(1)} + \nu_j \min_i\{\mu_i\} \le \lambda^{(j)} \le \lambda^{(1)} + \nu_j \max_i\{\mu_i\}$$

where μ_i are the eigenvalues of D.

Proof: Suppose that $\dot{\Phi}_1 = A(t)\Phi_1$. Then $\Phi_j = e^{t\nu_j D}\Phi_1$ satisfies $\dot{\Phi}_j = (A(t) + \nu_j D)\Phi_j$, using the fact that $A(t)$ and D commute. Thus, for any $v \in \mathrm{R}^m$,

$$\begin{aligned} ||\Phi_j v|| &= ||e^{t\nu_j D}\Phi_1 v|| \\ &\le ||e^{t\nu_j D}||\; ||\Phi_1 v||. \end{aligned}$$

Suppose that the eigenvalues of D are given by μ_i. Since D is symmetric, we then have that

$$\begin{aligned} ||e^{t\nu_j D}|| &= \rho[(e^{t\nu_j D})^T e^{t\nu_j D}]^{1/2} \\ &= \rho[e^{t\nu_j D^T} e^{t\nu_j D}]^{1/2} \\ &= \rho[e^{2t\nu_j D}]^{1/2} \\ &= [\max_i\{e^{2t\nu_j \mu_i}\}]^{1/2} \\ &= \max_i\{e^{t\nu_j \mu_i}\}. \end{aligned}$$

Therefore we may choose $v_j \in (E_1^{(1)} - E_1^{(2)}) \cap (E_j^{(1)} - E_j^{(2)})$, where $E_1^{(1)} = E_j^{(1)} = \mathrm{R}^m$, such that

$$\begin{aligned}
\lambda^{(j)} &= \lim_{t\to\infty} \frac{1}{t} \ln ||\Phi_j v_j|| \\
&\leq \lim_{t\to\infty} \frac{1}{t} (\ln ||\Phi_1 v_j|| + \ln ||e^{t\nu_j D}||) \\
&= \lambda^{(1)} + \nu_j \max_i \{\mu_i\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
||\Phi_1 v|| &= ||e^{-t\nu_j D} \Phi_j v|| \\
&\leq ||e^{-t\nu_j D}|| \; ||\Phi_j v|| \\
&= \max_i \{e^{-t\nu_j \mu_i}\} \; ||\Phi_j v||
\end{aligned}$$

and so

$$\begin{aligned}
\lambda^{(1)} &= \lim_{t\to\infty} \frac{1}{t} \ln ||\Phi_1 v_j|| \\
&\leq \lim_{t\to\infty} \frac{1}{t} (\ln ||\Phi_j v_j|| + \ln \max_i \{e^{-t\nu_j \mu_i}\}) \\
&= \lambda^{(j)} + \nu_j \max_i \{-\mu_i\} \\
&= \lambda^{(j)} - \nu_j \min_i \{\mu_i\}.
\end{aligned}$$

Combining these two inequalities gives the desired result. $\square$

Corollary 4.3 *(i) If $\nu_j \max_i\{\mu_i\} < 0$, then $\lambda^{(j)} < \lambda^{(1)}$ and if $\nu_j \min_i\{\mu_i\} > 0$, then $\lambda^{(j)} > \lambda^{(1)}$.*

(ii) If $D = kI$, then $\lambda^{(j)} = \lambda^{(1)} + k\nu_j$.

If either of the conditions in Corollary 4.3 hold, then a precise ordering on the dominant Lyapunov exponents associated with different diagonal blocks can be determined *a priori*. The bifurcations described in Sec. 3.2 are then ordered similarly. Also, in case (ii), there is no need to calculate any Lyapunov exponents apart from $\lambda^{(1)}$, related to the matrix $\mathcal{D}_1^e$ which is associated with the fully symmetric flow.

If a particular block $\mathcal{D}_i^e$ has a positive Lyapunov exponent for some $i \neq 1$, then the corresponding bifurcation of attractors can still be determined. This is achieved by applying Proposition 3.1 with W defined by

$$W = \{e_i \otimes x : x \in \mathbf{R}^m\}$$

where e_i is the eigenvector of the coupling matrix C associated with the i^{th} eigenvalue. However, we note that in this context of coupled oscillators, there is another type of bifurcation of attractors that can occur other than those described in Sec. 3. This arises when the two block diagonalisations $\mathcal{D}^s$ and $\mathcal{D}^e$ are different. In particular, suppose that the first diagonal block $\mathcal{D}_1^s$ associated with the trivial irreducible representation in the symmetry adapted decomposition, decomposes into r distinct blocks in the eigenvalue decomposition $\mathcal{D}^e$, one of which will be associated with the zero eigenvalue. There are now two distinct types of motion which are possible. In the first, all the oscillators are in phase but display chaotic motion. We call this a *uniform* attractor. Secondly, it is possible for the oscillators to have chaotic motion within the fixed point space of the group Γ but where they are not all behaving identically. We refer to this as a *symmetric* attractor. The transition from the uniform attractor to the symmetric attractor will occur if one of the blocks in the diagonalisation $\mathcal{D}^e$ which occurs as part of $\mathcal{D}_1^s$ has a Lyapunov exponent which changes from negative to positive. This type of bifurcation is then analogous to the steady state transcritical bifurcation from a trivial solution, as no symmetry is broken and yet there is a transition from a "trivial" (uniform) state, in which all the oscillators are in phase, to a nontrivial state in which they are not all in phase. An example of this type of bifurcation is described in Sec. 5.

5 Examples

In this section, we illustrate the results of the previous section on coupled oscillators by considering two examples. In the first, we consider a system of four coupled Lorenz systems and show that a symmetry breaking bifurcation of the chaotic attractors occurs. In the second, we use a system of three coupled Duffing oscillators and give an example of a bifurcation which does not break symmetry but which gives rise to an attractor for which all the oscillators are not in phase.

5.1 Coupled Lorenz Systems

We consider the Lorenz equations

$$\begin{aligned}
\dot{X} &= \sigma(Y - X) \\
\dot{Y} &= rX - Y - XZ \\
\dot{Z} &= -\beta Z + XY
\end{aligned}$$

and couple together p such systems with linear coupling as in (4.2) with $x_i = [X_i, Y_i, Z_i]^T$, $i = 1, \ldots, p$.

We first show the existence of an invariant ball B as required earlier. The phase space for the single oscillator is $\mathbf{R}^3$ and for p coupled systems is $\mathbf{R}^{3p}$. Lorenz [20] showed that there is a sphere in $\mathbf{R}^3$ on which the flow is invariant. This was done by defining $Q = (X^2+Y^2+(Z-r-\sigma)^2)/2$ and showing that $\dot{Q} = -\sigma X^2 - Y^2 + \beta Z(-Z+r+\sigma) < 0$ on the boundary of a sphere $S \in \mathbf{R}^3$ defined by $Q =$constant. We extend this method for the coupled Lorenz systems.

Proposition 5.1 *Let $\mathcal{B}$ be the matrix with blocks $B_{i,j}$ derived from the coupling of p Lorenz systems. If $\mathcal{B}$ is negative semi-definite then there is a ball $B \subset \mathbf{R}^{3p}$ which is invariant under the flow of the coupled system.*

Proof: Let $u_i = x_i - v$ where $v = [0, 0, r+\sigma]^T$ and define

$$P = \frac{1}{2}\sum_{i=1}^{p} u_i^T u_i$$

where the index i refers to the i^{th} oscillator. Then

$$\begin{aligned} \dot{P} &= \sum_{i=1}^{p} u_i^T \dot{u}_i \\ &= \sum_{i=1}^{p} u_i^T \dot{x}_i \\ &= \sum_{i=1}^{p} u_i^T \left(f(x_i) + \sum_{j=1}^{p} B_{i,j} x_j \right) \\ &= \sum_{i=1}^{p} \dot{Q}_i + \sum_{i=1}^{p}\sum_{j=1}^{p} (u_i^T B_{i,j} u_j + u_i^T B_{i,j} v). \end{aligned}$$

where $\dot{Q}_i = -\sigma X_i^2 - Y_i^2 + \beta Z_i(-Z_i + r + \sigma) < 0$. Since $B_{i,j} = c_{i,j} D$, the last term becomes

$$\begin{aligned} \sum_{i=1}^{p}\sum_{j=1}^{p} u_i^T B_{i,j} v &= \sum_{i=1}^{p}\sum_{j=1}^{p} c_{i,j} u_i^T D v \\ &= \sum_{i=1}^{p} u_i^T D v \sum_{j=1}^{p} c_{i,j} \\ &= 0 \end{aligned}$$

using (4.4).

We must now show that $\dot{P} < 0$ on the boundary of an appropriate region. We take B to be the ball in $\mathbf{R}^{3p}$ defined by $P =$constant. Now the maximum value that $\dot{Q}_i$

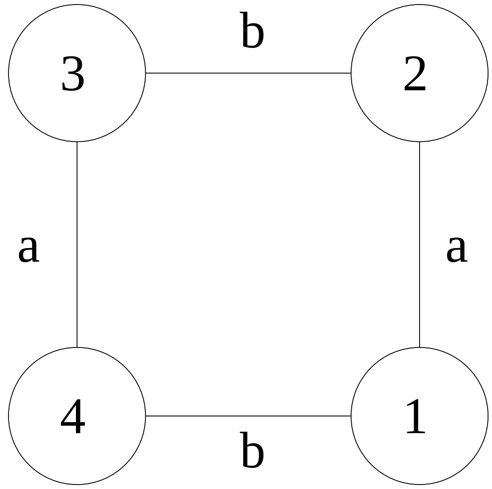

Figure 1: Coupling arrangement for the 4 Lorenz systems

can take occurs when $X_i = Y_i = 0$ and $Z_i = (r+\sigma)/2$. However, even if all the Z_i components are given the value $(r+\sigma)/2$, by taking the ball sufficiently large, the X_i any Y_i components will be sufficiently large to make the first term in $\dot{P}$ negative. Also, if $\mathcal{B}$ is negative semi-definite, then the second term in $\dot{P}$ will be less than or equal to zero. Since the third term is zero, combining these results gives $\dot{P} < 0$ ensuring that the ball $\mathcal{B}$ is invariant under the flow of the coupled system. $\square$

We consider two different couplings of the Lorenz equations, both of which involve four such systems as shown in Fig. 1 where a and b are the relative strengths of the couplings between the different systems. For each coupling, we use a diagonal 3×3 matrix D as the relative strength of the couplings between the X, Y and Z components and the coupling matrix $\mathcal{B}$ is then given by

$$\mathcal{B} = C \otimes D$$

where

$$C = \begin{bmatrix} -(a+b) & a & 0 & b \\ a & -(a+b) & b & 0 \\ 0 & b & -(a+b) & a \\ b & 0 & a & -(a+b) \end{bmatrix}.$$

To ensure the existence of an invariant ball B, by Proposition 5.1 we must ensure that the coupling matrix $\mathcal{B}$ is negative semi-definite. This is the case if for all $x \in \mathrm{R}^{mp}$

$$\begin{aligned} 0 &\geq \sum_{i,j} x_i^T B_{i,j} x_j \\ &= \sum_{i,j} c_{i,j} x_i^T D x_j \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} c_{i,j}\left(d_1 X_i X_j + d_2 Y_i Y_j + d_3 Z_i Z_j\right) \\
&= d_1 \sum_{i,j} c_{i,j} X_i X_j + d_2 \sum_{i,j} c_{i,j} Y_i Y_j + d_3 \sum_{i,j} c_{i,j} Z_i Z_j
\end{aligned}
$$

using the fact that D is a diagonal matrix. Clearly this relation will hold if $d_i > 0,\ i = 1, 2, 3$ and the matrix C is negative semi-definite.

For the first coupling arrangement, we take $a = 1$ and $b = 2$. The coupled system then has symmetry $\mathrm{Z}_2 \times \mathrm{Z}_2$ generated by the two reflections $S_1 = S$ and $S_2 = SR^2$ where S and R are defined as in Example 2.10 except that I is now the 3×3 identity matrix. The matrices which commute with this group action have the form

$$
\begin{bmatrix}
A & B & E & F \\
B & A & F & E \\
E & F & A & B \\
F & E & B & A
\end{bmatrix}
$$

and it is easily seen that the coupling matrix $\mathcal{B}$ has precisely this structure. The block diagonalisation, using the four one-dimensional irreducible representations of $\mathrm{Z}_2 \times \mathrm{Z}_2$, in the order $(i)\ S_1 = S_2 = I$, $(ii)\ -S_1 = S_2 = I$, $(iii)\ S_1 = -S_2 = I$, $(iv)\ S_1 = S_2 = -I$, is then

$$
\begin{bmatrix}
A+B+E+F & 0 & 0 & 0 \\
0 & A-B-E+F & 0 & 0 \\
0 & 0 & A+B-E-F & 0 \\
0 & 0 & 0 & A-B+E-F
\end{bmatrix}.
$$

Using the Jacobian matrix of the coupled system as defined in (4.5), the block diagonalisation becomes

$$
\begin{bmatrix}
A(t) & 0 & 0 & 0 \\
0 & A(t)-2D & 0 & 0 \\
0 & 0 & A(t)-4D & 0 \\
0 & 0 & 0 & A(t)-6D
\end{bmatrix}. \tag{5.1}
$$

In the second coupling arrangement, we take $a = b = 1$. The coupled system then has $\mathbf{D}_4$ symmetry generated by the rotation R and the reflection S as given in Example 2.10, again with I as the 3×3 identity matrix. The matrices which commute with this action and the corresponding block diagonalisation are given in Example 2.10. Again, using the Jacobian (4.5) of the coupled system leads to the block diagonalisation

$$
\begin{bmatrix}
A(t) & 0 & 0 & 0 \\
0 & A(t)-4D & 0 & 0 \\
0 & 0 & A(t)-2D & 0 \\
0 & 0 & 0 & A(t)-2D
\end{bmatrix}. \tag{5.2}
$$

Remark 5.2 (a) Note that the diagonal blocks in this case also occur in the previous coupling arrangement and so computations for the $\mathbf{Z}_2 \times \mathbf{Z}_2$ system also apply to the $\mathbf{D}_4$ system.
(b) We do not make any mention of the Z_2 symmetry of the individual Lorenz systems since the chaotic motion we consider does not have this (pointwise) symmetry and so it does not contribute towards the block diagonalisation.

To obtain numerical results, we take $\sigma = 10$ and $r = 28$ and use the parameter β as the bifurcation parameter. Initially, we take $D = 0.2I$. If σ_i, $i = 1, \ldots, 4$, is the dominant Lyapunov exponent for each of the four blocks in (5.1), then we know from Proposition 4.2 that $\sigma_i - \sigma_{i+1} = 0.4$ and so the precise order of any bifurcations of chaotic attractors is known *a priori*.

As a useful way of visualising the symmetry of the attractors, we define the distance functions

$$\begin{aligned} d(S_1) &= (d_{1,2} + d_{3,4})^{1/2} \\ d(S_2) &= (d_{2,3} + d_{1,4})^{1/2} \\ d(S_{12}) &= (d_{1,3} + d_{2,4})^{1/2} \\ d(S_1, S_2) &= (d_{1,2} + d_{2,3})^{1/2} \end{aligned}$$

where $d_{i,j} = (X_i - X_j)^2 + (Y_i - Y_j)^2 + (Z_i - Z_j)^2$. These measure the "distance from a fixed point space". Thus, if the attractor is in the fixed point space generated by the reflection S_1, then $d(S_1) = 0$. Note that the distance function $d(S_1, S_2)$ measures the distance from the fixed point space with the full $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry since, if the first, second and third oscillators are in phase, the fourth must be also.

A single Lorenz system was integrated using the standard fourth order Runge-Kutta method with $h = 0.0001$ and the Lyapunov exponents σ_i, $i = 1, \ldots, 4$, were computed and are shown in Fig. 2. As noted earlier, it is sufficient to calculate only σ_1 but calculating each of the σ_i values confirms the theory.

In the discussion that follows, we will refer to a chaotic attractor with particular symmetries. By this, we mean only that the motion is attracting in an appropriate fixed point space. If the motion is attracting in the whole space, then we refer to it as *stable*.

For low values of β, the dominant Lyapunov exponent σ_1 of $A(t)$ is zero, indicating that the solution of the fully symmetric system is not chaotic. The package DSTOOL was used to investigate the behaviour of the coupled system. In this case, it was found that all the oscillators were in phase on a (non-symmetric) periodic orbit. Small perturbations putting the separate systems out of phase are soon damped out.

When $\beta = 0.625$, we have $\sigma_1 > 0$ but $\sigma_i < 0$ for $i = 2, 3, 4$. This indicates that the oscillators are chaotic but are all in phase as the $Z_2 \times Z_2$ symmetry has not been

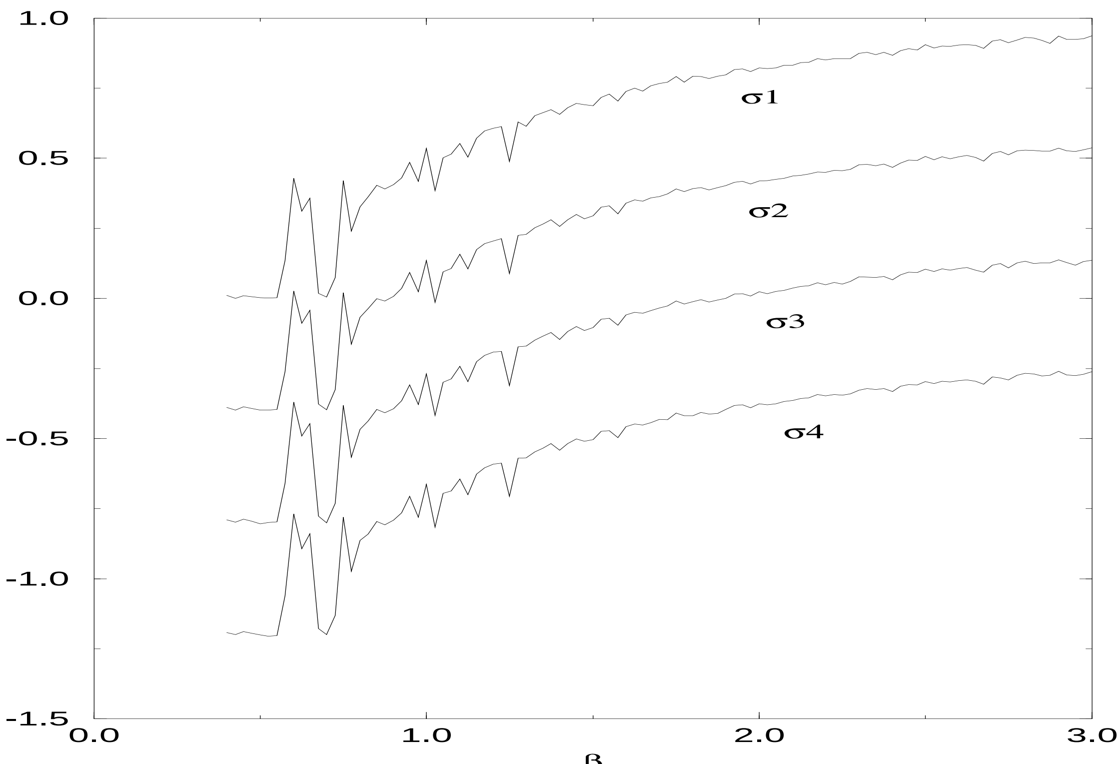

Figure 2: Dominant Lyapunov Exponents of the Blocks in (5.1) with $D = 0.2I$

broken. This was confirmed with DSTOOL and the attractor is shown in Figs. 3 and 4. Again, symmetry breaking perturbations are damped out as expected.

When $\beta = 1.5$, the dominant Lyapunov exponent σ_2 of the second block of the Jacobian has passed through zero and is now positive. In this case, only the S_2 symmetry is kept while the S_1 symmetry is broken. The attractor is shown in Figs. 5 and 6. Due to the S_2 symmetry, systems 1 and 4 are in phase together with systems 2 and 3. The individual attractors for the 4 oscillators look very similar indicating that the system still has full symmetry on average. Thus the attractor has precisely the symmetry of the kernel of the irreducible representation (cf. Theorem 3.5, Corollary 3.7). The plot of the distance function $d(S_1)$ is shown in Fig. 7 which clearly shows that the S_1 symmetry has been broken. Clearly $d(S_2) = 0$ and so when $d(S_1)$ is close to zero, the attractor is close to having full symmetry. This is another indicator that the attractor has full symmetry on average. We remark that a sophisticated way of exploiting such symmetries on average is provided by the concept of *detectives* (see [6, 9, 14]).

When $\beta = 2.2$, the third Lyapunov exponent σ_3 has also passed through zero becoming positive so that the attractor with S_2 symmetry is now unstable to perturbations which break this symmetry by Proposition 3.11. Numerical results confirm that the attractor, shown in Figs. 8 and 9, has no pointwise symmetry. However, the four in-

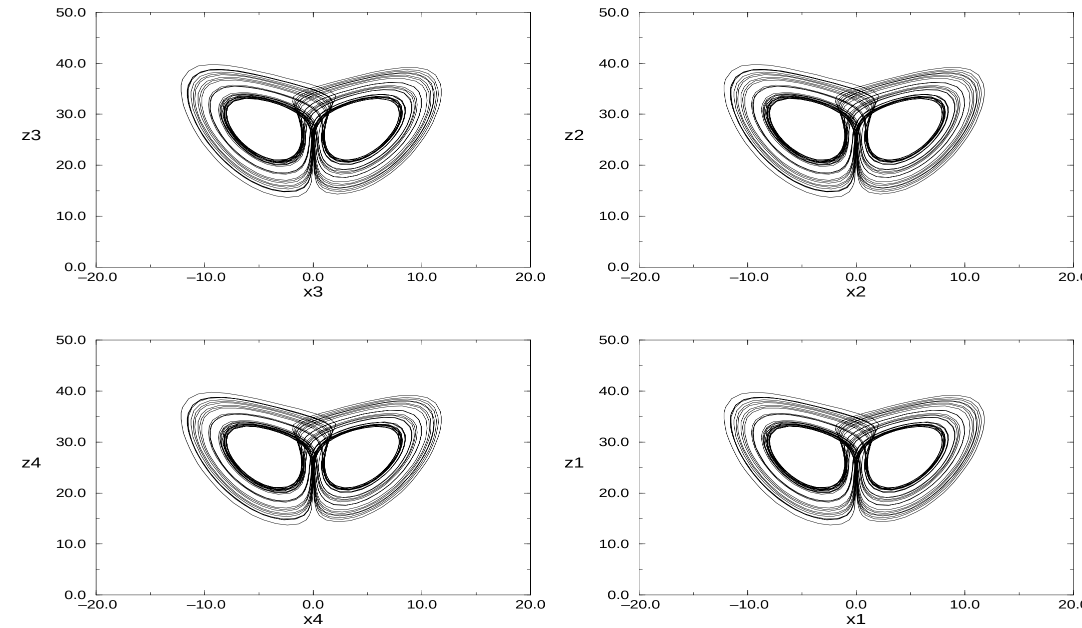

Figure 3: Fully Symmetric Chaotic Attractor at $\beta = 0.625$

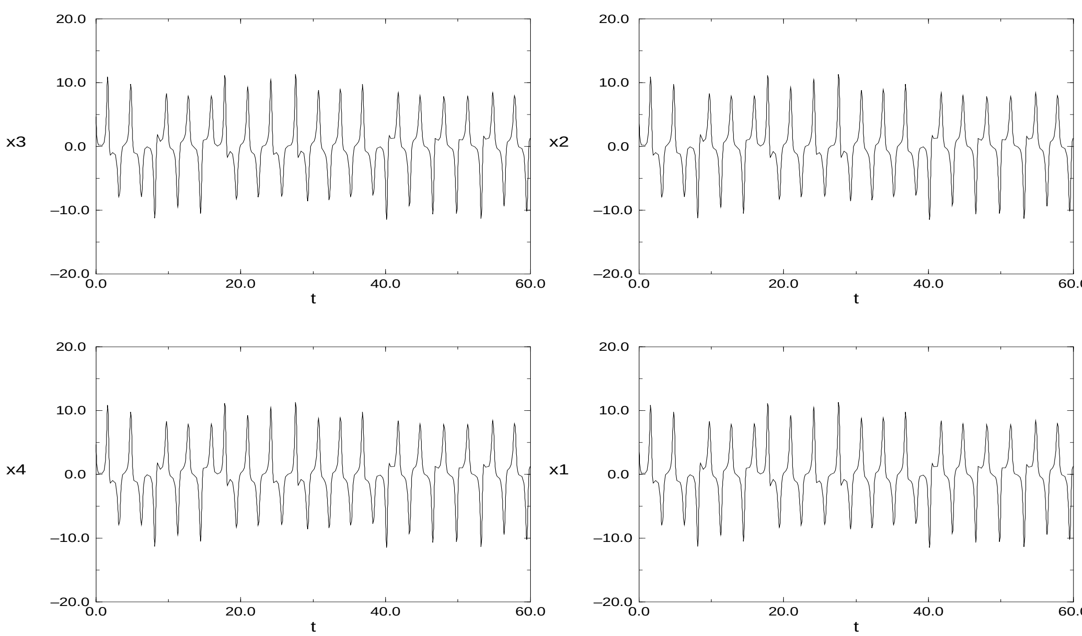

Figure 4: Fully Symmetric Chaotic Attractor at $\beta = 0.625$

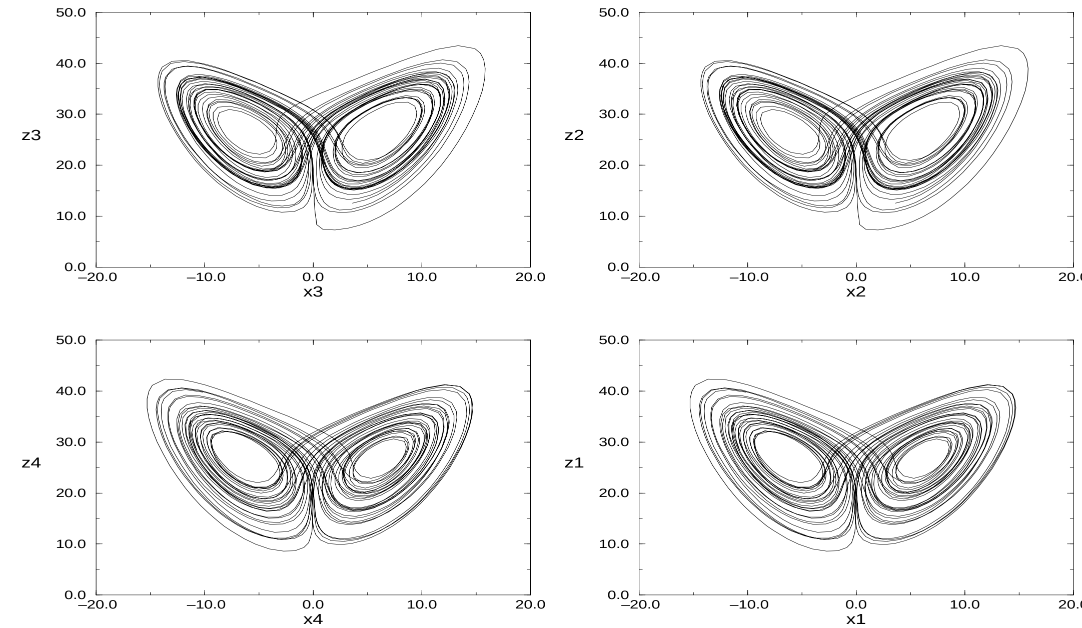

Figure 5: S_2 Symmetric Chaotic Attractor at $\beta = 1.5$

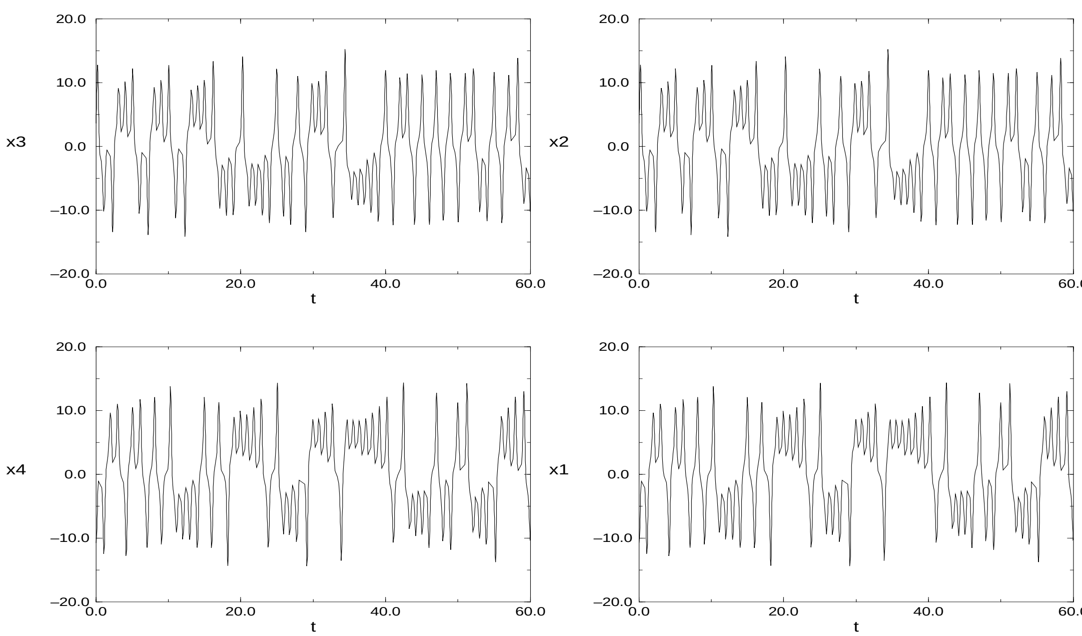

Figure 6: S_2 Symmetric Chaotic Attractor at $\beta = 1.5$

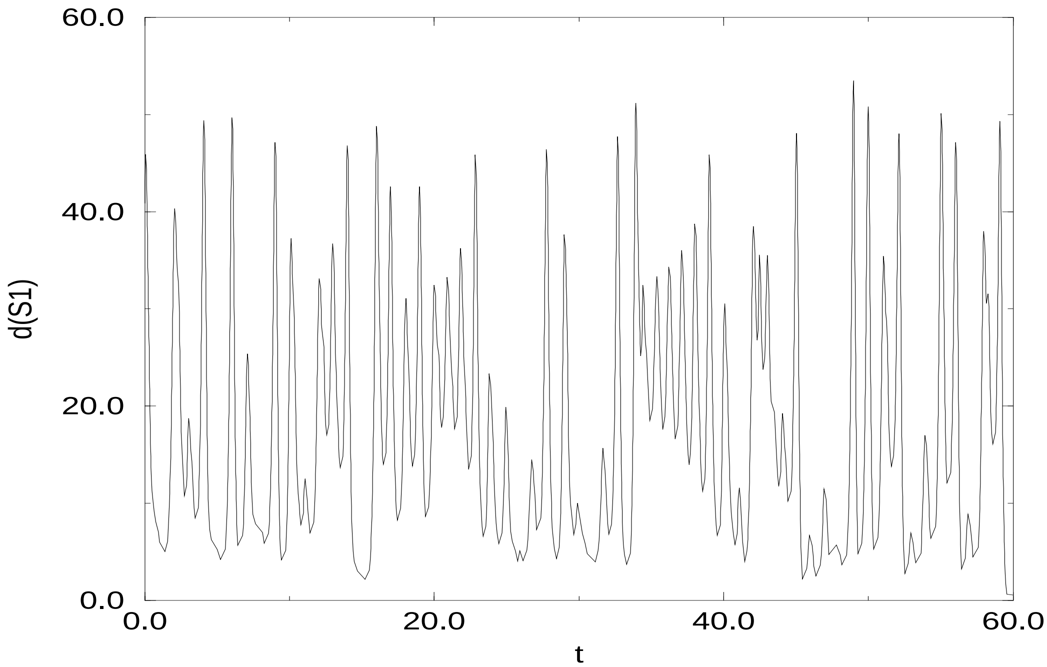

Figure 7: The $d(S_1)$ Distance Function for the Symmetric Chaotic Attractor at $\beta = 1.5$

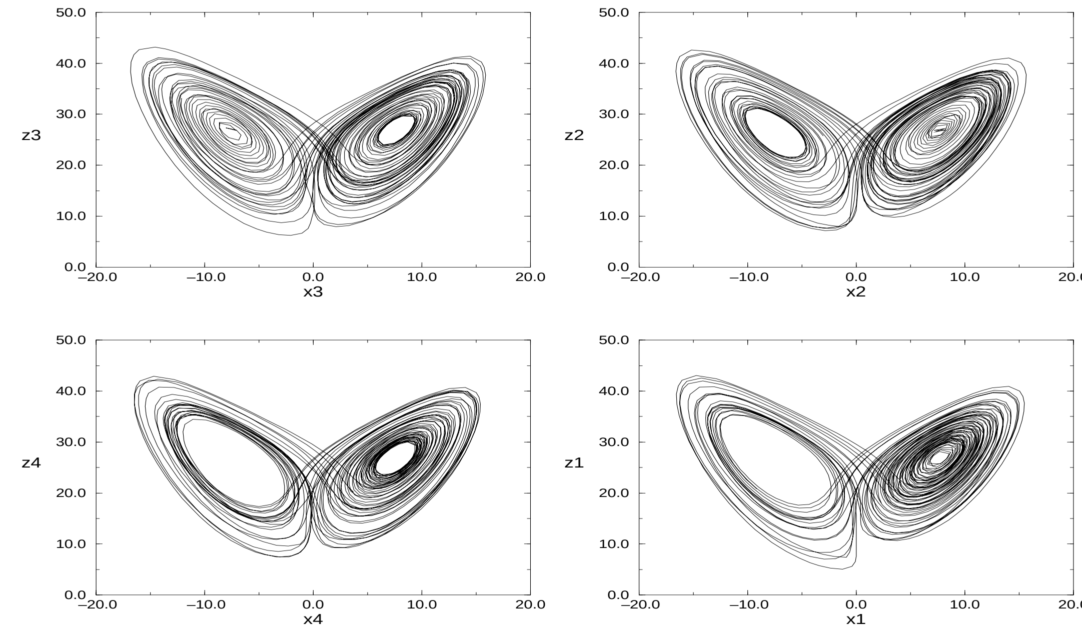

Figure 8: Nonsymmetric Chaotic Attractor at $\beta = 2.2$

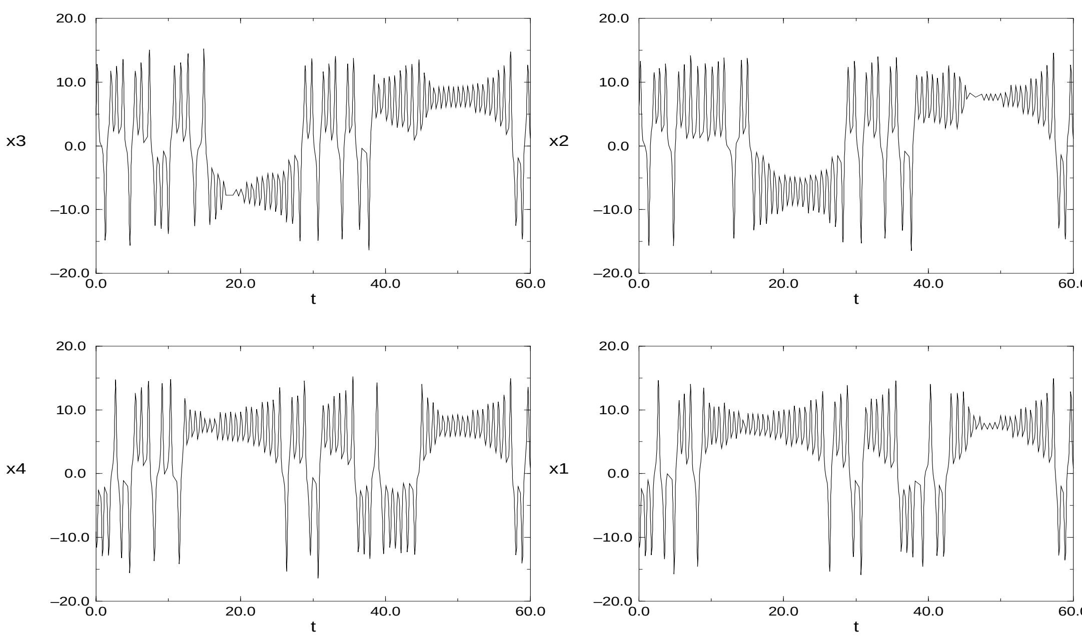

Figure 9: Nonsymmetric Chaotic Attractor at $\beta = 2.2$

dividual attractors all look very similar after a long period of integration, indicating that there is full symmetry on average. Again, detectives could be used to exploit this

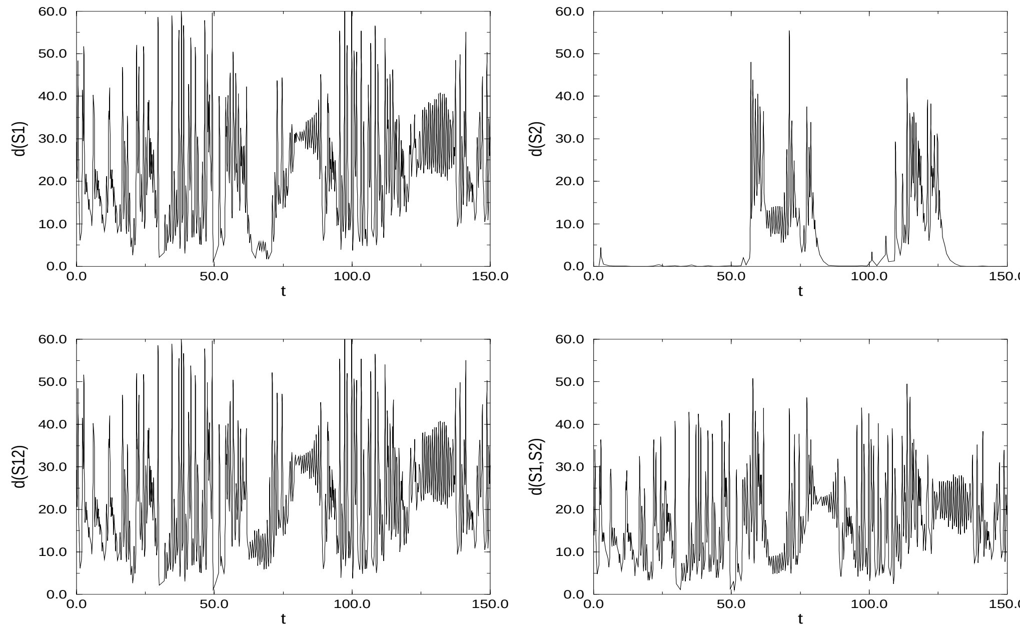

Figure 10: Plot of the Distance Functions with $\beta = 2.2$

in more detail. There are also chaotic attractors in some of the fixed point spaces. In particular, there is an attractor with full symmetry as well as two other attractors with either S_1 or S_2 symmetry. We note that there is no attractor with S_{12} symmetry only. This is due to the fact that $\sigma_4 < 0$ and so small perturbations from the fully symmetric attractor into this space are damped out and the motion returns to the fully symmetric attractor (cf. Corollary 3.10). Perturbations which do not preserve either full symmetry or S_{12} symmetry cause the attractor to lose symmetry, returning to the one shown in Figs. 8 and 9. The behaviour of this non-symmetric attractor can be seen more clearly by considering the distance functions which are plotted in Fig. 10. These diagrams show that the attractor comes very near to the fixed point space generated by the reflection S_2, sometimes for quite long periods of time. However, since $\sigma_3 > 0$, eventually the motion is forced away from this fixed point space. Note also that the attractor comes close to having full $Z_2 \times Z_2$ symmetry at certain times, although it stays near this fixed point space for much shorter times due to the larger exponent σ_2.

If σ_3 also became positive, we would not expect any qualitative change to occur in the chaotic motion, since there is no symmetry left which could be broken. However, another chaotic attractor with symmetry S_{12} only could be expected. We have not observed this situation.

Finally, we note that there is another chaotic attractor at this value of β which does not have the full symmetry on average. This is shown in Figs. 11 and 12, from which it can be observed that the attractor does have a Z_2 symmetry on average. Note however, that this symmetry combines a permutation of the oscillators with the reflectional symmetry of the individual Lorenz systems.

Taking the second coupling arrangement, the system has $\mathbf{D}_4$ symmetry and the block decomposition is given in (5.2). In this case, the dominant Lyapunov exponents of the different blocks are precisely σ_1, σ_3, σ_2 and σ_2 as in the previous system. These are shown in Fig. 2. The behaviour of this system is qualitatively similar to the behaviour of the previous one for $\beta = 0.5$ and $\beta = 0.625$. However, in this case, σ_2 is a multiple Lyapunov exponent associated with the two-dimensional irreducible representation of $\mathbf{D}_4$. Thus, when it passes through zero, the attractor is not expected to have any remaining symmetry (cf. Theorem 3.5) and this is observed using DSTOOL although again, there is full symmetry on average. When observing the appropriate distance functions, the motion occasionally comes near to a fixed point space associated with either $\mathbf{D}_4$ or one of its subgroups, but it very quickly moves away again. There is very little change at $\beta = 2.2$ when σ_3 is also positive since there was no symmetry left which could be broken.

Now taking the matrix $D = \mathrm{diag}(0.1, 0.2, 0.3)$ and repeating the computations, we obtain the results shown in Fig. 13. The behaviour of the coupled system is qualitatively similar to the previous system with diagonal D except that the final bifurcation does not occur since σ_3 stays negative. Note that the ordering of the Lyapunov exponents is very similar to the previous case also.

5.2 Coupled Duffing Oscillators

We consider Duffings equation

$$\ddot{x} + k\dot{x} + x^3 - x = A\cos\omega t$$

and couple three such systems together where the coupling involves only the x component of each system. Rewriting the second order equation as the first order system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -ky - x^3 + x + A\cos\omega t, \end{aligned}$$

the coupling matrix D is then given by

$$D = \begin{bmatrix} 0 & 0 \\ d & 0 \end{bmatrix}$$

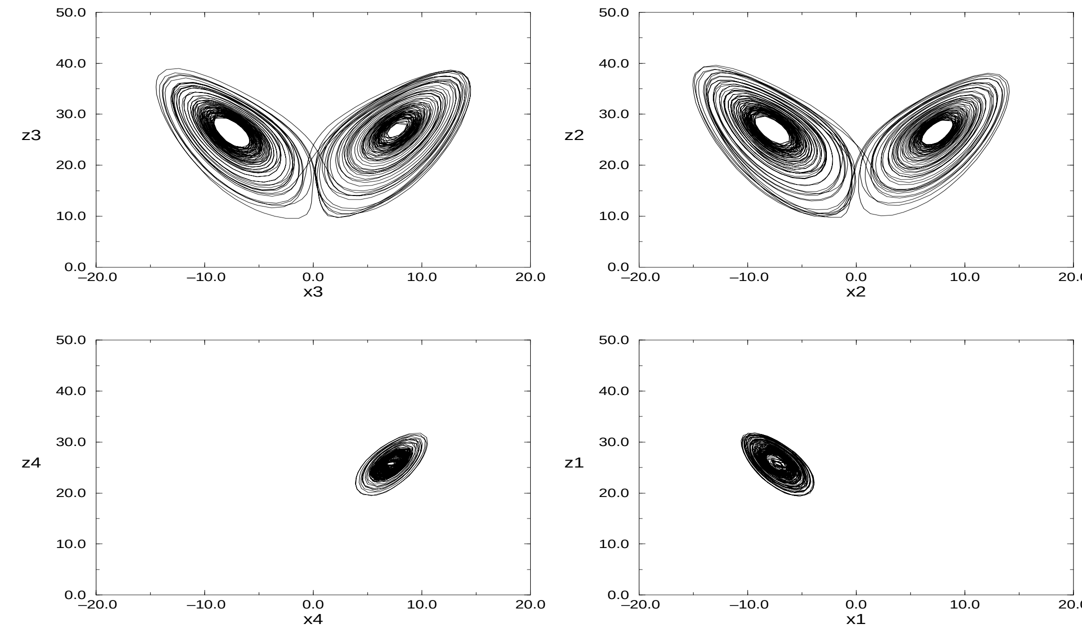

Figure 11: Nonsymmetric Chaotic Attractor at $\beta = 2.2$ without Full Symmetry on Average

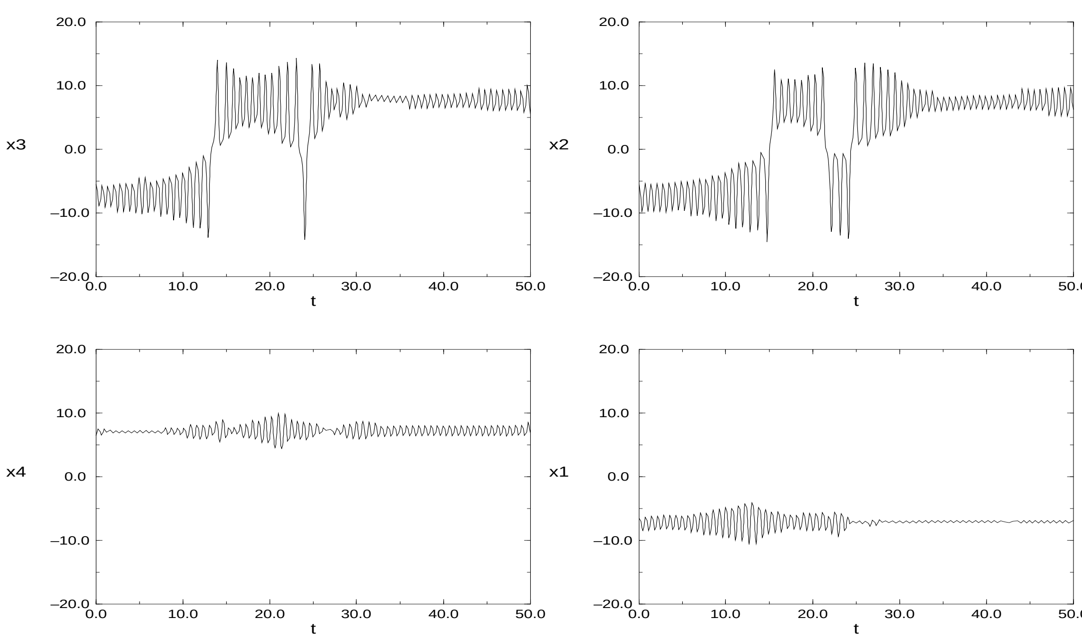

Figure 12: Nonsymmetric Chaotic Attractor at $\beta = 2.2$ without Full Symmetry on Average

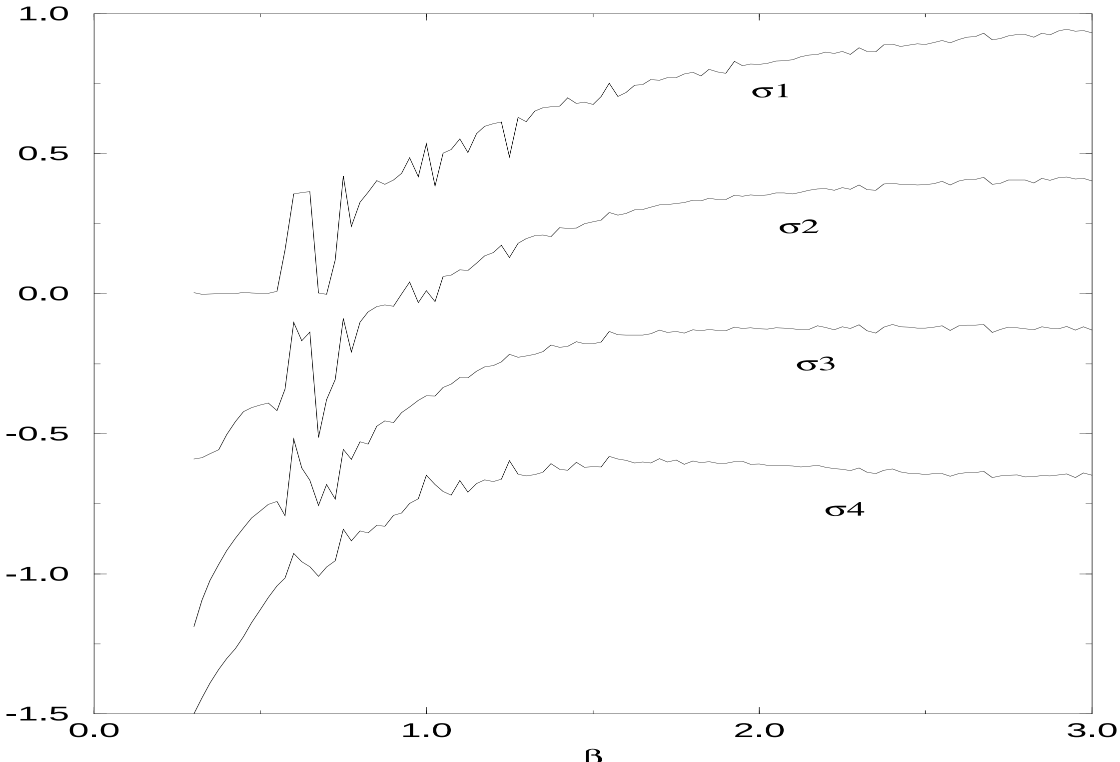

Figure 13: Dominant Lyapunov Exponents of the Blocks in (5.1) for $D = \mathrm{diag}(0.1, 0.2, 0.3)$

for some d. We note that in this case, D is not symmetric and so Proposition 4.2 does not hold. The three oscillators are coupled as shown in Fig. 14 where a and b are the relative strengths of the couplings between the oscillators. The coupling matrix in this case is then

$$\mathcal{B} = C \otimes D$$

where

$$C = \begin{bmatrix} -2a & a & a \\ a & -a-b & b \\ a & b & -a-b \end{bmatrix}.$$

If $a = b$, then this system has $\mathbf{D}_3$ symmetry and the situation is very similar to that for the coupled Lorenz systems with $\mathbf{D}_4$ symmetry. However, if $a \neq b$, then the system has only a $\mathbf{Z}_2$ symmetry generated by the reflection

$$S = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{pmatrix}$$

where I is the 2×2 identity matrix. In this case, the commuting matrices have the

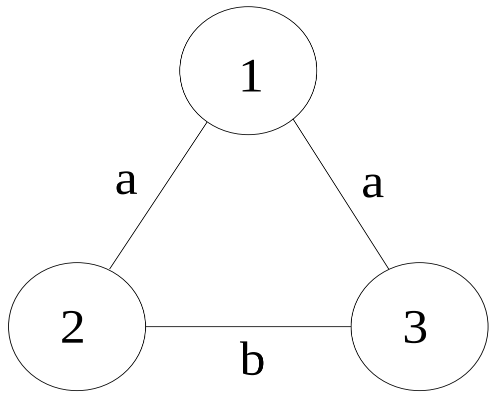

Figure 14: Coupling arrangement for the 3 Duffing oscillators

form

$$\begin{pmatrix} A & B & B \\ F & G & E \\ F & E & G \end{pmatrix}$$

and it is easily seen that the coupling matrix $\mathcal{B}$ is of this type.

For this example, symmetric solutions are those for which the second and third oscillators behave identically while the motion of the first can be different. However, there is also the possibility of all three oscillators being in phase giving rise to a uniform attractor (see definition at the end of Sec. 4). Thus, in this situation, the bifurcation of attractors from a uniform attractor to a symmetric attractor is in principle possible.

Now the block diagonalisation of the coupling matrix C determines the block structure of the Jacobian matrix of the coupled system (see Proposition 4.1). As C acts on $X = \mathbf{R}^3$, we can use the reflection S to decompose this space into the symmetric and antisymmetric spaces

$$\begin{aligned} X_s &= \{(x_1, x_2, x_2)^t \in \mathbf{R}^3 \ : \ x_1, x_2 \in \mathbf{R}\} \\ X_a &= \{(0, x_3, -x_3)^t \in \mathbf{R}^3 \ : \ x_3 \in \mathbf{R}\} \end{aligned}$$

corresponding to the two one-dimensional irreducible representations of Z_2. Thus, the block diagonalisation of C based on the symmetry is achieved using the transformation matrix

$$\tilde{Q}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

giving

$$\tilde{Q}_1^{-1} C \tilde{Q}_1 = \begin{pmatrix} -2a & \sqrt{2}a & 0 \\ \sqrt{2}a & -a & 0 \\ 0 & 0 & -a-2b \end{pmatrix}.$$

Since $\dim X_s = 2$ and $\dim X_a = 1$, this decomposition gives, as expected, a 2×2 symmetric block and a 1×1 antisymmetric block.

To obtain the further decomposition of the symmetric block, we consider the eigenvalues of C which are 0, $-3a$ and $-a-2b$. Taking the matrix of eigenvectors

$$\tilde{Q}_2 = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

leads to the decomposition of C as

$$\tilde{Q}_2^{-1} C \tilde{Q}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3a & 0 \\ 0 & 0 & -a-2b \end{pmatrix}.$$

Thus the three diagonal blocks of the Jacobian matrix, evaluated at a solution for which all three oscillators are in phase, are $\mathcal{D}_1^e = A(t)$, $\mathcal{D}_2^e = A(t) - 3aD$ and $\mathcal{D}_3^e = A(t) - (a+2b)D$ where $A(t)$ is the Jacobian matrix for a single oscillator.

Assuming that there is a uniform chaotic attractor, then if the dominant Lyapunov exponent of $\mathcal{D}_2^e$ goes from negative to positive, a bifurcation to an attractor within the symmetric subspace will occur so that the second and third oscillators remain in phase, but the first has different motion. However, if the dominant Lyapunov exponent of $\mathcal{D}_3^e$ changes from negative to positive, a symmetry breaking bifurcation of attractors will occur in which the resulting motion will have all three oscillators behaving differently.

We illustrate these two situations by taking two different coupling arrangements of this system with $k = 0.2$, $\omega = 1$ and $d = 0.1$. For the first, we take $a = 2$ and $b = 1$ giving the two symmetric eigenvalues of C as 0 and -6 and the antisymmetric eigenvalue as -4. In the second arrangement, we take $a = 4/3$ and $b = 7/3$ so that the symmetric eigenvalues of C are 0 and -4 while the antisymmetric eigenvalue is -6. Thus, only one calculation of the Lyapunov exponents is required to deal with both possibilities since the blocks are the same in both cases, although of course the interpretation of the results is different in each case. The dominant Lyapunov exponents σ_1, σ_2 and σ_3 for the blocks $A(t)$, $A(t) - 4D$ and $A(t) - 6D$ respectively are shown in Fig. 15.

We first consider the arrangement with $a = 2$ and $b = 1$. At $A = 0.36$, only σ_1 is positive and so we expect a uniform chaotic attractor. This was verified with DSTOOL

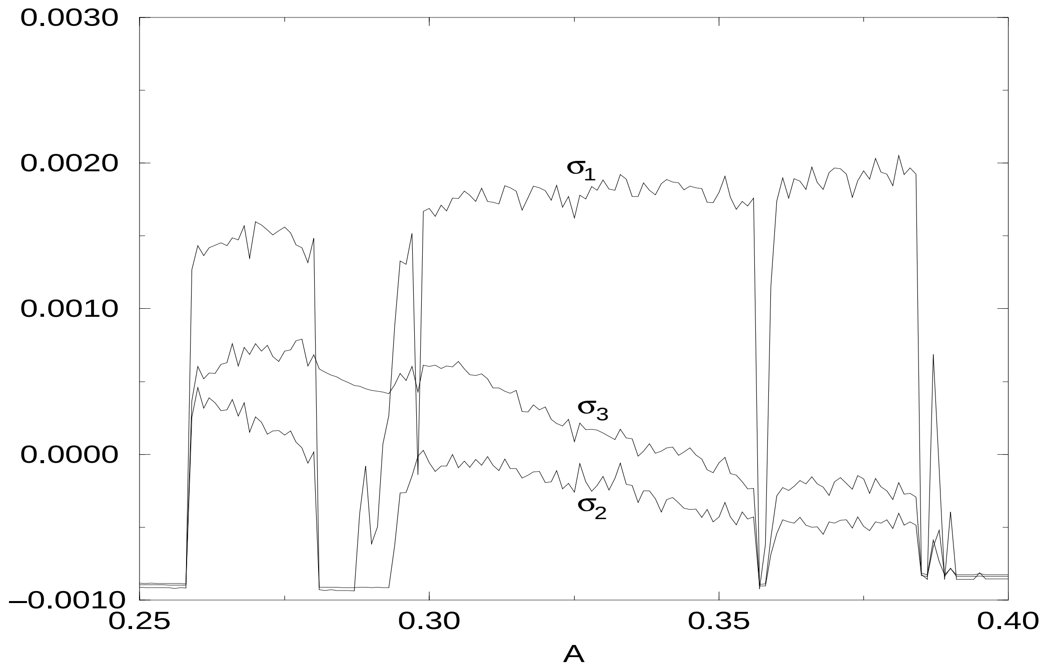

Figure 15: Dominant Lyapunov Exponents for the System of 3 Coupled Duffing Oscillators

and the attractor is shown in Figs. 16 and 17. Any type of small perturbation from this uniform attractor is damped out.

When $A = 0.33$, both σ_1 and σ_3 are positive. For this configuration, σ_3 is associated with the symmetric subspace and so we expect an attractor which is symmetric, but not uniform. This attractor is shown in Figs. 18 and 19.

For the second configuration, with $a = 4/3$ and $b = 7/3$, the diagonal blocks are the same as in the previous case and so the Lyapunov exponents for this system are shown in Fig. 15. Thus, at $A = 0.36$, the uniform chaotic attractor is stable as only σ_1 is positive. However, in this case, σ_3 is associated with the antisymmetric subspace and so at $A = 0.33$, we expect the uniform invariant set to be unstable with respect to symmetry breaking perturbations. Using DSTOOL, it is found that a perturbation in the solution of the first oscillator is soon damped out since this is not a symmetry breaking perturbation and the solution returns to the uniform motion. However, if a symmetry breaking perturbation is introduced to the second or third oscillator, then the resulting attractor is not symmetric. This motion is shown in Figs. 20 and 21.

Finally, we note that if σ_1, σ_2 and σ_3 are all positive, then three separate attractors

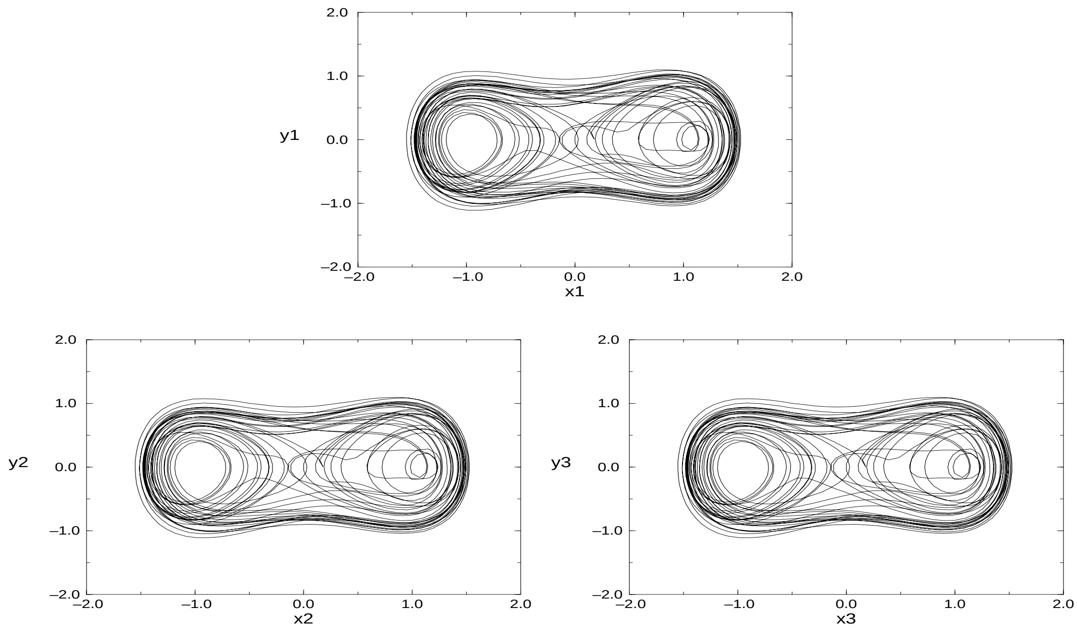

Figure 16: Uniform Chaotic Attractor at $A = 0.36$ with $a = 2$, $b = 1$

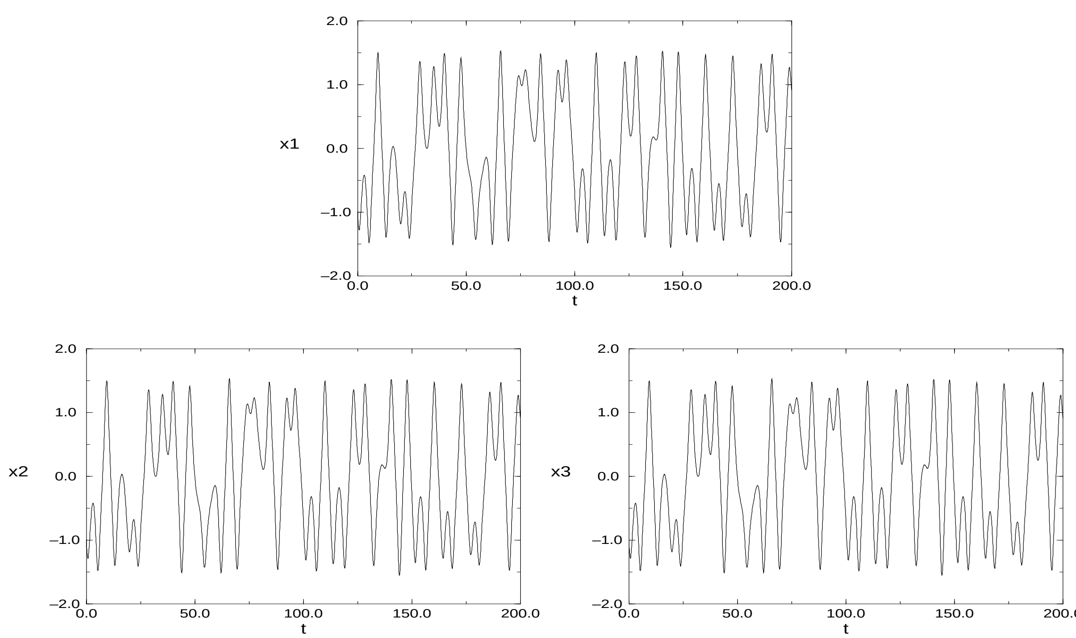

Figure 17: Uniform Chaotic Attractor at $A = 0.36$ with $a = 2$, $b = 1$

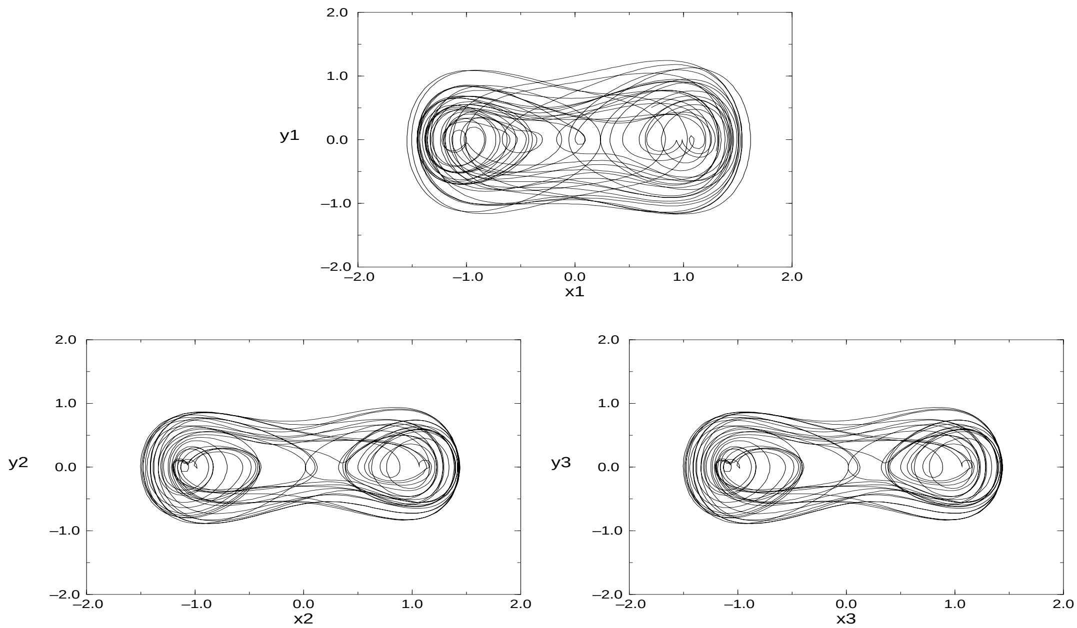

Figure 18: Symmetric Chaotic Attractor at $A = 0.33$ with $a = 2$, $b = 1$

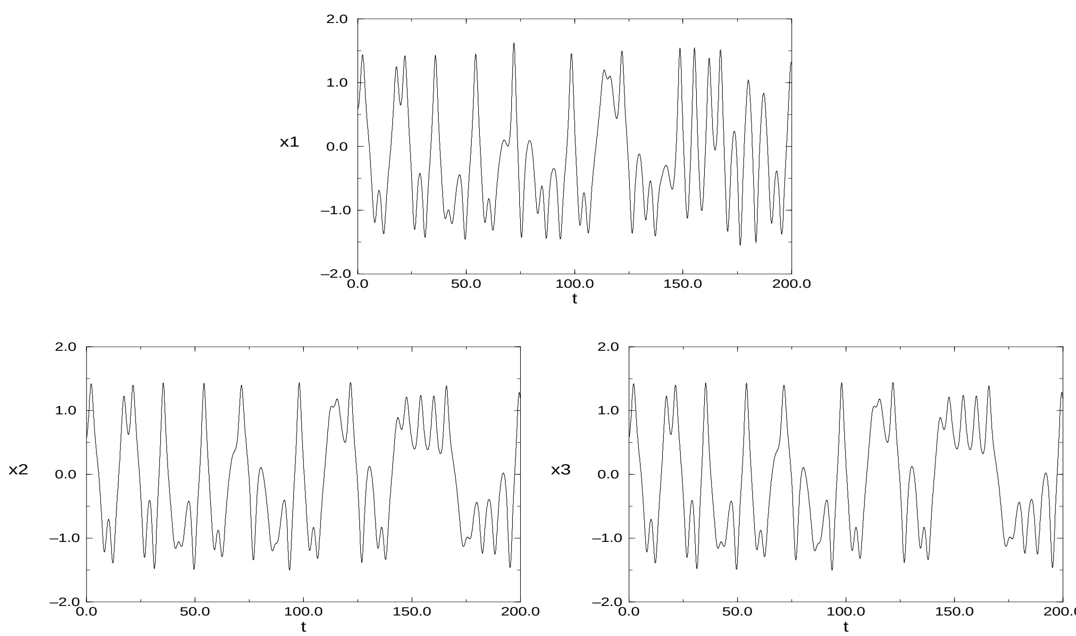

Figure 19: Symmetric Chaotic Attractor at $A = 0.33$ with $a = 2$, $b = 1$

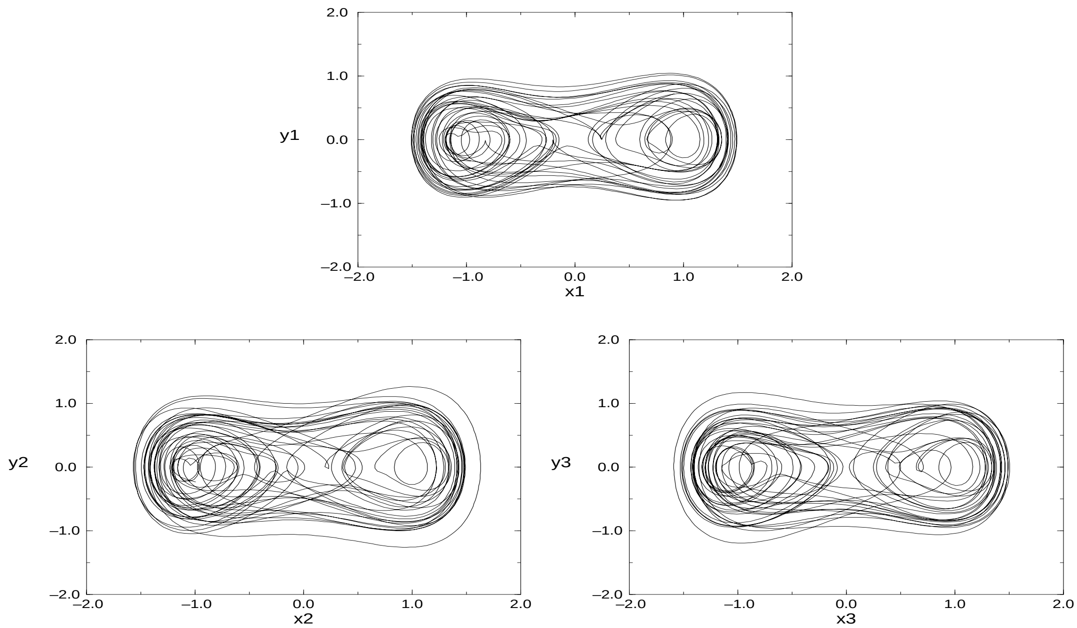

Figure 20: Nonsymmetric Chaotic Attractor at $A = 0.33$ with $a = 4/3$, $b = 7/3$

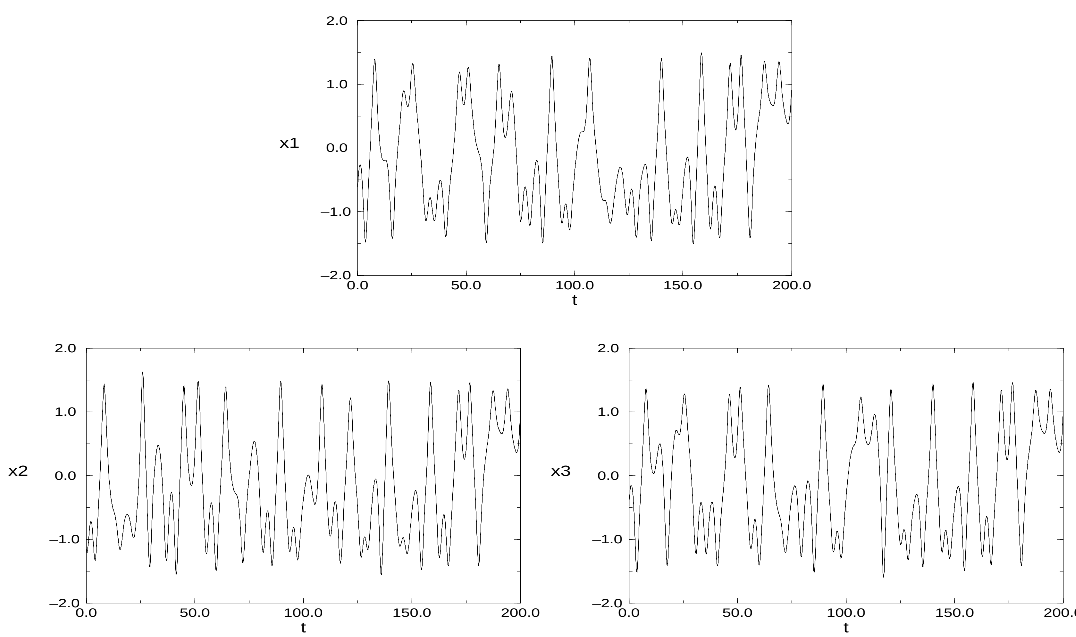

Figure 21: Nonsymmetric Chaotic Attractor at $A = 0.33$ with $a = 4/3$, $b = 7/3$

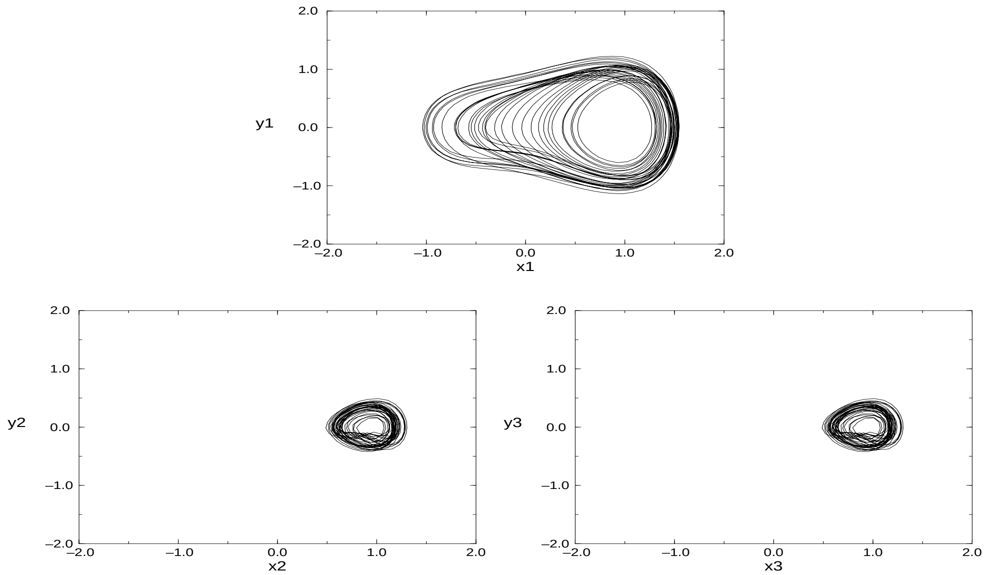

Figure 22: Symmetric Attractor at $A = 0.27$ with $a = 2$, $b = 1$

(inside appropriate subspaces) may exist. These consist of a uniform attractor, a symmetric attractor and a nonsymmetric attractor. If the symmetric and nonsymmetric attractors have full symmetry on average, then the nonsymmetric attractor will be the stable one (see Proposition 2.4). However, if the symmetric attractor does not have full symmetry on average, then it could be stable in which case there may not be a nonsymmetric attractor. Also, note that the symmetric and nonsymmetric attractors need not be chaotic.

To illustrate these phenomena, we take $A = 0.27$ at which point all three dominant Lyapunov exponents are positive. What we find in this case is a uniform chaotic attractor and a symmetric attractor. There is no nonsymmetric attractor in this case and the symmetric solution is stable. This is not possible if it has the full symmetry on average. However, in this situation, it is the Z_2 symmetry in the Duffing equation itself, given by $(x, y) \to (-x, -y)$ and $t \to t + \frac{\pi}{\omega}$ which is important. The uniform attractor does not have this symmetry pointwise and so it does not contribute to the block diagonalisation of the Jacobian matrix. However, it does have this symmetry on average. The reason that this symmetric attractor is stable is that it does not have this Z_2 symmetry on average. Thus, the positive Lyapunov exponent in the antisymmetric direction, associated with breaking the symmetric configuration of the oscillators, does not apply in this case. The symmetric attractor with $a = 2$, $b = 1$ is shown in Fig. 22.

By varying the damping coefficient k, we find that when $A = 0.27$ and $k = 0.15$, there

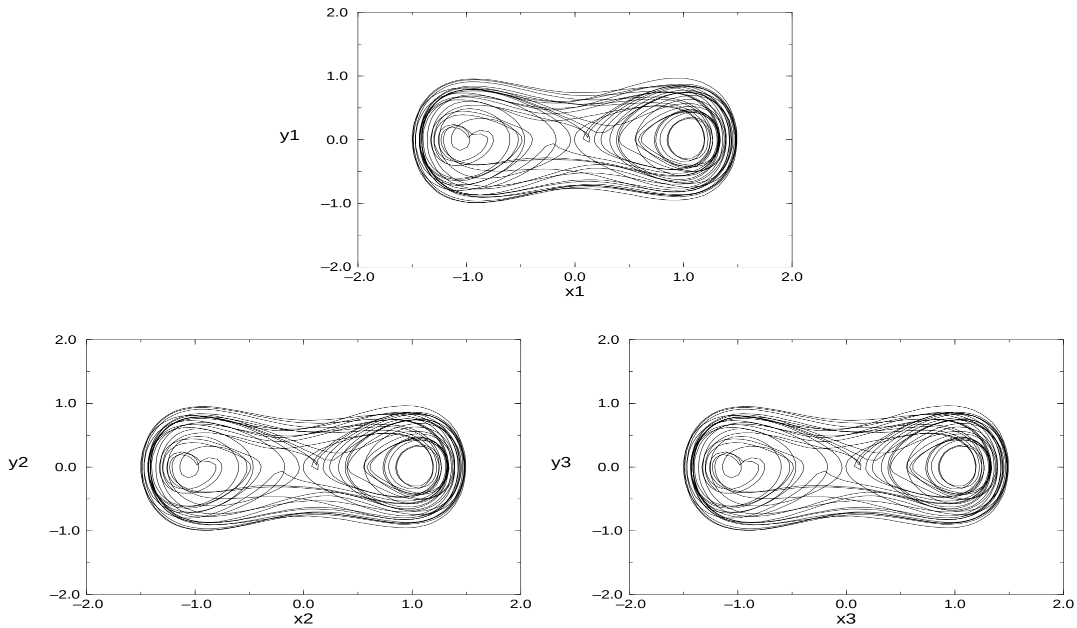

Figure 23: Uniform Attractor with $A = 0.27$, $k = 0.15$

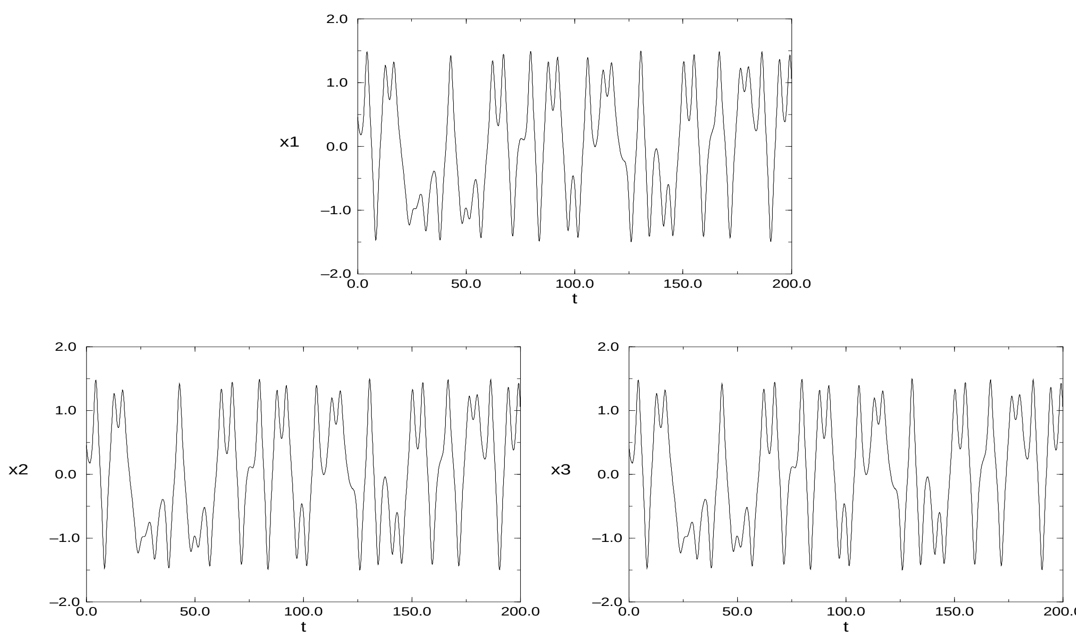

Figure 24: Uniform Attractor with $A = 0.27$, $k = 0.15$

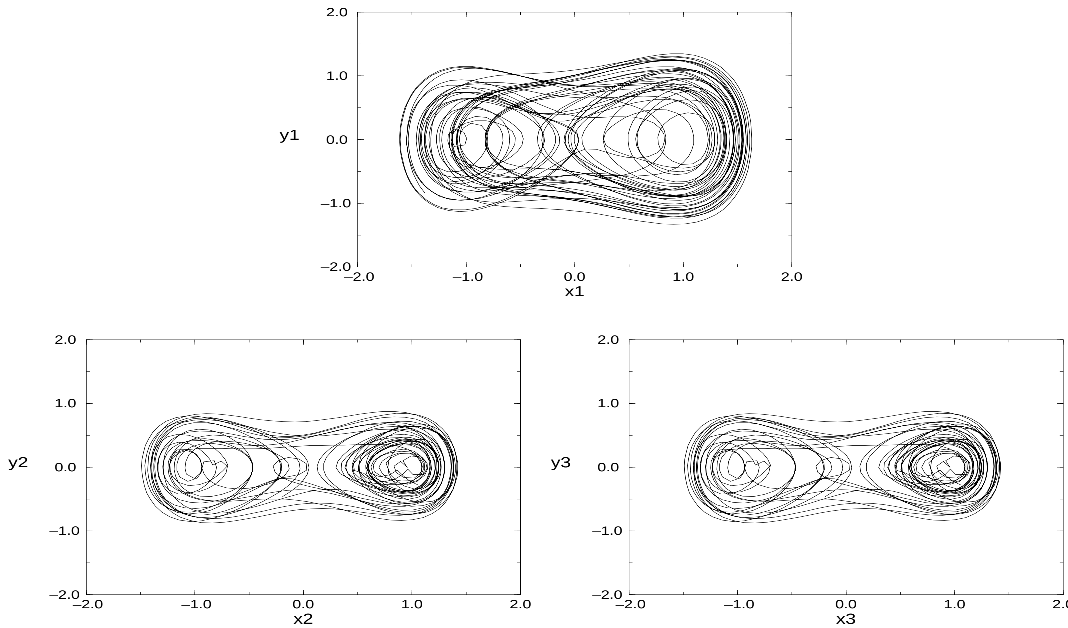

Figure 25: Symmetric Attractor with $A = 0.27$, $k = 0.15$

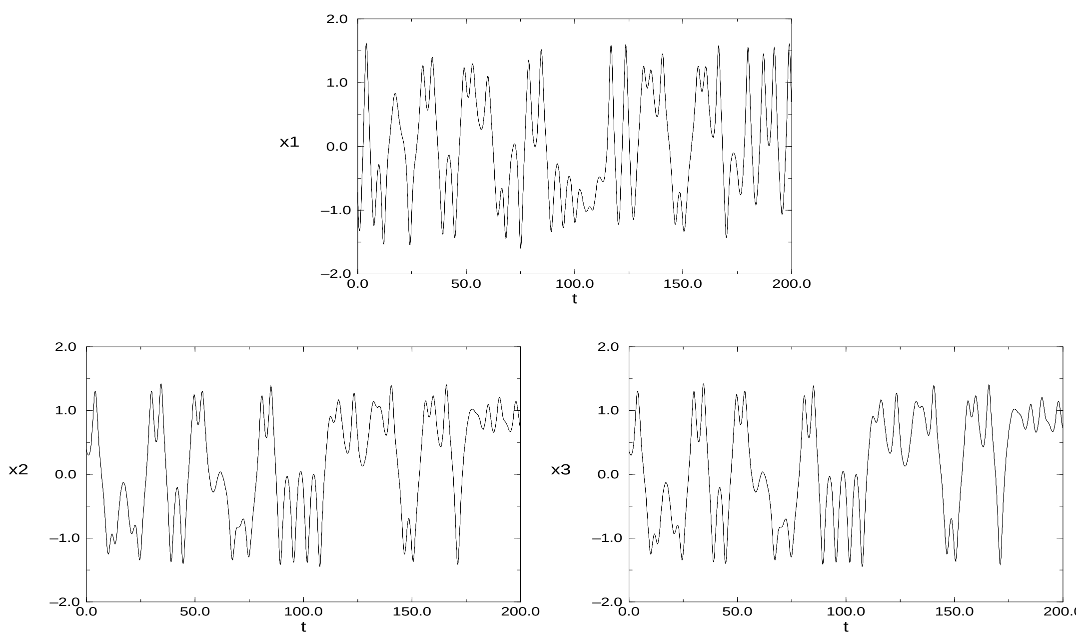

Figure 26: Symmetric Attractor with $A = 0.27$, $k = 0.15$

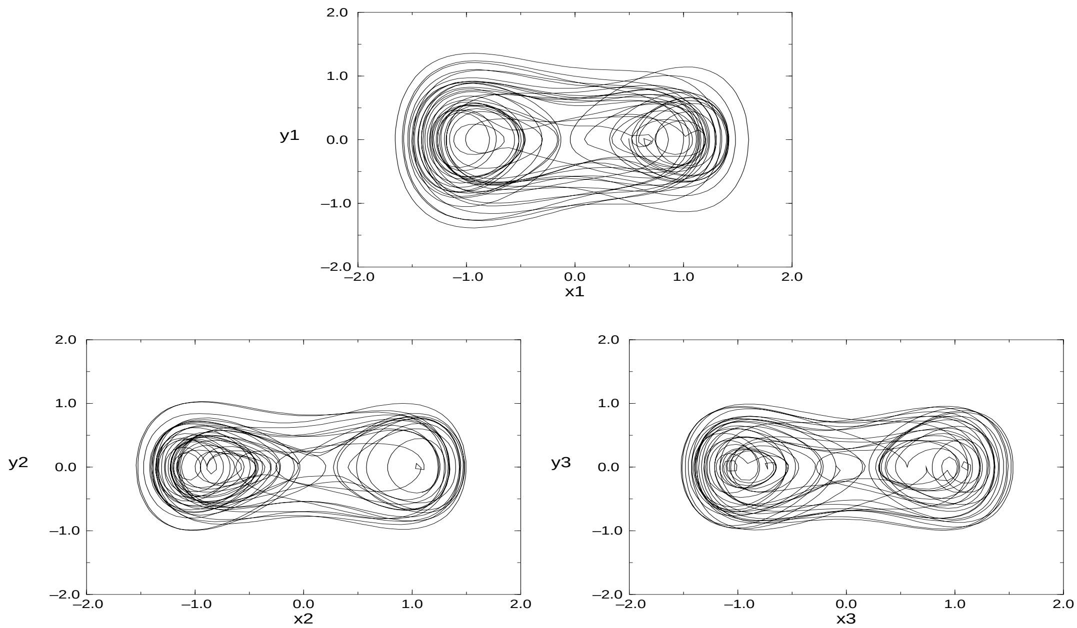

Figure 27: Nonsymmetric Attractor with $A = 0.27$, $k = 0.15$

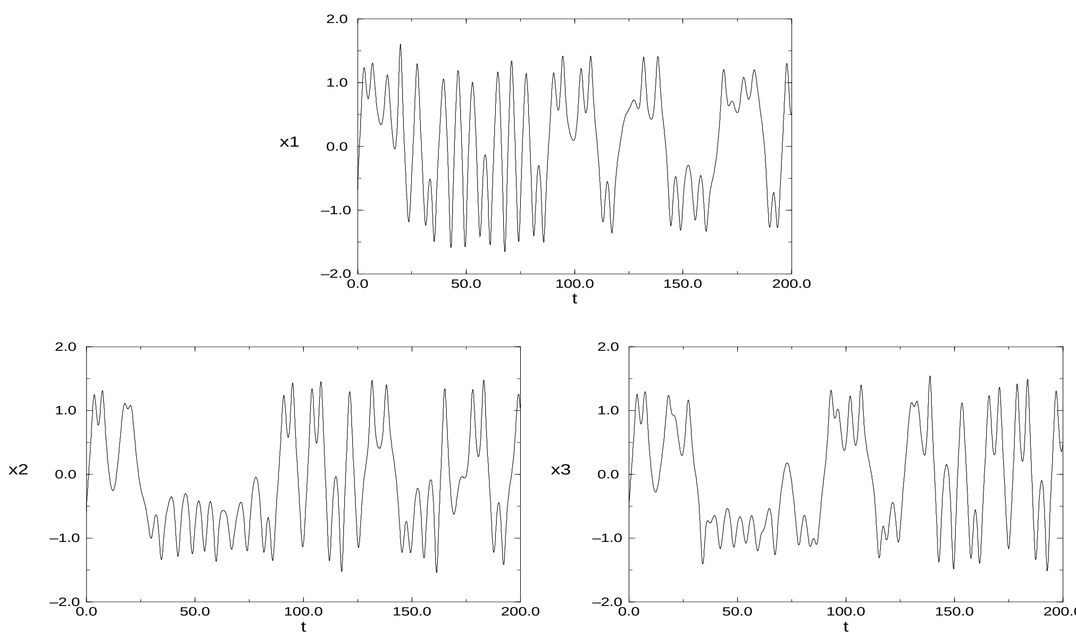

Figure 28: Nonsymmetric Attractor with $A = 0.27$, $k = 0.15$

are three chaotic attractors as described above. They all have the full symmetry on average, including the Z_2 symmetry from the Duffing equation, and it is easily verified that the nonsymmetric attractor is the stable one. These attractors are shown, for $a = 2$ and $b = 1$, in Figs. 23 to 28.

Acknowledgements

We wish to thank Marty Golubitsky for directing our attention to some papers on synchronous chaos in coupled oscillators. MD acknowledges support from the University of Surrey for a visit when this work was started.

References

[1] J.C. Alexander, I.Kan, J.A. Yorke and Z. You. Riddled basins, *Int. J. Bif. Chaos* **2** (1992), 795-813.

[2] P. Ashwin, J. Buescu and I. Stewart. Bubbling of attractors and synchronisation of chaotic oscillators, *Phys. Lett. A* **193** (1994), 126-139.

[3] P. Ashwin and I. Melbourne. Symmetry groups of attractors, *Arch. Rational Mech. Anal.* **126** (1994), 59-78.

[4] P.J. Aston. Analysis and computation of symmetry-breaking bifurcation and scaling laws using group theoretic methods, *SIAM J. Math. Anal.* **22** (1991), 181-212.

[5] P.J. Aston. Bifurcation and chaos in iterated maps with O(2) symmetry, *Int. J. Bif. Chaos* (1994), to appear.

[6] E. Barany, M. Dellnitz and M. Golubitsky. Detecting the symmetry of attractors, *Physica D* **67** (1993), 66-87.

[7] P. Chossat and M. Golubitsky. Symmetry increasing bifurcation of chaotic attractors, *Physica D*, **32** (1988), 423-436.

[8] M. Dellnitz, M. Golubitsky and I. Melbourne. Mechanisms of symmetry creation, in: *Bifurcation and Symmetry* (E. Allgower, K. Böhmer, M. Golubitsky eds.), ISNM **104** (Birkhäuser, 1992), 99-109.

[9] M. Dellnitz, M. Golubitsky and M. Nicol. Symmetry of attractors and the Karhunen-Loève decomposition, in: *Trends and Perspectives in Applied Mathematics* (L. Sirovich ed.) (Springer, 1994).

[10] J.-P. Eckmann and D. Ruelle. Ergodic theory of chaos and strange attractors, *Reviews of Modern Physics*, **57** (3) I (1985), 617-656.

[11] M. Field and M. Golubitsky. *Symmetry in Chaos*, (Oxford University Press, Oxford, 1992).

[12] H. Fujisaka and T. Yamada. Stability theory of synchronised motion in coupled oscillator systems, *Prog. Theor. Phys.*, **69** (1983), 32-47.

[13] K. Gatermann and A. Hohmann. Symbolic exploitation of symmetry in numerical pathfollowing, Konrad-Zuse-Zentrum für Informationstechnik Berlin, Preprint SC 90-11 (1990).

[14] M. Golubitsky and M. Nicol. Symmetry detectives for SBR attractors, preprint (1994).

[15] M. Golubitsky, I. Stewart and D. Schaeffer. *Singularities and Groups in Bifurcation Theory*, Vol. 2 (Springer, 1988).

[16] J.F. Heagy, T.L. Carroll and L.M. Pecora. Synchronous chaos in coupled oscillator systems, submitted (1994).

[17] T. Healey. Numerical bifurcation with symmetry: Diagnosis & computation of singular points, in: *Bifurcation Theory and its Numerical Analysis* (eds.: Li Kaitai, J.E. Marsden, M. Golubitsky and G. Iooss), (Xi'an Jiatong University Press, 1989), 218-227.

[18] G. King and I. Stewart. Symmetric chaos, in: *Nonlinear Equations in the Applied Sciences* (W.F. Ames and C.F. Rogers eds.), (Academic Press, 1991), 257-315.

[19] A.A Kirillov. *Elements of the Theory of Representations*, (Springer, 1976).

[20] E.N. Lorenz. Deterministic nonperiodic flow, *J. Atmos. Sci.*, **20** (1963), 130-141.

[21] R. Mañé. *Ergodic Theory and Differentiable Dynamics*, (Springer, 1987).

[22] I. Melbourne, M. Dellnitz and M. Golubitsky. The structure of symmetric attractors, *Arch. Rational Mech. Anal.* **123** (1993), 75-98.

[23] V.I. Oseledec. A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, *Trudy Mosk. Mat. Obsc.*, **19** (1968), 197-231.

[24] T.S. Parker and L.O. Chua. *Practical Numerical Algorithms for Chaotic Systems*, Springer, New York, (1989).

[25] C. Pugh and M. Shub. Ergodic attractors. *Trans. Amer. Math. Soc.* **312** (1989), 1-54.

[26] D. Ruelle. Small random perturbations of dynamical systems and the definition of attractors, *Commun. Math. Phys.*, **82** (1981), 137-151.

[27] N.F. Rulkov, M.M. Sushchik, L.S. Tsimring and H.D.I. Abarbanel. Generalised synchronisation of chaos in directionally coupled chaotic systems, preprint (1994).

[28] K.G. Szabo and T. Tel. On the symmetry-breaking bifurcation of chaotic attractors, *J. Stat. Phys.*, **54** (1989), 925-948.

[29] B. Werner. Eigenvalue problems with the symmetry of a group and bifurcations, in: *Continuation and Bifurcations: Numerical Techniques and Applications* (eds.: D. Roose, B. De Dier and A. Spence), NATO ASI Series C, **313** (1990), 71-88.

[30] B. Werner and A. Spence. The computation of symmetry-breaking bifurcation points, *SIAM J. Numer. Anal.*, **21** (1984), 388-399.

[31] L.-S. Young. Ergodic theory of chaotic dynamical systems, (1992), preprint.