The Sequential Knapsack Polytope

Y. Pochet[*] R. Weismantel[†]

[*] CORE and IAG, Université Catholique de Louvain, Belgium
[†] Konrad-Zuse-Zentrum für Informationstechnik Berlin, Germany

Abstract

In this paper we describe the convex hull of all solutions of the integer bounded knapsack problem in the special case when the weights of the items are divisible. The corresponding inequalities are defined via an inductive scheme that can also be used in a more general setting.

Keywords: complete description, integer programming, knapsack polytope, knapsack problem, separation.

1 Introduction

In this paper we deal with the integer bounded knapsack problem

$$(SKP) \qquad \begin{aligned} \max \quad & \sum_{i=1}^{n} \gamma_i x_i, \\ & \sum_{i=1}^{n} a_i x_i \le a_0, \\ & x_i \in \{0, \ldots, s_i\} \quad \text{for } i = 1, \ldots, n, \end{aligned}$$

where $0 < a_1 \le a_2 \le \ldots \le a_n$, $a_0, a_i, s_i \in \mathbb{N}$ for $i = 1, \ldots, n$ and the numbers a_i are divisible, i.e., $\frac{a_i}{a_{i-1}} \in \mathbb{N}$ for $i = 2, \ldots, n$. In this case we say that the knapsack problem has the *divisibility property*. It is also called the *sequential knapsack problem* (see [1]). Whenever we are given a knapsack problem having the divisibility property, we will assume without loss of generality that $a_1 = 1$.

Our main result is the construction of the system of inequalities that describes the convex hull of all solutions in this special case.

Since 30 years the knapsack polytope is of particular interest for researchers in polyhedral combinatorics. This is due to several reasons: one is the increasing number of applications like in circuit design, telecommunication, vehicle routing and scheduling that involve the knapsack problem as a subproblem. In order to apply polyhedral methods to such complex problems, a good understanding of the knapsack polytope is important. Secondly, the knapsack problem is the "easiest" case of a number dependent problem. A slight change of the weights of the items might change the inequalities that describe the polyhedron drastically. Therefore, it is important to understand "general principles" according to which valid inequalities are constructed. Examples in this direction are, for instance, Gomory cutting planes [2], covers [12], $(1, k)$-configurations [8], the concept of lifting [7], the weight reduction principle [10] or inequalities based on the Hilbert basis of a cone of exchange vectors [11]. The knapsack polytope is one of the very interesting and challenging polyhedra for which beautiful results can be discovered.

We present an inductive scheme to construct valid inequalities for the knapsack polytope and show, in case that the weights of the items have the divisibility property, that we obtain the complete description of the associated polyhedron. The special case of the knapsack problem with the divisibility property has been studied in the literature by several authors.

Hartmann and Olmstead [4] give an $O(n\ log\ n)$ algorithm for optimizing a linear objective function whose bottleneck operation is sorting the ratios $\frac{\gamma_i}{a_i}$, $i \in N$. The case of the sequential knapsack problem when $s_i = \infty$ for all $i \in N$ has been considered by Marcotte [6]. He shows that an optimum solution can be found in linear time and applies his algorithm to the cutting stock problem. Pochet and Wolsey [9] give an explicit description of the knapsack polyhedron with the divisibility property when there are no bounds on the variables. They also refer to applications in local area networking.

In Section 2 we present a transformation of any given sequential knapsack problem to a special one such that in terms of feasible solutions and optimization both formulations are equivalent. In Section 3 we outline a decomposition result for all the optimal solutions of such a transformed sequential knapsack problem. Our main result is contained in Section 4. Here we present an inductive scheme to generate valid inequalities for the sequential knapsack problem. Given an objective function, we construct an inequality via this scheme whose induced face contains the set of all optimal solutions. This sufficies to show that our inductive class of inequalities describes the sequential knapsack polyhedron. How inequalities defined via our inductive scheme can be interpreted combinatorially is the issue of Section 5. The discussions end in Section 6 with some extensions.

Throughout the paper we use the following notation.

For $v \in \mathbb{R}$, we set $v^+ := \max\{v, 0\}$, $\lceil v \rceil := \min\{j \in \mathbb{N} : j \geq v\}$ and $\lfloor v \rfloor := \max\{j \in \mathbb{N} : j \leq v\}$.

The constraint $\sum_{i=1}^n a_i x_i \leq a_0$ is called the *knapsack inequality*. The number $a_i \in \mathbb{N}$ is termed the *weight* of item i and $a_0 \in \mathbb{N}$ is called the *knapsack capacity*. We set $N := \{1, \ldots, n\}$ and we always assume that $0 < a_1 \leq \ldots \leq a_n \leq a_0$. An integer vector that satisfies the knapsack constraint and the lower and upper bound constraints is called *feasible*.

We say, F_c is a face of some polytope $\mathcal{P}$ induced by the valid inequality $c^T x \leq \gamma$, if $F_c = \{x \in \mathcal{P} \mid c^T x = \gamma\}$. Every $x \in F_c$ is also called a *root* of $c^T x \leq \gamma$. The inequalities $x_i \leq s_i$, $i \in N$ and $x_i \geq 0$, $i \in N$ are called *trivial*. For real numbers τ_j, $j = 1, \ldots, n$ we define $\sum_{j=v}^w \tau_j := 0$ if $v > w$ and, for $I \subseteq \{1, \ldots, n\}$ we use the notation $\tau(I) := \sum_{i \in I} \tau_i$ with $\tau(\emptyset) = 0$.

2 A transformation

In this section we present a transformation of the given sequential knapsack problem to a special sequential knapsack problem that satisfies certain requirements. We show that in terms of polyhedra and in terms of optimization both formulations are equivalent. We start by introducing the notion of *blocks*.

Definition 2.1 Let $B := \{i_1, \ldots, i_l\}$, $i_1 < \ldots < i_l$ be a subset of items. B is called a block if, for every $j \in \{2, \ldots, l\}$, $a_{i_j} \leq \sum_{v=1}^{j-1} s_{i_v} a_{i_v} + a_{i_1}$ holds.

Let B be a block. The above definition implies that for every number $\tau \in \{a_{i_1}, 2a_{i_1}, \ldots, \sum_{v=1}^l s_{i_v} a_{i_v}\}$ there exists a subset $W \subseteq B$ such that $\sum_{k \in W} \lambda_k a_k = \tau$ where $0 < \lambda_k \leq s_k$ for all $k \in W$. The number $u_B := \frac{\sum_{v=1}^l s_{i_v} a_{i_v}}{a_{i_1}}$ is called the *multiplicity* of block B. We replace block B by a single item B with *weight* a_{i_1} and multiplicity (upper bound) $u_B = \frac{\sum_{v=1}^l s_{i_v} a_{i_v}}{a_{i_1}}$. The objective function coefficient of B is the number γ_{i_1}.

Let $B_1, \ldots, B_m$ be a partition of N into blocks and denote by f_w, c_w, u_w the weight, objective function coefficient, multiplicity of block B_w, respectively, $w =$

$1, \ldots, m$. Now consider the knapsack problem where every block is replaced by a single item:

$$
(MSKP) \qquad \begin{aligned} \max \quad & \sum_{w=1}^{m} c_w z_w, \\ & \sum_{w=1}^{m} f_w z_w \le a_0, \\ & z_w \in \{0, \ldots, u_w\} \quad \text{for } w = 1, \ldots, m. \end{aligned}
$$

From the construction of the blocks it is clear that (MSKP) is a sequential knapsack problem (MSKP stands for modified sequential knapsack problem). We now show that there is a many to one correspondence between the feasible solutions of the original problem (SKP) and the feasible solutions of its modified version (MSKP). For ease of notation we assume that $f_1 \le f_2 \le \ldots \le f_m$, and in case $f_w = f_{w+1}$, then $c_w \ge c_{w+1}$ holds. By $\mathcal{P}_{SKP}$ and $\mathcal{P}_{MSKP}$ we denote the convex hull of all feasible vectors of the problem (SKP) and (MSKP), respectively.

Let $z \in \mathbb{R}^m$ be a feasible solution of (MSKP), i.e., $0 \le z_w \le u_w$, z_w integer for all $w = 1, \ldots, m$. By Definition 2.1, for every $w \in \{1, \ldots, m\}$ there exist integers $0 \le \lambda_j \le s_j$, $j \in B_w$ such that $\sum_{j \in B_w} a_j \lambda_j = f_w z_w$. In fact, for all subsets I_w of items in B_w with $\sum_{j \in I_w} a_j \lambda_j = f_w z_w$, $0 \le \lambda_j \le s_j$, $\lambda_j \in \mathbb{N}$, the vector $x \in \mathbb{R}^n$ defined via $x_j = \lambda_j$ if $j \in I_w$ for some $w = 1, \ldots, m$ and $x_j = 0$, otherwise, is feasible for problem (SKP).
Conversely, with every vector $x \in \mathbb{R}^n$ that is feasible for problem (SKP) we associate a vector $z \in \mathbb{R}^m$ by setting $z_w := \frac{\sum_{j \in B_w} a_j x_j}{f_w}$, $w = 1, \ldots, m$. Then, $\sum_{i=1}^n a_i x_i = \sum_{w=1}^m f_w z_w$.
It follows that an integer vector z with $z_w \in \{0, \ldots, u_w\}$ for $w = 1, \ldots, m$ is feasible for (MSKP) if and only if there exist feasible vectors of (SKP) with the same total weight as z.

Now suppose that $\sum_{w=1}^m b_w z_w \le b_0$ is a valid inequality for the polytope $\mathcal{P}_{MSKP}$. By setting $\beta_i := b_w \frac{a_i}{f_w}$ if item i belongs to block B_w, the inequality $\sum_{i=1}^n \beta_i x_i \le b_0$ is valid for $\mathcal{P}_{SKP}$. This statement follows from the fact that if x is feasible for (SKP) then $z = (z_1, \ldots, z_m)^T$ defined via $z_w = \frac{\sum_{j \in B_w} a_j x_j}{f_w}$, $w = 1, \ldots, m$ is feasible for (MSKP) and satisfies $\sum_{i \in B_w} \beta_i x_i = b_w z_w$. This shows that valid inequalities for $\mathcal{P}_{MSKP}$ can be transformed into valid inequalities for $\mathcal{P}_{SKP}$.

In the following we focus on a special partition of the set N into blocks $B_1, \ldots, B_m$. For an item i of (SKP), its gain per unit is defined as $\frac{\gamma_i}{a_i}$. Let $g_1, \ldots, g_v$ denote the different values of gains per unit for all items of (SKP) (clearly, $v \le n$). We partition each set $V_g := \{i \in N : \frac{\gamma_i}{a_i} = g\}$, $g \in \{g_1, \ldots, g_v\}$ into blocks $B_1^g, \ldots, B_{n_g}^g$ such that $B_i^g \cup B_j^g$ is not a block anymore, for $i, j \in \{1, \ldots, n_g\}$, $i \ne j$.

Let $B_1, \ldots, B_m$ denote the final blocks constructed this way. Each block B_i, $i = 1, \ldots, m$ is called a *maximal block* and, by definition, all items belonging to the block B_i have the same gain per unit.

Example 2.2. Consider the instance of the sequential knapsack problem

$$\begin{array}{ll} \max & x_1 + 3x_2 + 6x_3 + 18x_4 + 6x_5 + 50x_6 + 200x_7, \\ & x_1 + 5x_2 + 10x_3 + 30x_4 + 30x_5 + 120x_6 + 360x_7 \leq 396, \end{array}$$

with upper bounds s_i on the variables x_i as follows: $s_1 = 4$, $s_2 = 4$, $s_3 = 20$, $s_4 = 4$, $s_5 = 2$, $s_6 = 1$ and $s_7 = 1$. The set of items is partitioned into the 5 maximal blocks: $V_1 = B_1 = \{1\}$, $V_2 = B_2 = \{2, 3, 4\}$, $V_3 = B_3 = \{5\}$, $V_4 = B_4 = \{6\}$ and $V_5 = B_5 = \{7\}$. After transformation we obtain the instance of the sequential knapsack problem:

$$\begin{array}{ll} \max & z_1 + 3z_2 + 6z_3 + 50z_4 + 200z_5, \\ & z_1 + 5z_2 + 30z_3 + 120z_4 + 360z_5 \leq 396, \end{array}$$

with upper bounds $u_1 = 4$, $u_2 = 68$, $u_3 = 2$, $u_4 = 1$ and $u_5 = 1$ on the variables z_i, $i = 1, \ldots, 5$. ■

For a given sequential knapsack problem, the aggregation of items into maximal blocks is unique. If $V_g = \{i_1, \ldots, i_l\}$ with $i_1 < i_2 < \ldots < i_l$ is the set of all items in N with gain per unit equal to g, then the unique maximal block containing i_1 is $B_1^g = \{i_1, \ldots, i_t\}$ where $t + 1 = \min\{j \in \{2, \ldots, l + 1\} : \ a_{i_j} > \sum_{v=1}^{j-1}(s_{i_v} a_{i_v}) + a_{i_1}\}$ and $a_{i_{l+1}}$ is defined as $\sum_{v=1}^{l}(s_{i_v} a_{i_v}) + a_{i_1} + \epsilon$ with $\epsilon > 0$. No item in this subset $\{i_1, \ldots, i_t\}$ can belong to some maximal block containing an item i_j, $l \geq j > t$, because $a_{i_j} \geq a_{i_{t+1}} > \sum_{v=1}^{t}(s_{i_v} a_{i_v}) + a_{i_1}$. By removing B_1^g from V_g and iteratively using the same argument, the unique partition of V_g into maximal blocks $B_1^g, \ldots, B_{n_g}^g$, with $B_j^g = \{i_{s(j)}, \ldots, i_{e(j)}\}$ for $j = 1, \ldots, n_g$, $s(1) = 1$, $e(n_g) = l$, $e(j-1) + 1 = s(j)$ for $j = 2, \ldots, n_g$ can be constructed easily. This argument applies to all numbers $g \in \{g_1, \ldots, g_v\}$.

From the above discussions follows that, if we define (MSKP) using the unique partition into maximal blocks, a vector z is feasible for (MSKP) if and only if the associated vectors x are feasible for (SKP). As each maximal block contains items in N with the same gain per unit we obtain in addition: a vector $x \in \mathbb{R}^n$ is optimal with respect to (SKP) if and only if the associated vector $z \in \mathbb{R}^m$ is optimal with respect to (MSKP) and vice versa, a vector $z \in \mathbb{R}^m$ is optimal with respect to (MSKP) if and only if all of its associated vectors $x \in \mathbb{R}^n$ are optimal solutions to (SKP).

To simplify notation, we always assume, when transforming (SKP) to (MSKP) using maximal blocks, that $f_1 \leq f_2 \leq \ldots \leq f_m$, and in case $f_w = f_{w+1}$, then $c_w > c_{w+1}$. Moreover, the above arguments for the construction of the unique partition into maximal blocks show that for the transformed problem (MSKP) the following property always holds:

$$f_w > \sum_{i=1, \frac{c_i}{f_i} = \frac{c_w}{f_w}}^{w-1} f_i u_i \quad \text{for } w = 2, \ldots, m.$$

This property will be used in the next section to derive a decomposition scheme of all optimal solutions of (MSKP).

3 Decomposition of optimum solutions

In this section, we characterize the optimal solutions of a problem (MSKP) obtained by the maximal block transformation of an initial (SKP) problem presented in the previous section.

Let positive rational numbers $c_1, \ldots, c_m$ and positive integers $u_1, \ldots, u_m$, $f_1, \ldots, f_m$ be given such that

$$1 = f_1 \leq f_2 \leq \ldots \leq f_m;$$

$$f_j = f_{j+1} \text{ implies } c_j > c_{j+1};$$

$$\frac{f_{j+1}}{f_j} \in \mathbb{N}, \text{ for } j = 1, \ldots m-1.$$

We also assume that for every $j \in \{2, \ldots, m\}$, $f_j > \sum_{i=1, \frac{c_i}{f_i} = \frac{c_j}{f_j}}^{j-1} f_i u_i$ holds.

For every $F \in \mathbb{N}$ and $j \in M = \{1, \ldots, m\}$, we denote by $\mathcal{P}_F(j)$ the convex hull of all solutions of the following (MSKP) problem with knapsack capacity F and restricted to the variables 1 to j.

$$\mathcal{P}_F(j) = \Big\{ z \in \mathbb{R}^j : \begin{array}{l} \sum_{i=1}^{j} f_i z_i \leq F \\ 0 \leq z_i \leq u_i \text{ and } z_i \text{ integer} \qquad \text{for } i = 1, \ldots, j \end{array} \Big\}.$$

The optimization problem $OP_F(j)$ is the program

$$(OP_F(j)) \qquad \max \sum_{i=1}^{j} c_i z_i \quad \text{such that} \quad z \in \mathcal{P}_F(j).$$

Note that in this section we only consider optimization problems $OP_F(j)$ with positive objective coefficients. Using this notation we have that $\mathcal{P}_{MSKP} = \mathcal{P}_{a_0}(m)$ and $MSKP = OP_{a_0}(m)$. By $O_F(j)$ we denote the set of all optimal solutions to $OP_F(j)$. Finally, for an item $i \in M$, we define $\Delta_i = \left\{u \in \{1, \ldots, i-1\} : \frac{c_u}{f_u} > \frac{c_i}{f_i}\right\}$, i.e., Δ_i is the set of all items before i whose gain per unit is strictly better than the one of i. Let $f(\Delta_i) = \sum_{j \in \Delta_i} f_j u_j$ be the total weight of items in Δ_i.

For every F and j, we now construct a decomposition tree whose paths from the root node to the leaves contain all the optimal solutions of $OP_F(j)$. The key for this result is the next lemma showing that for every optimum solution $z \in O_F(j)$ the component z_j can attain at most two different values.

Lemma 3.1. For the optimization problem $OP_F(j)$ with positive cost coefficients the following statement is true:

$$\begin{aligned} z \in O_F(j) \quad \text{implies} \quad \text{that} \quad & z_j \geq \min\left\{u_j \;;\; \left\lfloor \frac{(F - f(\Delta_j))^+}{f_j} \right\rfloor \right\} \\ \text{and} \quad & z_j \leq \min\left\{u_j \;;\; \left\lceil \frac{(F - f(\Delta_j))^+}{f_j} \right\rceil \right\}. \end{aligned}$$

Proof. We prove this result by contradiction using standard exchange arguments. Several cases are distinguished.

(i) When $f(\Delta_j) \geq F$, the lemma states that $z_j = 0$ for all $z \in O_F(j)$. By contradiction, suppose that there exists $z \in O_F(j)$ with $z_j > 0$. As $\sum_{l \in \Delta_j} f_l z_l + f_j z_j \leq F \leq f(\Delta_j)$ and $z_j > 0$, we have $\sum_{l \in \Delta_j} f_l(u_l - z_l) \geq f_j z_j > 0$. By the divisibility of the weights, there exist integers $\lambda_l \in \{0, \ldots, u_l - z_l\}$ for all $l \in \Delta_j$ such that $\sum_{l \in \Delta_j} f_l \lambda_l = f_j z_j$. We now define a solution z' with $z'_l = z_l$ for $l \in \{1, \ldots, j-1\} \setminus \Delta_j$, $z'_l = z_l + \lambda_l \leq u_l$ for $l \in \Delta_j$ and $z'_j = 0$. Then, $z' \in \mathcal{P}_F(j)$ because $\sum_{i=1}^{j} f_i z_i = \sum_{i=1}^{j} f_i z'_i$ and the solution z' has strictly better objective value than z by definition of Δ_j. This contradicts the optimality of z.

(ii) When $f(\Delta_j) + f_j u_j \leq F$, we obtain $u_j \leq \left\lfloor \frac{(F - f(\Delta_j))^+}{f_j} \right\rfloor$ because u_j is integral. In this case the lemma states that $z_j = u_j$ for all $z \in O_F(j)$. By contradiction, suppose that there exists $z \in O_F(j)$ with $z_j < u_j$ and set $\delta = \sum_{i \in \{1, \ldots, j-1\} \setminus \Delta_j} f_i z_i$.

If $\delta < f_j(u_j - z_j)$, the new solution z' with $z'_i = 0$ for $i \in \{1, \ldots, j-1\} \setminus \Delta_j$, $z'_i = z_i$ for $i \in \Delta_j$ and $z'_j = u_j$ belongs to $\mathcal{P}_F(j)$ and has strictly better objective value than z, because $\frac{c_j}{f_j} \geq \frac{c_i}{f_i}$ for all $i \in \{1, \ldots, j-1\} \setminus \Delta_j$, a contradiction.

Hence, we can assume that $\delta \geq f_j(u_j - z_j)$. By the divisibility of the weights, there exist integers $\lambda_l \in \{0, \ldots, z_l\}$ for all $l \in \{1, \ldots, j-1\} \setminus \Delta_j$ with $\sum_{l \in \{1,\ldots,j-1\} \setminus \Delta_j} f_l \lambda_l = f_j(u_j - z_j)$. The new solution z' with $z'_l = z_l - \lambda_l$ for $l \in \{1, \ldots, j-1\} \setminus \Delta_j$, $z'_l = z_l$ for $l \in \Delta_j$ and $z'_j = u_j$ belongs to $\mathcal{P}_F(j)$. Let $W = \{i \in \{1, \ldots, j-1\} \setminus \Delta_j : \lambda_i > 0\}$. As $\sum_{i \in W} f_i u_i \geq \sum_{i \in W} f_i \lambda_i = f_j(u_j - z_j) \geq f_j > \sum_{i=1, \frac{c_i}{f_i} = \frac{c_j}{f_j}}^{j-1} f_i u_i$ (where the last inequality holds by assumption), there exists $i \in W$ with $\frac{c_i}{f_i} < \frac{c_j}{f_j}$. Then the solution z' has strictly better objective value than z, again a contradiction.

In the remaining cases we have that $F - f_j u_j < f(\Delta_j) < F$ and the lemma states $z_j \geq \left\lfloor \frac{F - f(\Delta_j)}{f_j} \right\rfloor$ and $z_j \leq \left\lceil \frac{F - f(\Delta_j)}{f_j} \right\rceil$.

(iii) When $F - f_j u_j < f(\Delta_j) < F$, suppose, by contradiction, that there exists $z \in O_F(j)$ with $z_j = \left\lfloor \frac{F - f(\Delta_j)}{f_j} \right\rfloor - \epsilon$, $\epsilon \in \mathbb{N}$ and $\epsilon > 0$. As $f(\Delta_j) + \left\lfloor \frac{F - f(\Delta_j)}{f_j} \right\rfloor f_j \leq F$, a similar argument as in case (ii) shows that there exists a solution $z' \in \mathcal{P}_F(j)$ with $z'_i = z_i$ for $i \in \Delta_j$, $z'_j = \left\lfloor \frac{F - f(\Delta_j)}{f_j} \right\rfloor = z_j + \epsilon$, $\sum_{i \in \{1,\ldots,j-1\} \setminus \Delta_j} f_i z'_i = \left[\sum_{i \in \{1,\ldots,j-1\} \setminus \Delta_j} f_i z_i - \epsilon f_j \right]^+$ and with a strictly better objective value than z, a contradiction.

(iv) When $F - f_j u_j < f(\Delta_j) < F$, suppose, by contradiction, that there exists $z \in O_F(j)$ with $z_j = \left\lceil \frac{F - f(\Delta_j)}{f_j} \right\rceil + \epsilon$, $\epsilon \in \mathbb{N}$ and $\epsilon > 0$. As $\sum_{l \in \Delta_j} f_l z_l + f_j z_j \leq F \leq f(\Delta_j) + \left\lceil \frac{F - f(\Delta_j)}{f_j} \right\rceil f_j$, a similar argument as in case (i) shows that there exists a solution $z' \in \mathcal{P}_F(j)$ with $z'_i = z_i$ for $i \in \{1, \ldots, j-1\} \setminus \Delta_j$, $z'_j = \left\lceil \frac{F - f(\Delta_j)}{f_j} \right\rceil = z_j - \epsilon$, $\sum_{i \in \Delta_j} f_i z'_i = \left[\sum_{i \in \Delta_j} f_i z_i + \epsilon f_j \right]$ and with a strictly better objective value than z, a contradiction. ■

Lemma 3.1 can be applied inductively to build a binary decomposition tree containing all potential optimal solutions in $O_F(j)$. We illustrate this on an example.

Example 2.2 Continued. The modified sequential knapsack problem $\mathcal{P}_{396}(5)$ using the maximal block transformation was defined as

$$
\begin{aligned}
\max \quad & z_1 + 3z_2 + 6z_3 + 50z_4 + 200z_5, \\
& z_1 + 5z_2 + 30z_3 + 120z_4 + 360z_5 \leq 396,
\end{aligned}
$$

with upper bounds $u_1 = 4$, $u_2 = 68$, $u_3 = 2$, $u_4 = 1$ and $u_5 = 1$ on the variables z_i, $i = 1, \ldots, 5$.

We have $\frac{c_1}{f_1} = 1$, $\frac{c_2}{f_2} = 0.6$, $\frac{c_3}{f_3} = 0.2$, $\frac{c_4}{f_4} = 0.42$, $\frac{c_5}{f_5} = 0.56$, and hence $\Delta_1 = \emptyset$, $\Delta_2 = \{1\}$, $f(\Delta_2) = 4$, $\Delta_3 = \Delta_4 = \Delta_5 = \{1, 2\}$, $f(\Delta_3) = f(\Delta_4) = f(\Delta_5) = 344$.

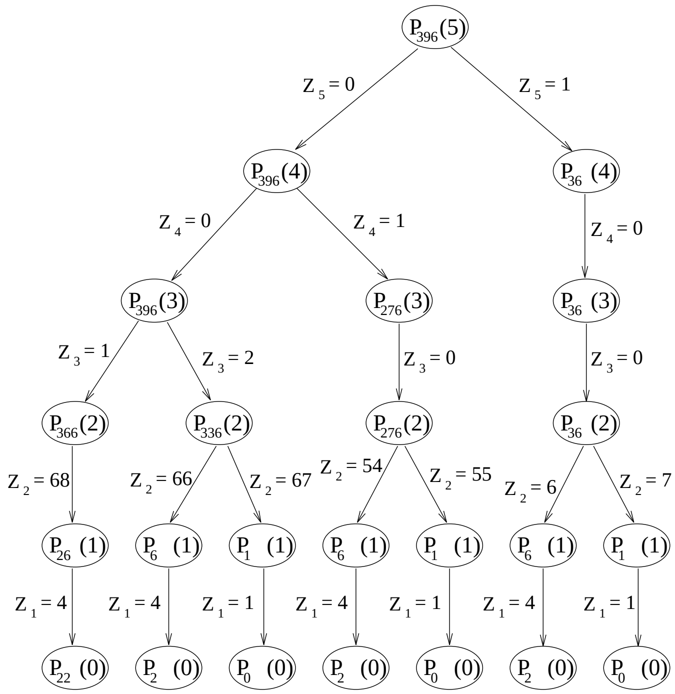

Figure 1: Decomposition of Optimal Solutions for Example 2.2

Figure 1 illustrates the decomposition tree that we obtain from applying Lemma 3.1 iteratively. The node labels identify the problems $\mathcal{P}_F(j)$ to be solved and the value of z_j is fixed on the corresponding branches. For example, Lemma 3.1 applied to $\mathcal{P}_{396}(5)$ yields $z_5 = 0$ or $z_5 = 1$. If $z_5 = 0$ we are left with problem $\mathcal{P}_{396}(4)$, and if $z_5 = 1$ we are left with problem $\mathcal{P}_{36}(4)$. Potential optimal solutions of problems $\mathcal{P}_{396}(4)$ and $\mathcal{P}_{36}(4)$ are further decomposed using Lemma 3.1.

The set $S_{396}(5)$ of potential optimal solutions to $\mathcal{P}_{396}(5)$ is defined by all the paths from the leaves to the root node in the decomposition tree, that is

$$\begin{aligned} S_{396}(5) \quad = \Big\{ \quad & (4,68,1,0,0), \quad (4,66,2,0,0), \quad (1,67,2,0,0), \\ & (4,54,0,1,0), \quad (1,55,0,1,0), \quad (4,6,0,0,1), \\ & (1,7,0,0,1) \Big\} \end{aligned}$$

and, by Lemma 3.1, $O_{396}(5) \subseteq S_{396}(5)$. ■

For a given problem $\mathcal{P}_F(j)$ and its associated decomposition tree, we define in the next section valid inequalities that are satisfied at equality by all solutions in this decomposition tree, and thus by all optimal solutions in $O_F(j)$.

4 The convex hull of all solutions to the sequential knapsack problem

Let (SKP) be a sequential knapsack problem and suppose that $\mathcal{C}$ is a class of valid inequalities for $\mathcal{P}_{SKP}$. The technique that we use in order to show that $\mathcal{C}$ describes $\mathcal{P}_{SKP}$ is due to Lovasz [5]: for every objective function γ we prove that the set of optimal solutions to (SKP) belongs to the face induced by some inequality in $\mathcal{C}$. This suffices to show that $\mathcal{C}$ describes $\mathcal{P}_{SKP}$, because when an objective function γ is parallel to a facet defining inequality, then the only inequality satisfied at equality by all optimal points in (SKP) is this facet defining inequality. Hence, $\mathcal{C}$ contains all the facet defining inequalities.

We first consider the case that all objective function coefficients are positive. As outlined in Section 2, we partition N into maximal blocks $B_1, \ldots, B_m$ and construct the modified sequential knapsack problem (MSKP). Associated with the transformed problem (MSKP), we use the notations $\mathcal{P}_F(j)$, $OP_F(j)$, $O_F(j)$, Δ_i and $f(\Delta_i)$ introduced in Section 3.

For every knapsack capacity $F \in \mathbb{N}$ and for every $j \in M = \{1, \ldots, m\}$, we now define an inequality $I_F(j)$ satisfying the conditions (i), (ii), (iii) listed below:

(i) The left hand sides of inequalities $I_F(j)$ and $I_{F'}(j)$ are equal if F modulo f_j $= F'$ modulo f_j holds.

(ii) $I_F(j)$ is a valid inequality for $\mathcal{P}_F(j)$.

(iii) The set of optimal solutions $O_F(j)$ is contained in the face induced by the inequality $I_F(j)$.

The inequalities $I_F(j)$ are defined inductively on j.

$j = 1$. We define the inequality $I_F(1)$ as $z_1 \leq \min\{F, u_1\}$. This inequality clearly satisfies all the properties (i) – (iii).

$j-1 \to j$. Let some number r between 0 and $f_j - 1$ be given and assume that for every number $F \in \mathbb{N}$ with F modulo $f_j = r$ there exists an inequality $I_F(j-1)$ that satisfies the properties (i) – (iii).

In particular, property (i) guarantees that this family of inequalities is of the form $\sum_{i=1}^{j-1} d_i z_i \leq g_{F,j-1}$, where d_i, $i \in \{1, \dots, j-1\}$, are the coefficients of the inequalities $I_F(j-1)$ for all F with F modulo $f_j = r$. With the parameter r we associate a number F_r. We set $F_r := r$ if $f(\Delta_j) < r$. Otherwise,

$$F_r := \max\{F \in \mathbb{N} \mid F \leq f(\Delta_j) \text{ and } F \text{ modulo } f_j = r\},$$

i.e., if $r \leq f(\Delta_j)$, then F_r is the largest number of residuum class r with respect to f_j not exceeding the sum of weights in $\{1, \dots, j-1\}$ that have a better gain per unit than j.

For every $F \in \mathbb{N}$ with F modulo $f_j = r$ the left hand side of the inequality $I_F(j)$ is of the form $\sum_{i=1}^{j} d_i z_i$ with $d_1, \dots, d_{j-1}$ defined as in $I_F(j-1)$ and d_j defined by

$$d_j := g_{F_r+f_j,j-1} - g_{F_r,j-1}.$$

In order to define the corresponding right hand side – that we denote by $g_{F,j}$ – we need to distinguish several cases.

First, write $F \in \mathbb{N}$ via $F = F_r + sf_j$ where $s := \frac{(F-F_r)}{f_j}$ is an integer.

We set

$$g_{F,j} := \begin{cases} g_{F_r,j-1} + sd_j, & \text{if } 1 \leq s \leq u_j; \\ g_{F,j-1}, & \text{if } s \leq 0; \\ g_{F-f_j u_j,j-1} + u_j d_j, & \text{if } s > u_j. \end{cases}$$

Under these assumptions the inequality $I_F(j)$ defined via $\sum_{i=1}^{j} d_i z_i \leq g_{F,j}$ satisfies the three properties (i), (ii) and (iii). These statements are shown below. We first illustrate this (inductive) construction on the initial example. Then three technical lemmas are proved and afterwards applied to show that $I_F(j)$ satisfies (i)-(iii).

Example 2.2 continued.

$$\begin{aligned} \max \quad & z_1 + 3z_2 + 6z_3 + 50z_4 + 200z_5, \\ & z_1 + 5z_2 + 30z_3 + 120z_4 + 360z_5 \leq 396, \end{aligned}$$

with upper bounds $u_1 = 4$, $u_2 = 68$, $u_3 = 2$, $u_4 = 1$ and $u_5 = 1$ on the variables z_i, $i = 1, \dots, 5$.

The construction of the inequalities is defined for any value of F. If we are only interested in the inequality $I_{396}(5)$, then we need not find $I_F(j)$ for all values of

F. The node labels in Figure 1 represent the subproblems we have to solve in order to obtain an optimum solution for the original problem $OP_{396}(5)$. They also give the F and r values we must consider in order to construct $I_{396}(5)$.

$$I_F(1): \quad z_1 \le g_{F,1} = \min\{F,4\}$$

$$\begin{array}{lll}
I_F(2) & \text{with} & r = F \text{ modulo } f_2 = 396 \text{ modulo } 5 = 1, \\
 & & \Delta_2 = \{1\}, f(\Delta_2) = 4, F_r = 1, F_r + f_j = 6. \\
 & I_1(2): & z_1 + (g_{6,1} - g_{1,1})z_2 \le g_{1,2} := g_{1,1} + 0(g_{6,1} - g_{1,1}) \\
 & & z_1 + 3z_2 \le 1 \\
 & I_6(2): & z_1 + (g_{6,1} - g_{1,1})z_2 \le g_{6,2} := g_{1,1} + 1(g_{6,1} - g_{1,1}) \\
 & & z_1 + 3z_2 \le 4 \\
 & I_{1+5s}(2): & z_1 + 3z_2 \le 1 + 3s \quad \text{for } 1 \le s \le 68 \\
 & I_{1+5s}(2): & z_1 + 3z_2 \le g_{1+5(s-68),1} + 68 * 3 = 208 \quad \text{for } s > 68
\end{array}$$

$$\begin{array}{lll}
I_F(3) & \text{with} & r = F \text{ modulo } f_3 = 396 \text{ modulo } 30 = 6, \\
 & & \Delta_3 = \{1,2\}, f(\Delta_3) = 344, F_r = 336, F_r + f_j = 366. \\
 & I_{336}(3): & z_1 + 3z_2 + (g_{366,2} - g_{336,2})z_3 \le g_{336,3} := g_{336,2} \\
 & & z_1 + 3z_2 + 6z_3 \le 202 \\
 & I_{366}(3): & z_1 + 3z_2 + 6z_3 \le g_{366,3} := g_{336,2} + 1(g_{366,2} - g_{336,2}) \\
 & & z_1 + 3z_2 + 6z_3 \le 208 \\
 & I_{396}(3): & z_1 + 3z_2 + 6z_3 \le 214
\end{array}$$

$$\begin{array}{lll}
I_F(4) & \text{with} & r = F \text{ modulo } f_4 = 396 \text{ modulo } 120 = 36, \\
 & & \Delta_4 = \{1,2\}, f(\Delta_4) = 344, F_r = 276, F_r + f_j = 396. \\
 & I_{276}(4): & z_1 + 3z_2 + 6z_3 + (g_{396,3} - g_{276,3})z_4 \le g_{276,4} := g_{276,3} = g_{276,2} \\
 & & z_1 + 3z_2 + 6z_3 + 48z_4 \le 166 \\
 & I_{396}(4): & z_1 + 3z_2 + 6z_3 + 48z_4 \le g_{396,4} := g_{276,3} + 1(g_{396,3} - g_{276,3}) \\
 & & z_1 + 3z_2 + 6z_3 + 48z_4 \le 214
\end{array}$$

$$\begin{array}{lll}
I_F(5) & \text{with} & r = F \text{ modulo } f_5 = 396 \text{ modulo } 360 = 36, \\
 & & \Delta_5 = \{1,2\}, f(\Delta_5) = 344, F_r = 36, F_r + f_j = 396. \\
 & I_{36}(5): & z_1 + 3z_2 + 6z_3 + 48z_4 + (g_{396,4} - g_{36,4})z_5 \le g_{36,5} := g_{36,4} \\
 & & z_1 + 3z_2 + 6z_3 + 48z_4 + 192z_5 \le 22 \\
 & I_{396}(5): & z_1 + 3z_2 + 6z_3 + 48z_4 + 192z_5 \le g_{396,4} := g_{36,3} + g_{396,4} - g_{36,4} \\
 & & z_1 + 3z_2 + 6z_3 + 48z_4 + 192z_5 \le 214
\end{array}$$

The inequality $I_{396}(5)$ is satisfied at equality by all solutions in $S_{396}(5)$ containing all optimal solutions in $O_{396}(5)$. ■

Lemma 4.1. Let F and G be natural numbers such that $F \le G$ and F modulo $f_j = r = G$ modulo f_j. Then, $g_{F+f_j,j} - g_{F,j} \ge g_{G+f_j,j} - g_{G,j}$ holds.

Proof. For $j = 1$ the statement is certainly true. So assume, it holds for all numbers that are less or equal than $j-1$. We show that it is true for j as well.

We write $F = F_r + sf_j$ and $G = F_r + tf_j$. Since $F \leq G$, we know that $s \leq t$. We define

$$F' := \begin{cases} F_r, & \text{if } 0 \leq s \leq u_j; \\ F, & \text{if } s < 0; \\ F - f_j u_j, & \text{if } s > u_j. \end{cases}$$

$$G' := \begin{cases} F_r, & \text{if } 0 \leq t \leq u_j; \\ G, & \text{if } t < 0; \\ G - f_j u_j, & \text{if } t > u_j. \end{cases}$$

Checking all cases we notice that $F' \leq G'$, F' modulo $f_j = r = G'$ modulo f_j, $g_{F+f_j,j} - g_{F,j} = g_{F'+f_j,j-1} - g_{F',j-1}$ and $g_{G+f_j,j} - g_{G,j} = g_{G'+f_j,j-1} - g_{G',j-1}$ holds. As F' modulo $f_{j-1} = G'$ modulo f_{j-1}, by assumption of the induction

$$g_{F'+f_j,j-1} - g_{F',j-1} \geq g_{G'+f_j,j-1} - g_{G',j-1},$$

and the claim follows. ■

Lemma 4.2. Let F and G be natural numbers such that $F \leq G$ and F modulo $f_j = r = G$ modulo f_j. Then, $g_{G,j} + \sigma(g_{F+f_j,j} - g_{F,j}) \geq g_{G+\sigma f_j,j}$ holds for every $\sigma \in \mathbb{N}$.

Proof. $g_{G+\sigma f_j,j} = g_{G+(\sigma-1)f_j,j} + [g_{G+\sigma f_j,j} - g_{G+(\sigma-1)f_j,j}]$. Applying Lemma 4.1 yields $[g_{G+\sigma f_j,j} - g_{G+(\sigma-1)f_j,j}] \leq [g_{F+f_j,j} - g_{F,j}]$. Therefore, $g_{G+\sigma f_j,j} \leq g_{G+(\sigma-1)f_j,j}$ $+ [g_{F+f_j,j} - g_{F,j}]$. Iterating this argument proves Lemma 4.2. ■

Accordingly, we obtain Lemma 4.3.

Lemma 4.3. Let F and G be natural numbers such that $F \leq G$ and F modulo $f_j = r = G$ modulo f_j. Then, $g_{F-\sigma f_j,j} + \sigma(g_{G,j} - g_{G-f_j,j}) \leq g_{F,j}$ holds for every $\sigma \in \mathbb{N}$ with $F - \sigma f_j \geq 0$.

Proof. $g_{F,j} = g_{F-f_j,j} + [g_{F,j} - g_{F-f_j,j}]$. By Lemma 4.1, we conclude that $[g_{F,j} - g_{F-f_j,j}] \geq [g_{G,j} - g_{G-f_j,j}]$. Therefore, $g_{F,j} \geq g_{F-f_j,j} + [g_{G,j} - g_{G-f_j,j}]$. Iterating this argument proves Lemma 4.3. ■

Using Lemmas 4.1 - 4.3 we are now able to prove the following theorem.

Theorem 4.4. Given a modified sequential knapsack problem with positive objective function obtained from the maximal block transformation. If the inequalities $I_F(j-1)$ satisfy the three conditions (i), (ii), (iii) with $k = j-1$ for all $F \in \mathbb{N}$, so do the inequalities $I_F(j)$ with $k = j$.

(i) The left hand sides of two inequalities $I_F(k)$ and $I_{F'}(k)$ are identical whenever F modulo $f_k = F'$ modulo f_k holds;

(ii) $I_F(k)$ is valid for $\mathcal{P}_F(k)$;

(iii) Every optimum solution to problem $OP_F(k)$ is contained in the face induced by the inequality $I_F(k)$;

Proof. We write $I_F(j)$ as $\sum_{i=1}^{j} d_i z_i \leq g_{F,j}$.

(i) Let F and F' be two natural numbers satisfying F modulo $f_j = r = F'$ modulo f_j. As Δ_j and F_r are uniquely defined by the residuum class r and the objective function we obtain – per definition – that the left hand sides of the two inequalities $I_F(j)$ and $I_{F'}(j)$ are the same.

(ii) The inequality $I_F(j)$ is valid for the polyhedron $\mathcal{P}_F(j)$. Let $z \in \mathcal{P}_F(j)$ be a feasible point, then $\sum_{i=1}^{j} d_i z_i \leq g_{F-z_j f_j, j-1} + z_j d_j$ because, by assumption, $\sum_{i=1}^{j-1} d_i z_i \leq g_{G,j-1}$ is a valid inequality for all values of G with G modulo $f_{j-1} = F$ modulo f_{j-1} and $(F - z_j f_j)$ modulo $f_{j-1} = F$ modulo f_{j-1}. Again, we write $F = F_r + s f_j$ and distinguish several cases.

(ii) (a) $F \leq F_r$. Then $s \leq 0$. If $z_j = 0$ it follows from the definition of $g_{F,j} = g_{F,j-1}$ that the inequality is valid. Suppose that $z_j > 0$. Then $\sum_{i=1}^{j} d_i z_i \leq g_{F-z_j f_j, j-1} + z_j d_j = g_{F-z_j f_j, j-1} + z_j (g_{F_r+f_j, j-1} - g_{F_r, j-1}) \leq g_{F,j-1} = g_{F,j}$, where the last inequality follows from Lemma 4.3, and the statement follows.

(ii) (b) $F > F_r + f_j u_j$. Then $s > u_j$. If $z_j = u_j$, it follows from the definition of $g_{F,j} = g_{F-f_j u_j, j-1} + d_j u_j$ that the inequality is valid. Suppose that $z_j < u_j$. By applying Lemma 4.2, we obtain: $\sum_{i=1}^{j} d_i z_i \leq g_{F-z_j f_j, j-1} + z_j d_j = g_{F-z_j f_j, j-1} + z_j (g_{F_r+f_j, j-1} - g_{F_r, j-1}) \leq g_{F-u_j f_j, j-1} + (u_j - z_j)(g_{F_r+f_j, j-1} - g_{F_r, j-1}) + z_j (g_{F_r+f_j, j-1} - g_{F_r, j-1}) = g_{f-u_j f_j, j-1} + u_j d_j = g_{F,j}$.

(ii) (c) What remains is the case where $F_r < F \leq F_r + u_j f_j$. Then, $1 \leq s \leq u_j$ holds and we obtain

$$(\star) \quad \sum_{i=1}^{j} d_i z_i \leq g_{F-z_j f_j, j-1} + z_j d_j = g_{F_r-(z_j-s) f_j, j-1} + (z_j - s) d_j + s d_j.$$

If $z_j = s$, as $g_{F,j} = g_{F_r, j-1} + s d_j$, the inequality is valid by construction. Otherwise, if $z_j > s$, then Lemma 4.3 implies that $g_{F_r-(z_j-s) f_j, j-1} + (z_j - s) d_j \leq g_{F_r, j-1}$ and together with $(\star)$ we have that $I_F(j)$ is valid. Finally, if $z_j < s$, Lemma 4.2 implies that $g_{F_r-(z_j-s) f_j, j-1} + (z_j - s) d_j = g_{F_r+(s-z_j) f_j, j-1} - (s - z_j) d_j \leq g_{F_r, j-1}$ which again shows that the inequality $I_F(j)$ is valid.

(iii) It remains to be shown that the set of optimal solution $O_F(j)$ is contained in the face induced by the inequality $I_F(j)$.

By definition of F_r, we can always write $F = F_r + sf_j$. Let $z \in O_F(j)$, then by Lemma 3.1 and by definition of F_r we have $\frac{F-f(\Delta_j)}{f_j} = s + \frac{F_r - f(\Delta_j)}{f_j}$. If $r \leq f(\Delta_j)$, then F_r is the unique number such that $F_r \leq f(\Delta_j) < F_r + f_j$ and F modulo $f_j = r$. In this case Lemma 3.1 implies that

$$z_j = \min\{u_j, (s-1)^+\} \text{ or } z_j = \min\{u_j, s^+\}.$$

If $r > f(\Delta_j)$, then $F_r = r < f_j$, $s \geq 0$ and Lemma 3.1 yields

$$z_j = \min\{u_j, s\} \text{ or } z_j = \min\{u_j, s+1\}.$$

In this case, $z_j = s+1$ is impossible, because $F - (s+1)f_j = F_r - f_j < 0$.

Hence, $r > f(\Delta_j)$ implies that $z_j = \min\{u_j, s\}$.
Summarizing all cases yields $z_j = \min\{u_j, (s-1)^+\}$ or $z_j = \min\{u_j, s^+\}$.

In case $s \leq 0$, i.e., $F \leq F_r$, we have $z_j = 0$ in every optimum solution. Therefore by assumption of the induction, every optimum solution to problem $OP_F(j)$ is contained in the face $\sum_{i=1}^{j-1} d_i z_i = g_{F,j-1}$. Since $g_{F,j} = g_{F,j-1}$ in this case, the claim follows.

In case $s \geq u_j + 1$, i.e., $F > F_r + u_j f_j$, every element in the set $O_F(j)$ satisfies $z_j = u_j$. By assumption of the induction, every optimum solution z to problem $OP_F(j)$ satisfies $\sum_{i=1}^{j-1} d_i z_i = g_{F-f_j u_j, j-1}$ and as $z_j = u_j$, we obtain $\sum_{i=1}^{j} d_i z_i = g_{F-f_j u_j, j-1} + d_j u_j = g_{F,j}$. This proves the claim in this case.

Finally, we have $1 \leq s \leq u_j$. Then every optimum solution z of $OP_F(j)$ satisfies either $z_j = s$ or $z_j = s-1$. By assumption of the induction we obtain that (a) $\sum_{i=1}^{j-1} d_i z_i = g_{F-sf_j, j-1} = g_{F_r, j-1}$, if $z_j = s$ and (b) $\sum_{i=1}^{j-1} d_i z_i = g_{F-(s-1)f_j, j-1} = g_{F_r+f_j, j-1}$, if $z_j = s-1$. This yields in case (a): $\sum_{i=1}^{j} d_i z_i = g_{F_r, j-1} + s d_j = g_{F,j}$. In case (b) we obtain: $\sum_{i=1}^{j} d_i z_i = g_{F_r+f_j, j-1} + (s-1)d_j = g_{F_r, j-1} + (g_{F_r+f_j, j-1} - g_{F_r, j-1}) + (s-1)d_j = g_{F_r, j-1} + s d_j = g_{F,j}$. This shows that in both cases the inequality $I_F(j)$ is satisfied at equality by all optimal points. ∎

Let us now present the final theorem describing $\mathcal{P}_{SKP}$ as a system of inequalities. Let $W \subseteq N$ be a subset of items in N, let $\mathcal{B} = \{B_1, \ldots, B_m\}$ be a partition of W into blocks and let π be a permutation of $\{1, \ldots, m\}$. Let $f_i' := \min_{l \in B_i}\{a_l\}$ be the weight of block B_i, $u_i := \sum_{l \in B_i} \frac{a_l s_l}{f_i'}$ be the multiplicity of block B_i and assume that $f_1' \leq \ldots \leq f_m'$. We set $f_j = \frac{f_j'}{f_1'}$ and $\Delta_j = \{i \in \{1, \ldots, j-1\} : \pi(i) < \pi(j)\}$,

$j = 1, \ldots, m$.
Denote by $\mathcal{P}_{\lfloor \frac{a_0}{f_1'} \rfloor}(m)$ the modified knapsack polytope defined with the block partition $\mathcal{B}$ of W, weights $f_1, \ldots, f_m$, multiplicities $u_1, \ldots, u_m$ and knapsack capacity $\lfloor \frac{a_0}{f_1'} \rfloor$. That is

$$\mathcal{P}_{\lfloor \frac{a_0}{f_1'} \rfloor}(m) = \Big\{ z \in \mathbb{R}^m : \begin{array}{l} \sum_{i=1}^{m} f_i z_i \leq \lfloor \frac{a_0}{f_1'} \rfloor \\ 0 \leq z_i \leq u_i \text{ and } z_i \text{ integer} \qquad \text{for } i = 1, \ldots, m \end{array} \Big\}$$

If the inequality $I_{\lfloor \frac{a_0}{f_1'} \rfloor}(m)$, written as $\sum_{j=1}^{m} d_j z_j \leq g_{\lfloor \frac{a_0}{f_1'} \rfloor, m}$, denotes the valid inequality developed in this section for $\mathcal{P}_{\lfloor \frac{a_0}{f_1'} \rfloor}(m)$ using the sets Δ_j induced by the permutation π, then the inequality $K(W, \mathcal{B}, \pi)$ is defined as

$$\sum_{j=1}^{m} \sum_{i \in B_j} d_j \frac{a_i}{f_j f_1'} x_i \leq g_{\lfloor \frac{a_0}{f_1'} \rfloor, m}.$$

Theorem 4.5. Given an instance of (SKP), the following system of inequalities describes the polyhedron $\mathcal{P}_{SKP}$:

$$\begin{array}{ll} 0 \leq x_i, & \text{for } i = 1, \ldots, n; \\ K(W, \mathcal{B}, \pi), & \text{for all } W \subseteq N, \ \text{ all partitions } \mathcal{B} = \{B_1, \ldots, B_m\} \text{ of} \\ & W \text{ into blocks, and all permutations } \pi \text{ of } \{1, \ldots, m\}. \end{array}$$

Proof. We first show validity of the inequalities $K(W, \mathcal{B}, \pi)$. Given W, $\mathcal{B} = \{B_1, \ldots, B_m\}$ and π. It is easy to check that there exists an objective function $\gamma \in \mathbb{R}^{|W|}$ for which $\mathcal{B}$ is the partition of W into maximal blocks and there exists π such that

$$\Delta_j = \{ i \in \{1, \ldots, j-1\} : \frac{\gamma_i}{f_i} > \frac{\gamma_j}{f_j} \}.$$

Then, the inequality $I_{\lfloor \frac{a_0}{f_1'} \rfloor}(m)$ is valid for the polyhedron $\mathcal{P}_{\lfloor \frac{a_0}{f_1'} \rfloor}(m)$ by Theorem 4.4 (i) and (ii). By the arguments on the transformation of valid inequalities for $\mathcal{P}_{MSKP}$ to valid inequalities for $\mathcal{P}_{SKP}$ (see Section 2), the inequality $K(W, \mathcal{B}, \pi)$ is valid for the polyhedron

$$\text{conv}\ \{ x \in \mathbb{R}^n : \sum_{i \in W} \frac{a_i}{f_1'} x_i \leq \lfloor \frac{a_0}{f_1'} \rfloor,\ 0 \leq x_i \leq s_i,\ x_i \in \mathbb{N} \text{ for } i \in W \}.$$

This polyhedron is a relaxation of $\mathcal{P}_{SKP}$, because f_1' is the smallest weight among all items in W. As $K(W, \mathcal{B}, \pi)$ is valid for this relaxation of $\mathcal{P}_{SKP}$, it is certainly valid for $\mathcal{P}_{SKP}$.

Now given any objective function $\gamma = (\gamma_1, \ldots, \gamma_n)^T$, we construct an inequality satisfied at equality by all optimal solutions of (SKP). If $\gamma_i < 0$ for some $i \in N$, then $x_i = 0$ for every optimal solution. Otherwise, $\gamma \geq 0$ and we set $W := \{i \in N : \gamma_i > 0\}$. Let

$\mathcal{B} = \{B_1, \ldots, B_m\}$ be the partition of W into maximal blocks and let (MSKP) denote the modified sequential knapsack problem of Section 2. From Theorem 4.4 (iii) we know that $I_{\lfloor \frac{a_0}{f_1'} \rfloor}(m)$ is satisfied at equality by all optimal solutions of (MSKP). By the arguments on the equivalence of optimal solutions beween problems (SKP) and (MSKP) (see Section 2), $K(W, \mathcal{B}, \pi)$ is satisfied at equality by all optimal solutions of

$$\begin{aligned} \max \quad & \sum_{i \in W} \gamma_i x_i, \\ & \sum_{i \in W} \frac{a_i}{f_1'} x_i \leq \Big\lfloor \frac{a_0}{f_1'} \Big\rfloor, \\ & x_i \in \{0, \ldots, s_i\} \qquad \text{for } i \in W. \end{aligned}$$

Now if x is an optimal solution of the original problem with $K(W, \mathcal{B}, \pi)$ not satisfied at equality (because some $i \in N \setminus W$ has value $x_i > 0$), then a solution with strictly better objective function value can be found by setting $x_i = 0$ for all $i \in N \setminus W$, a contradiction. This completes the proof. ■

5 Explicit Inequalities

In the previous section we have inductively defined a class of inequalities that depends on the choice and ordering of the blocks. Can we find a more explicit or combinatorial formulation for those inequalities? This question is addressed now.

Given a sequential knapsack problem of the form

$$(SKP) \quad \sum_{i \in N_1} x_i + \sum_{i \in N_2} f_2 x_i + \ldots + \sum_{i \in N_k} f_k x_i \leq F, \; 0 \leq x \leq u, \; x \text{ integer},$$

where u is the vector of upper bounds on the variables and $1 < f_2 \leq f_3 \leq \ldots \leq f_k$.

A large class of inequalities for the associated polyhedron $\mathcal{P}_{SKP}$ can be described as follows:
Let r_i denote the residuum of the capacity F modulo f_i. We choose sets $S_i \subseteq N_i$, $T_i \subseteq N_i \setminus S_i$, $i = 1, \ldots, k$ with the following properties:

$\sum_{j=1}^{k} f_j u(S_j) = F$,

$0 < u(T_1) < f_2$,

$T_k = N_k \setminus S_k$.

Setting $b_1 := 1$ and, for $j \geq 2$,

$$b_j = \sum_{w=1}^{j-1} b_w u(T_w) \quad \text{if} \sum_{w=1}^{j-1} f_w u(T_w) < f_j$$

$$b_j = \frac{f_j}{f_{j-1}} b_{j-1} \quad \text{otherwise,}$$

the inequality

$$(\star) \quad \sum_{j=1}^{k} b_j \sum_{i \in S_j \cup T_j} x_i \ \leq \ \sum_{j=1}^{k} u(S_j) b_j$$

is valid for $\mathcal{P}_{SKP}$. This statement can be verified by applying our inductive scheme: we define a modified sequential knapsack problem and, for every item i in this modified problem, we choose a set Δ_i such that the inequality constructed via our inductive scheme coincides with $(\star)$.

We first consider the case where $\sum_{w=1}^{j-1} f_w u(T_w) < f_j$ for all $j \geq 2$, i.e., $b_j = \sum_{w=1}^{j-1} b_w u(T_w)$ for all $j \geq 2$. Here we define the transformation to (MSKP) by considering k blocks $B_1, \ldots, B_k$ with $B_j = S_j \cup T_j$ for $i = 1, \ldots, k$. Thus, the modified sequential knapsack is of the form

$$\sum_{j=1}^{k} f_j z_j \leq F, \quad 0 \leq z_j \leq u(S_j) + u(T_j) \text{ and } z_j \text{ integer for } j = 1, \ldots, k.$$

The ordering of blocks is defined by $\Delta_1 = \emptyset$ and $\Delta_j = \{1, \ldots, j-1\}$ for $j = 2, \ldots, k$ with $f(\Delta_j) = \sum_{i=1}^{j-1} f_i(u(S_i) + u(T_i))$. Let $\sum_{j=1}^{k} d_j z_j \leq g_{F,k}$ denote the inequality $I_F(k)$ for this modified problem (MSKP) with sets $\Delta_j, j = 1, \ldots, k$. We now show that $(\star)$ coincides with the inequality $\sum_{j=1}^{k} \sum_{i \in S_j \cup T_j} d_j x_i \leq g_{F,k}$ that is obtained by transforming $I_F(k)$ to a valid inequality for (SKP) (see Section 2).

As $\sum_{j=1}^{k} f_j u(S_j) = F$, we have that, for any $j \geq 2$, $r_j = F \text{ modulo } f_j = (\sum_{i=1}^{j-1} f_i u(S_i)) \text{ modulo } f_j$ and thus $\sum_{i=1}^{j-1} f_i u(S_i) = r_j + n_j f_j$ for some $n_j \in \mathbb{N}$. To derive $I_F(k)$ using our inductive scheme, we have to compute the numbers $F_{r_j} := \max\{G \in \mathbb{N} : G \leq f(\Delta_j), G \text{ modulo } f_j = r_j\}$. As $f(\Delta_j) =$

$\sum_{i=1}^{j-1} f_i(u(S_i) + u(T_i)) = r_j + n_j f_j + \sum_{i=1}^{j-1} f_i u(T_i) < r_j + n_j f_j + f_j$, we obtain $F_{r_j} = \sum_{i=1}^{j-1} f_i u(S_i)$.

Starting from $d_1 = 1$, $g_{F,1} = \min\{F, u(S_1) + u(T_1)\}$ and going through the inductive scheme (see Section 4) we obtain for each $j = 1, \ldots, k-1$

$$g_{[\sum_{i=1}^{j} f_i u(S_i)],j} = \sum_{i=1}^{j} d_i u(S_i),$$

$$g_{[\sum_{i=1}^{j} f_i u(S_i)]+f_{j+1},j} = g_{[\sum_{i=1}^{j} f_i(u(S_i)+u(T_i))],j} = \sum_{i=1}^{j} d_i(u(S_i) + u(T_i)),$$

$$d_{j+1} = g_{[\sum_{i=1}^{j} f_i u(S_i)]+f_{j+1},j} - g_{[\sum_{i=1}^{j} f_i u(S_i)],j} = \sum_{i=1}^{j} d_i u(T_i)$$

So, for $j = 1, \ldots, k$, we obtain $d_j = b_j$ and finally $g_{F,k} = g_{[\sum_{i=1}^{k} f_i u(S_i)],k} = \sum_{i=1}^{k} d_i u(S_i)$ which shows that the inequality $(\star)$ is obtained via our inductive scheme and thus is valid for $\mathcal{P}_{SKP}$ when $\sum_{w=1}^{j-1} f_w u(T_w) < f_j$ for all $j = 2, \ldots, k$.

When $\sum_{w=1}^{j-1} f_w u(T_w) \geq f_j$ for some $j \in \{2, \ldots, k\}$, then we consider $S_{j-1} \cup T_{j-1} \cup S_j \cup T_j$ as a single block. Performing this for all j with $\sum_{w=1}^{j-1} f_w u(T_w) \geq f_j$, generating the corresponding modified knapsack problem, constructing the valid inequality using our inductive scheme and transforming it to a valid inequality for (SKP) yields the inequality $(\star)$.

The inequalities of the form $(\star)$ are already a strong generalization of other known inequalities:
In case that $k = 2$ and if u is the vector of all ones (the 0/1 case), then the inequality $(\star)$ is of the form

$$\sum_{i \in S_1} x_i + \sum_{i \in N_2} l x_i \leq |S_1| + \lfloor \frac{F - |S_1|}{f_2} \rfloor l$$

where $\emptyset \neq S_1 \subseteq N_1$, $r_2 = F$ modulo f_2, $l = (|S_1| - r_2)$ modulo f_2 and $|S_1| + f_2|N_2| > F$. The latter class of inequalities plus the trivial inequalities $0 \leq x_i \leq 1$, $i \in N_1 \cup N_2$ plus the cover inequality $\sum_{i \in N_2} x_i \leq \lfloor \frac{F}{f_2} \rfloor$ describe the polyhedron conv $\{x \in \{0,1\}^{N_1 \cup N_2} : \sum_{i \in N_1} x_i + \sum_{i \in N_2} f_2 x_i \leq F,\ x \text{ integer }\}$. This result was shown in [11] and, independently in [3].
As a special case we obtain Padberg's result on $(1,k)$-configurations [8]: Suppose, we are given a knapsack problem such that the set of feasible solutions is equal to

$$x \in \{0,1\}^{N_1 \cup \{z\}} : \sum_{i \in N_1} x_i + f_z x_z \leq |N_1|.$$

The corresponding polyhedron is described by the lower and upper bound constraints plus the inequalities

$$\sum_{i \in S_1} x_i + (|S_1| + f_z - |N_1|)x_z \leq |S_1|$$

for all subsets $S_1 \subseteq N_1$, $|S_1| + f_z > |N_1|$.

Summarizing our discussions, the inequalities ($\star$) are only a subclass of the inequalities needed to describe a sequential knapsack polyhedron. Nevertheless, this subclass is quite large and extends all the explicitly known inequalities for special cases of the knapsack problem having the divisibility property.

6 Extensions

The previous sections deal exclusively with the sequential knapsack polytope which is still a restrictive assumption when considering integer programs in general. Can we use parts of this polyhedral knowledge presented so far and apply it within a more general framework? The answer is "yes" and we outline now some directions.

A first question in using our inductively defined inequalities computationally is whether we have a combinatorial algorithm for solving the separation problem, i.e., given a fractional solution y: does there exists an inequality that is violated by y and if so, then what is the inequality? We did not succeed in solving this separation problem. "Only" for the subclass of inequalities $\sum_{i \in S_1} x_i + \sum_{i \in N_2} l x_i \leq |S_1| + \lfloor \frac{F - |S_1|}{f_2} \rfloor l$ where $\emptyset \neq S_1 \subseteq N_1$, $l = (|S_1| - r_2)$ modulo f_2 and $|S_1| + f_2 |N_2| > F$, Hartmann [3] gives a linear time algorithm for solving the separation problem . The general problem is still open. However, we can use our inductive scheme as a separation heuristic. For instance, defining every item $i \in N$ as a single block, setting $\Delta_i = \{t \in N : \; f_t \leq f_i, \; y_t > y_i\}$, $i \in N$ and generating an inequality according to this ordering seems to be a promissing approach to end up with a violated inequality, if one exists. Other reasonable definitions of Δ_i might be to set $\Delta_i = \{t \in N : \; f_t \leq f_i, \; \frac{y_t}{f_t} > \frac{y_i}{f_i}\}$, $i \in N$. Whether those ideas work is certainly not clear, but similar "greedy type" of procedures work pretty well for the separation of cover- and $(1,k)$-configuration inequalities.

Given an integer programming problem $Ax \leq b$, $0 \leq x \leq u$, x integer with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $u \in \mathbb{N}^n$, $x \in \mathbb{R}^n$. If there exists some row $\sum_{j \in N} a_{ij} x_j \leq b_i$ such that a subset S of items in $\{j \in \{1, \ldots, n\} : \; a_{ij} > 0\}$ has the divisibility property, then we can investigate the polyhedron: $\text{conv}\{x \in \mathbb{R}^S : \; \sum_{i \in S} a_{ij} x_i \leq$

b_i, $0 \le x \le u$, x integer $\}$ and generate inequalities for this polyhedron. By computing lifting coefficients for the items in $N \setminus S$, we obtain a valid inequality for the overall polyhedron $\mathrm{conv}\{x \in \mathbb{R}^n : \ Ax \le b,\ 0 \le x \le u,\ x \text{ integer }\}$. This approach can always be used to apply knowledge about special integer programs to more general cases.

Another idea is to try to relax a given integer program as a sequential knapsack problem. Given a row $\sum_{j \in N} a_{ij} x_j \le b_i$ of an integer program, the easiest way to obtain a relaxation as a sequential knapsack problem is to choose, a priori, a set of divisible numbers $f_1, \ldots, f_k$, say. The sequential knapsack problem defined via the constraint

$$\sum_{j \in N} (\max_{i=1,\ldots,k} \{f_i : \ f_i \le a_{ij}\}) x_j \le \lfloor \frac{b_i}{f_1} \rfloor f_1$$

is certainly a relaxation of the given integer program.

A more specific relaxation is obtained by generalizing the concept of $(1,k)$-configurations. Consider the 0/1 knapsack problem defined by the constraint

$$\sum_{i \in N} f_i x_i \le F \ , \quad x_i \in \{0,1\} \text{ for } i \in N$$

with $N := \{1, \ldots, s+1\}, s \ge 3$ and $f_1 \le f_2 \le \ldots \le f_s \le f_{s+1}$. Let $S := \{1, \ldots, s\}$, assume that $f(S) \le F$, $f(S) + f_{s+1} > F$ and define $r := F - f(S)$.

Define indices $1 = i_1 < i_2 < \ldots < i_\tau < i_{\tau+1} = s+1$ such that $i_2 \ge 3$, $f_{i_j} \ge f_{i_j - 1} + f_{i_j - 2}$ for $j = 2, \ldots, \tau$ and define a partition $S_1, S_2, \ldots, S_{\tau+1}$ of the set N of items as

$$S_j := \{i_j, \ldots, i_{j+1} - 1\} \text{ for } j = 1, \ldots, \tau \quad \text{and } S_{\tau+1} = \{i_{\tau+1}\} = \{s+1\}.$$

Based on this partition, we define an inequality with the divisibility property that is valid for the given 0/1 knapsack problem. We set $b_1 := 1$ and, for $j = 2, \ldots, \tau$, we define

$$t_j := \max\{t = 1, \ldots, i_j - i_{j-1} : \sum_{w = i_j - t}^{i_j - 1} f_w \le f_{i_j}\},$$

$$b_j = b_{j-1} t_j.$$

Note that $t_j \le |S_{j-1}| = i_j - i_{j-1}$. We define finally

$$t_{\tau+1} := \max\{t = 1, \ldots, i_{\tau+1} - i_\tau : \sum_{w = i_{\tau+1} - t}^{i_{\tau+1} - 1} f_w < f_{i_{\tau+1}} - r + f_1\},$$

$$b_{\tau+1} = b_\tau t_{\tau+1}.$$

If such a $t_{\tau+1}$ exists (i.e. if $f_s < f_{s+1} - r + f_1$), then the inequality

$$(\star) \qquad \sum_{j=1}^{\tau+1} b_j \sum_{i\in S_j} x_i \le \sum_{j=1}^{\tau} b_j |S_j|$$

is valid for the 0/1 knapsack problem. Before verifying this statement, let us illustrate the above construction on an example.

Example 5.1. Consider the knapsack problem in 0/1 variables defined via the constraint

$$x_1 + x_2 + 2x_3 + 3x_4 + 3x_5 + 3x_6 + 4x_7 + 7x_8 + 7x_9 + 7x_{10} + 8x_{11} + 25x_{12} \le 49.$$

Set $S := \{1, \ldots, 11\}$, Then $f(S) = 46$ and $r = 3$. We choose $\tau = 3$, $i_1 = 1$, $i_2 = 4$, $i_3 = 8$ and $i_4 = 12$. This meets the requirements that the indices i_2, i_3 must satisfy, because $f_4 \ge f_3 + f_2$ and $f_8 \ge f_7 + f_6$. In this example, the inequality $(\star)$ is of the form

$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 + 2x_6 + 2x_7 + 4x_8 + 4x_9 + 4x_{10} + 4x_{11} + 12x_{12} \le 27$$

and it is valid for the given knapsack polytope. ■

Let us now show that the inequality $(\star)$ is always valid under the above assumptions. It is valid if and only if every subset $T \subseteq S$ with $f(T) \ge f_{s+1} - r$ satisfies $b(T) := \sum_{j=1}^{\tau} b_j |S_j \cap T| \ge b_{\tau+1}$, or equivalently if and only if the problem

$$f^\star := \max\{\sum_{i=1}^{s} f_i y_i : \ \sum_{j=1}^{\tau} b_j \sum_{i\in S_j} y_i \le b_{\tau+1} - 1, \ y \in \{0,1\}^s\}.$$

has an optimal value $f^\star < f_{s+1} - r$.

Setting $Y_j := \sum_{i\in S_j} y_i$ for $j = 1, \ldots, \tau$, we first show that there always exists an optimal solution to this problem with $Y_j < t_{j+1}$ for $j = 1, \ldots, \tau$. First, observe that $Y_\tau \ge t_{\tau+1}$ is infeasible for this problem because $b_{\tau+1} = t_{\tau+1} b_\tau$, so $Y_\tau < t_{\tau+1}$. Now, if $Y_{\tau-1} \ge t_\tau$, as by construction $\sum_{w=i_\tau - t_\tau}^{i_\tau - 1} f_w \le f_{i_\tau}$ and $b_\tau = t_\tau b_{\tau-1}$, the solution obtained by decreasing $Y_{\tau-1}$ by t_τ and increasing Y_τ by 1 is at least as good as the initial solution in terms of objective value $\sum_{i=1}^{s} f_i y_i$ and equivalently in terms of the knapsack constraint $\sum_{j=1}^{\tau} b_j Y_j \le b_{\tau+1} - 1$. So, any optimal solution with $Y_{\tau-1} \ge t_\tau$ can be transformed into an optimal solution with $Y_{\tau-1} < t_\tau$. Proceeding in this way for all $j = 1, \ldots, \tau$, we can produce an optimal solution with $Y_j < t_{j+1}$ for $j = 1, \ldots, \tau$.

The objective value of such a solution satisfies

$$\sum_{i \in S_j} f_i y_i \leq f_{i_{j+1}} - f_{i_j} \quad \text{for } j = 1, \ldots, \tau - 1,$$

because $Y_j < t_{j+1} \leq |S_j|$ implies that there exists $z \in S_j$ with $y_z = 0$ and $f_z \geq f_{i_j}$ such that $f_{i_j} + \sum_{i \in S_j} f_i y_i \leq f_z + \sum_{i \in S_j} f_i y_i \leq f_{i_{j+1}}$.

Summing these inequalities for $j = 1, \ldots, \tau - 1$, we obtain

$$\begin{aligned}
\sum_{j=1}^{\tau} \sum_{i \in S_j} f_i y_i &= \sum_{j=1}^{\tau-1} \sum_{i \in S_j} f_i y_i + \sum_{i \in S_\tau} f_\tau y_\tau \\
&\leq -f_1 + (f_{i_\tau} + \sum_{i \in S_\tau} f_\tau y_\tau) \\
&\leq -f_1 + (\sum_{w = i_{\tau+1} - t_{\tau+1}}^{i_{\tau+1}-1} f_w) \\
&< f_{i_{\tau+1}} - r = f_{s+1} - r
\end{aligned}$$

Hence $f^\star < f_{s+1} - r$ and the inequality is valid.

By construction, b_j is a multiple of b_{j-1} for all $j \geq 2$. It follows that $(\star)$ has the divisibility property and we can apply all of our information for the sequential knapsack polytope induced by inequality $(\star)$. In case that $\tau = 1$ and if we impose a "regularity condition" such as "every subset T in S with $b(T) = b(S) - b_{\tau+1}$ satisfies $f(T) + f_{s+1} \leq F$", then the corresponding inequality defines a facet of the 0/1 knapsack polytope [8].

For $\tau \geq 2$ one can also derive sufficient conditions under which inequality $(\star)$ defines a facet of the corresponding polytope. Yet, such conditions are quite technical and we refrain within this paper from explaining further details.

If one finds such generalized $(1, k)$-configurations or some subset of the items having the divisibility property with respect to some row of a given integer program $Ax \leq b$, then all the knowledge about the sequential knapsack polytope can be used. Together with lifting this yields a powerful tool that might help solving integer programs.

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