Complete Linear Descriptions For Special Instances of the Single Source Fixed Charge Network Flow Design Problem

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December 5, 1994

Abstract

The multicommodity linear formulation of the Fixed Charge Network Flow Design problem is known to have significantly sharp linear relaxation lower bounds. However the tradeoff is the introduction of a large amount of artificial variables. We exhibit a class of special instances for which the lower bound is tight. Further we completly describe the polyhedron in the space of the natural variables.

1 Introduction

1.1 The Network Flow Design problem

An instance of the Fixed Charge Single Source Network Flow Design (SSNFD) problem is constituted of directed graph with node set V and arc set A, denoted by (V, A), of a root node r and of a set of commodities M, where a destination node s^m ($s^m \in V$) and a non-negative volume d^m ($d^m \in \mathbb{R}_+$) are given for every commodity m in M. Let $G = (V, A, r, M)$ denote the instance. Furthermore, a non-negative variable cost coefficient c_a is given for every arc a in subset C of A and a non-negative fixed cost, or fixed charge, f_a is given for every arc a in a subset F of A.

A set of arcs Y ($Y \subseteq A$) is a *network* in G if and only if every commodity m in M can be shipped from the root node r to its specific destination node s^m by arcs of the network. In other terms a set of arcs Y is a network if it contains a directed path Z^m from r to s^m for every commodity m in M. A *solution* for the (SSNFD) problem in the instance G is a network $Y \subseteq A$ plus a collection of $r - s^m$ directed path Z^m ($m \in M$) included in Y. Let $X = (Y, Z^m : m \in M)$ denote a solution in G, and let $\mathcal{X}_G$ denote the set of solutions in the instance G.

The cost of a solution $X = (Y, Z^m : m \in M)$, denoted by $c(X)$, is the sum of the fixed costs f_a for every arc a of the network contained in F, plus the sum of the variable cost coefficient c_a times the total volume of the commodities flowing through a for every arcs in C. So we have

$$c(X) = \sum_{a \in Y \cap F} f_a + \sum_{m \in M} \sum_{a \in Z^m \cap C} c_a d^m.$$

The (SSNFD) problem is to select a network Y in G such that there is a solution $X = (Y, Z^m : m \in M)$ whose cost is minimal. Let W_G denote this cost. We have

$$\begin{array}{llll} (SSNFD) \quad W_G & = & \min & c(X) \\ & & \text{s.t.} & X \in \mathcal{X}_G. \end{array}$$

The (SSNFD) is a general network design model with numerous special cases such as facility location problems and production planning problems. Its generality resides in its ability to mix discrete costs (the fixed charges) and continuous costs (the variable costs). This permits to model tradeoffs between fixed costs and operating costs in decision for investments.

Considering fixed charges only, that is assuming $C = \emptyset$, the (SSNFD) problem reduces to a Directed Steiner Tree problem. As the latter is NP-complete for general graphs, the (SSNFD) problem is NP-complete for general graphs as well.

In [EMV87], a polynomial algorithm is presented, solving the problem on planar graphs with all demand nodes lying on a same face. It is shown in [Sch90] that the problem can be solved in polynomial time for any Series-Parallel graphs.

1.2 The multicommodity linear formulation

Consider an (SSNFD) instance G and a solution $X = (Y, Z^m : m \in M)$ in $\mathcal{X}_G$.

Let y_a be the $\{0,1\}$-variable indicating if arc a belongs to the network Y for every arc a in F and let x_a be the non-negative real variable holding the total volume of the commodities flowing through arc a, for every arc a in C. Finally let z_a^m be the $\{0,1\}$-variable indicating if arc a is used by commodity m in the solution X, for every arc a in A and every commodity m in M.

To the solution $X = (Y, Z^m : m \in M)$ corresponds the vector (x, y, z) in $I\!R_+^{|C|} \times I\!R_+^{|F|} \times I\!R_+^{|A||M|}$ defined by

$$\begin{array}{rcll} x_a & = & \sum_{Z^m \ni a} d^m & (a \in C) \\ y_a & = & 1 & (a \in Y \cap F) \\ & = & 0 & (a \in F \setminus Y) \\ z_a^m & = & 1 & (m \in M, a \in Z^m) \\ & = & 0 & (m \in M, a \notin Z^m). \end{array}$$

Let (x^X, y^X) denote the vector (x, y) associated with solution X. The cost of solution X is given by

$$c(X) = cx + fy.$$

Let N be the node-arc $|V| \times |A|$-incidence matrix of the directed graph (V, A), that is the matrix for which the entry (t, a) is -1 if node t is the tail of arc a, $+1$ if node t is the head of arc a and 0 otherwise. Let b^m be the $|V|$-vector for which the entry corresponding to node r is -1, the entry corresponding to node s^m is $+1$ and the other entries are zero. The vector z^m is a unit flow vector with for unique source node node r and with for unique sink node node s^m $(m \in M)$. It therefore satisfies the flow conservation constraints

$$N z^m = b^m (m \in M). \tag{1}$$

Clearly the volume constraints

$$x_a \geq \sum_{m \in M} z_a^m d^m (a \in C) \tag{2}$$

and the capacity constraints

$$y_a \geq z_a^m (a \in F, m \in M) \tag{3}$$

are satisfied by the vector (x, y, z) associated with any feasible solution.

Define the polyhedron

$$P_{xyz} = \{(x, y, z) \in I\!R_+^{|C|} \times I\!R_+^{|F|} \times I\!R_+^{|A||M|} \text{ sat. (1), (2) and (3)}\}.$$

The (SSNFD) problem can be linearly formulated as follows. Consider the Mixed Integer Program

$$\begin{array}{lrcll} (MIP) & W & = & \min & cx + fy \\ & & & \text{s.t.} & (x, y, z) \in P_{xyz} \\ & & & & y \in \{0,1\}^{|F|}. \end{array}$$

If the capacity variables y are fixed to some $\{0,1\}$-values, then the (MIP) has an optimal solution with variables z integral, to which corresponds a feasible solution $X = (Y, Z^m : m \in$

M) (see [Wol89]). The (MIP) is thus equivalent to the (SSNFD) problem, in the sense that both problems have same value ($W = W_G$) for any cost function c, f in $I\!R_+^{|C|}, I\!R_+^{|F|}$ respectively.

The linear relaxation

$$(LP) \quad Z_G \quad = \quad \begin{array}{ll} \min & cx + fy \\ \text{s.t.} & (x, y, z) \in P_{xyz} \end{array}$$

of the (MIP) gives a lower bounds to the optimal value of the network design problem:

$$Z_G \leq W_G.$$

The multicommodity linear formulation is known to give very good lower bounds. In this paper, we prove that it gives tight lower bounds for a particular class of (SSNFD) instances $\mathcal{G}$. So we prove

$$\forall G \in \mathcal{G}, \forall c \in I\!R_+^{|C|}, \forall f \in I\!R_+^{|F|} : W_G = Z_G.$$

That is we prove that the multicommodity linear formulation is a polyhedral characterization valid in $\mathcal{G}$ for the (SSNFD) problem.

In [Sch94] it is proved that the multicommodity linear formulation is a polyhedral characterization for the (SSNFD) problem on Series-Parallel graph, if it is valid for a well-structured class of elementary instances $\mathcal{G}^{ele}$. The class of graphs $\mathcal{G}$ considered here is a part of the class elementary instances $\mathcal{G}^{ele}$.

Let P_{xy} be the projection of the polyhedron P_{xyz} onto the space of the variables x, y:

$$P_{xy} = \{(x, y) : \exists z \text{ s.t.}(x, y, z) \in P_{xyz}\}.$$

For a collection of vectors $\{x^i : i \in I\}$ in $I\!R^n$, let $< x^i : i \in I >^+$ denote the dominant of the convex hull of the vectors $x^i (i \in I)$, that is the polyhedron

$$\{x \in I\!R^n : \exists \lambda^i \in I\!R_+ (i \in I) \text{ s.t. } x \geq \sum_{i \in I} \lambda^i x^i, 1 = \sum_{i \in I} \lambda^i\}.$$

Let P^I be the dominant of the convex hull of the characteristic vectors (x^X, y^X) of the feasible solutions X in $\mathcal{X}_G$:

$$P^I =< (x^X, y^X) : X \in \mathcal{X}_G >^+ .$$

A tight linear relaxation for any non-negative cost function means that the dominant of the polyhedron P_{xy} is the dominant of the convex hull of the characteristic vectors (x^X, y^X) of the feasible solutions. As the polyhedron P_{xy} has for characteristic cone the positive orthant, to prove a tight linear relaxation for every non-negative cost function is equivalent to prove that

$$P_{xy} = P^I$$

that is to prove that the extreme points of the polyhedron P_{xy} are characteristic vectors of feasible solutions in $\mathcal{X}_G$.

Finding classes of inequalities valid for P^I may permit us to solve the SSNFD problem by using a Branch and Cut scheme. In [VRW85], a collection of inequalities, the *Basic*

Network inequalities, is presented and their validity for P^I is proved. Then, in [RW90], these inequalities are generalised by a bigger class of valid inequalities, the *Dicut* inequalities.

Although no description of the polyhedron P_{xy} is known for general graphs, the class of Dicut inequalities inequalities is proved to be sufficient in [RW90].

As the number of variables of the multicommodity formulation can be very large for practical applications, an expression of the polyhedron P_{xy} in the space of the natural variables can be potentially useful. In this paper, we completely describe $P_{xy} = P^I$ for any instance in the particular class $\mathcal{G}$ considered, by giving the class of inequalities necessary and sufficient for the description of $P_{xy} = P^I$.

1.3 The class of instances

Suppose that a set of four natural numbers

$$N = (n^p \in I\!N : p \in \{u, v, \bar{u}, \bar{v}\}) \tag{4}$$

and a collection of positive volumes

$$D = (d^{p,n} \in I\!R^0_+ : p \in \{u, v\}, 1 \leq n \leq n^p) \tag{5}$$

are given.

For any set of datas (N, D) defined by (4), (5), we define an instance $G = (V, A, r, M)$ as follows.

Pose

$$M^p = \{(p, n) : 1 \leq n \leq n^p\}(p \in \{u, v\})$$
$$A^p = \{(p, n) : 1 \leq n \leq n^{\bar{p}}\}(p \in \{u, v\}).$$

Let the set of nodes of G be

$$V = \{r, u, v\} \cup \{s^m : m \in M^u \cup M^v\}.$$

Let the set of commodities of G be

$$M = M^u \cup M^v \cup \{(u, 0), (v, 0)\}$$

let s^m and d^m be the demand nodes and the volume associated with the commodity m for every commodity m in $M^u \cup M^v$ and let

$$s^{p,0} = p(p \in \{u, v\})$$

$$d^{p,0} = 0(p \in \{u, v\})$$

define the demand nodes and the volumes of commodities $(u, 0)$ and $(v, 0)$.

Let the set of arcs of G be

$$\begin{array}{rcccccc} A & = & A^u & \cup & \{(v, u)\} & \cup & \{(v, s^m) : m \in M^u \cup M^v\} \\ & = & A^v & \cup & \{(u, v)\} & \cup & \{(u, s^m) : m \in M^u \cup M^v\}. \end{array}$$

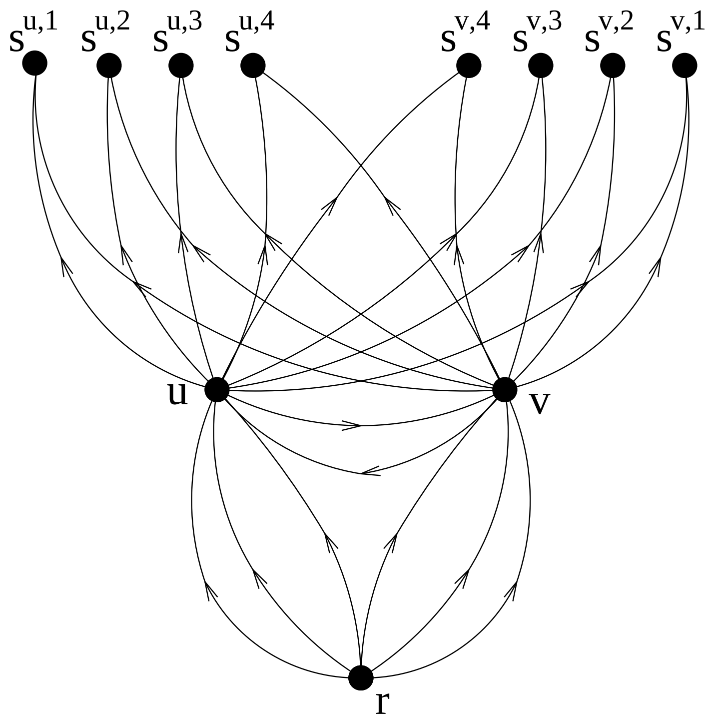
where A^u, A^v are sets of parallel arcs with as tail node node r and as head node nodes u, v respectively:

$$\forall a \in A^u : a = (r, u)$$

$$\forall a \in A^v : a = (r, v).$$

This defines the instance $G = (V, A, r, M)$. See figure 1.

Figure 1: The instance



A non-negative variable cost coefficient c_a is given for every arc a in the set

$$C = A^u \cup A^v$$

and a fixed charge f_a is given for every arc a in the set

$$\begin{array}{rcccccc} F & = & A^u & \cup & \{(v,u)\} & \cup & \{(v, s^m) : m \in M^u\} \\ & = & A^v & \cup & \{(u,v)\} & \cup & \{(u, s^m) : m \in M^v\}. \end{array}$$

In the following, the variables $y_{(v,u)}$, $y_{(v,s^m)}(m \in M^u)$, $y_{(u,v)}$, $y_{(u,s^m)}(m \in M^v)$ are denoted by $y^{u,0}$, $y^m(m \in M^u)$, $y^{v,0}$, $y^m(m \in M^v)$ respectively. Similarly, the fixed costs associated with these arcs are denoted by $f^{u,0}$, $f^m(m \in M^u)$, $f^{v,0}$, $f^m(m \in M^v)$ respectively.

For any set of commodities $N \subseteq M$, let d^N denote the total volume of the commodities in N:

$$d^N = \sum_{m \in N} d^m.$$

In particular we have

$$d^M = d^{M^u} + d^{M^v}.$$

The purpose of this paper is to prove for any set of values (N, D) defining an (SSNFD) instance G as specified above, the lower bound Z_G given by the linear relaxation is tight, that is, is equal to the optimal values W_G of the (SSNFD) problem defined on G. So we prove the following theorem.

Theorem 1 (Polyhedral Characterization) *For any set of datas (N, D), if G is the (SSNFD) instance defined by (N, D) as mentioned above, then we have*

$$\forall c \in I\!R_+^{|C|}, \forall f \in I\!R_+^{|F|} : W_G = Z_G.$$

Let P^I be the dominant of the convex hull of the characteristic vectors of the solutions in $\mathcal{X}_G$, that is the polyhedron defined by

$$P^I = < (x^X, y^X) : X \in \mathcal{X}_G >^+ .$$

As made explicit previously, theorem 1 is equivalent to the following theorem.

Theorem 2 (Polyhedral Characterization) *For any set of datas (N, D), if G is the (SSNFD) instance defined by (N, D) as mentioned above, then we have*

$$P_{xy} = P^I.$$

1.4 The set of combinatorial solutions

Consider an (SSNFD) instance G as defined in section 1.3 for some set of datas (N, D). We describe in this section four set of combinatorial solutions $\mathcal{X}^{A^u}$, $\mathcal{X}^{A^v}$, $\mathcal{X}^{B^u}$, $\mathcal{X}^{B^v}$, such that it is sufficient to consider the solutions in these sets to compute the optimal value of the (SSNFD) problem. We have

$$\mathcal{X}_G \supseteq \mathcal{X}^{A^u} \cup \mathcal{X}^{A^v} \cup \mathcal{X}^{B^u} \cup \mathcal{X}^{B^v}.$$

1.4.1 The solutions of type A^u

For any arc a^u in A^u, a^v in A^v, for any set of commodities I^u in M^u, consider the solution such that the commodities in I^u and $M \setminus I^u$ are served by a directed path containing arc a^u, a^v respectively. The solution is constructed as follows. Let

$$\begin{array}{rcccl} Y & = & \{a^u\} & \cup & \{(u, s^m) : m \in I^u\} \\ & \cup & \{a^v\} & \cup & \{(v, s^m) : m \in M \setminus I^u\} \end{array}$$

define the network, let

$$\begin{array}{rcll} Z^m & = & \{a^u, (u, s^m)\} & (m \in I^u) \\ & = & \{a^u\} & (m = (u, 0)) \\ & = & \{a^v, (v, s^m)\} & (m \in M \setminus I^u) \\ & = & \{a^v\} & (m = (v, 0)) \end{array}$$

define the collection of directed paths $Z^m (m \in M)$ and let X^{a^u, a^v, I^u} denote the corresponding solution in $\mathcal{X}_G$, that is

$$X^{a^u, a^v, I^u} = (Y, Z^m : m \in M).$$

We consider the set of solutions

$$\mathcal{X}^{A^u} = \{X^{a^u, a^v, I^u} : a^u \in A^u, a^v \in A^v, I^u \subseteq M^u\}.$$

The characteristic vector of a solution $X = X^{a^u, a^v, I^u}$ in $\mathcal{X}^{A^u}$, denoted by

$$(x^X, y^X) = (x_a : a \in C, y_a : a \in F)$$

is defined by

$$\begin{array}{lcll} x_{a^u} & = & d^{Iu} & \\ y_{a^u} & = & 1 & \\ x_{a^v} & = & d^M - d^{Iu} & \\ y_{a^v} & = & 1 & \\ y^m & = & 1 & (m \in M^u \setminus I^u) \\ x_a, y_a & = & 0 & \text{otherwise.} \end{array}$$

1.4.2 The solutions of type A^v

This class of solutions is defined symmetrically, by permuting the roles played by u and v in the definition of $\mathcal{X}^{A^u}$. So we have

$$\mathcal{X}^{A^v} = \{X^{a^u, a^v, I^v} : a^u \in A^u, a^v \in A^v, I^v \subseteq M^v\}.$$

Further let the set of solutions of type A be

$$\mathcal{X}^A = \mathcal{X}^{A^u} \cup \mathcal{X}^{A^v}.$$

1.4.3 The solutions of type B^u

For any arc a^u in A^u, consider the solution such that every commodity is served by a directed path containing arc a^u. The solution is constructed as follows. Let

$$\begin{array}{rcccl} Y & = & \{a^u\} & \cup & \{(u, s^m) : m \in M^u\} \\ & \cup & \{(u, v)\} & \cup & \{(v, s^m) : m \in M^v\} \end{array}$$

define the network, let

$$\begin{aligned} Z^m &= \{a^u, (u, s^m)\} && (m \in M^u) \\ &= \{a^u\} && (m = (u, 0)) \\ &= \{a^u, (u, v), (v, s^m)\} && (m \in M^v) \\ &= \{a^u, (u, v)\} && (m = (v, 0)) \end{aligned}$$

define the collection of directed paths $Z^m (m \in M)$ and let X^{a^u} denote the corresponding solution in $\mathcal{X}_G$, that is

$$X^{a^u} = (Y, Z^m : m \in M).$$

We consider the set of solutions

$$\mathcal{X}^{B^u} = \{X^{a^u} : a^u \in A^u\}.$$

The characteristic vector of any solution $X = X^{a^u}$ in $\mathcal{X}^{B^u}$, denoted by

$$(x^X, y^X) = (x_a : a \in C, y_a : a \in F)$$

is defined by

$$\begin{aligned} x_{a^u} &= d^M \\ y_{a^u} &= 1 \\ y^{u,0} &= 1 \\ x_a, y_a &= 0 \qquad \text{otherwise.} \end{aligned}$$

1.4.4 The solutions of type B^v

This class of solutions is defined symmetrically, by permuting the roles played by u and v in the definition of $\mathcal{X}^{B^u}$. So we have

$$\mathcal{X}^{B^v} = \{X^{a^v} : a^v \in A^v\}.$$

1.4.5 Interpretation

We claim that there is always an optimal solution of the (SSNFD) problem in $\mathcal{X}^A \cup \mathcal{X}^{B^u} \cup \mathcal{X}^{B^v}$. Therefore the optimal value W can be computed by considering the solutions in this latter set only. This can be achieved as follows.

Consider a cost function $(c_a : a \in C, f_a : a \in F)$ in $\mathbb{R}_+^{|C|} \times \mathbb{R}_+^{|F|}$.

For any solution in $\mathcal{X}^A$ using arcs a^u, a^v in A^u, A^v respectively, let d^u, d^v be the total volume of the commodities flowing from node r to nodes u, v respectively. The total cost in the arcs a^u, a^v is then

$$c_{a^u} d^u + f_{a^u} + c_{a^v} d^v + f_{a^v}.$$

As $d^u + d^v$ is the total volume of the commodities in G, i.e. d^M, the latter cost can be written as

$$(c_{a^u} - c_{a^v}) d^u + f_{a^u} + f_{a^v} + c_{a^v} d^M.$$

The cost function in the set of arcs $A^u \cup A^v$ can therefore be characterised by the concave piecewise linear cost function f of d^u defined by

$$f : d^u \to \min_{\substack{a^u \in A^u \\ a^v \in A^v}} (c_{a^u} - c_{a^v})d^u + f_{a^u} + f_{a^v} + c_{a^v} d^M.$$

For any solution X^{a^u,a^v,I^u} in $\mathcal{X}^{A^u}$, let K be the cost of the solution in $G \setminus A^u \setminus A^v$. We have

$$\begin{aligned} K &= \sum_{i \in M^u \setminus I^u} f^i \\ d^u &= d^{I^u} \\ d^v &= d^M - d^{I^u}. \end{aligned}$$

Analogously, a vector (d^u, d^v, K) can be defined for any solution in $\mathcal{X}^{A^v}$.

Plot on a (d^u, K)-graph the vectors defined above for every solution in $\mathcal{X}^A$. It is not difficult to see that only the vectors which are lying on the lower frontier of the convex hull of the vectors plotted on the graph are to be considered for computing the optimal value of the solutions in $\mathcal{X}^A$ (see [Sch90]). Let $g(d^u)$ be the convex piecewise linear function describing this lower frontier. So the optimal value for the solutions of type A is actually the minimum of a concave function plus a convex function. See figure 2.

Then the optimal value of the solution in $\mathcal{X}^A$, defined by

$$W^A = \min_{X \in \mathcal{X}^A} c(X)$$

is given by

$$W^A = \min_{d^u \in [0, d^M]} f(d^u) + g(d^u).$$

Let W^{B^u}, W^{B^v} be the optimal values of the solution in $\mathcal{X}^{B^u}$, $\mathcal{X}^{B^v}$ respectively. We have

$$\begin{aligned} W^{B^u} &= \min_{a^u \in A^u} c_{a^u} d^M + f_{a^u} + f^{u,0} \\ W^{B^v} &= \min_{a^v \in A^v} c_{a^v} d^M + f_{a^v} + f^{v,0}. \end{aligned}$$

Then the optimal value of the (SSNFD) problem is computed by

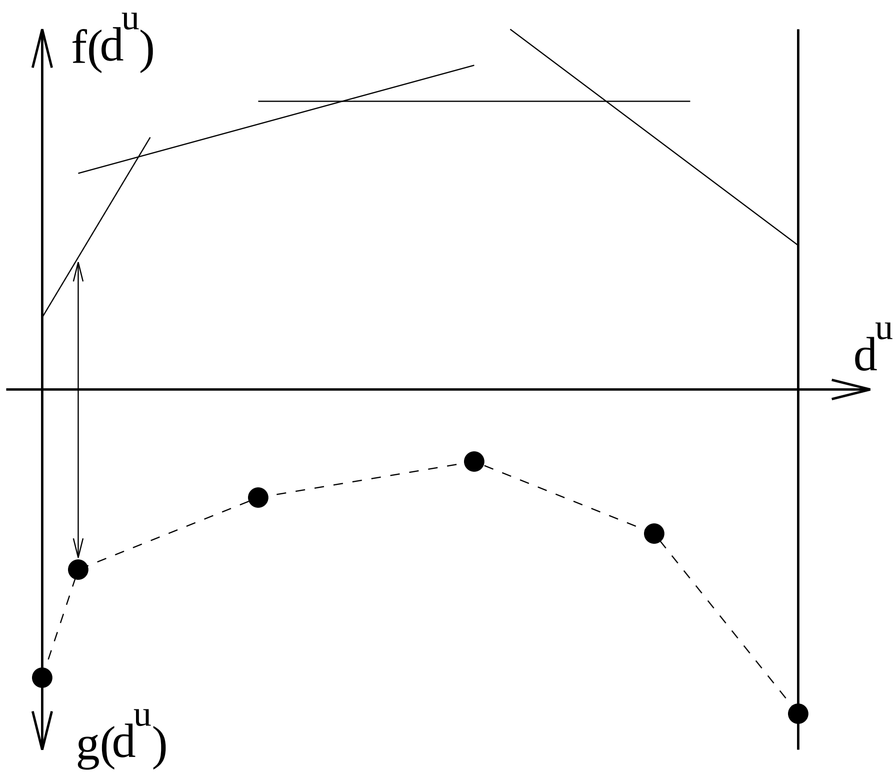
$$W = \min\{W^A, W^{B^u}, W^{B^v}\}.$$

2 The set of constraints

In section 1.3, an (SSNFD) instance G is defined for every set of datas (N, D).

In section 1.2, the multicommodity formulation is described, defining a polyhedron P_{xyz} in an extended space and its projection P_{xy} onto the space of the natural variables, that is the space of the variables $(x_a : a \in C, y_a : a \in F)$.

Figure 2: The optimal value for type A



The purpose of this paper is to prove that the polyhedron P_{xy} has for extreme points characteristic vectors of solutions in $\mathcal{X}_G$, that is to prove that P_{xy} is the polyhedron P^I associated with the (SSNFD) problem.

In this section, we describe a set of inequalities $\mathcal{C}$ and prove the validity of the inequalities in $\mathcal{C}$ for the polyhedron P_{xy}. Let P' be the polyhedron defined by the inequalities in $\mathcal{C}$. We prove thus $P_{xy} \subseteq P'$.

2.1 Characterization of the valid constraints

In this subsection, we characterize the valid inequalities of P_{xy}, as it is done in [RW90]. This characterization holds for any (SSNFD) instance.

A directed cut $Q = \delta^-(S)$ $(S \subseteq V)$ is *feasible* for the commodity m $(m \in M)$ if it separates the demand node s^m from the root node r. Let $\mathcal{S}^m$ be the set of feasible directed cuts for commodity m. We have

$$\mathcal{S}^m = \{\delta^-(S) : S \subseteq V, r \notin S, s^m \in S\}.$$

Every $r - s^m$ directed path intersects every directed cut Q in $\mathcal{S}^m$ $(m \in M)$. It follows that the constraints

$$\sum_{a\in Q} z_a^m \geq 1 (m \in M, Q \in \mathcal{S}^m) \tag{6}$$

are satisfied by the characteristic vector of the directed paths Z^m of every solution in $\mathcal{X}_G$.

Consider the polyhedron in the extended space

$$P'_{xyz} = \{(x,y,z) \in \mathbb{R}_+^{|C|} \times \mathbb{R}_+^{|F|} \times \mathbb{R}_+^{|A||M|} \text{ sat. (6), (2) and (3)}\}$$

and its projection onto the space of the natural variables

$$P'_{xy} = \{(x,y) : \exists z \text{ s.t.} (x,y,z) \in P'_{xyz}\}.$$

Clearly the flow vectors z^m in $\mathbb{R}_+^{|A|}$ $(m \in M)$ satisfying the flow conservation constraints (1) satisfy the cut constraints (6) as well. Indeed for any commodity m in M and for any cut $Q = \delta^-(S)$ in $\mathcal{S}^m$, aggregating the flow conservation constraints (1) over the set of nodes S gives

$$\sum_{a\in\delta^-(S)} z_a^m - \sum_{a\in\delta^+(S)} z_a^m = 1$$

with the validity of the cut constraint (6) as a consequence.

So we have

$$P_{xyz} \subseteq P'_{xyz}$$

which implies

$$P_{xy} \subseteq P'_{xy}. \tag{7}$$

In order to prove the validity of a constraint for P_{xy}, by (7), it is sufficient to prove its validity for P'_{xy}. Here follows a characterization of the constraints valid for P'_{xy}.

Lemma 1 (Characterization of valid constraints) *For any (SSNFD) instance G, the cost function (c,f) in $\mathbb{R}_+^{|C|} \times \mathbb{R}_+^{|F|}$ and the constant K define a valid constraint for the polyhedron P'_{xy}, i.e.*

$$\forall (x,y) \in P'_{xy} : cx + fy \geq K$$

if and only if there exist

$$\begin{array}{ll} \alpha_Q^m \in \mathbb{R}_+ & (m \in M, Q \in \mathcal{S}^m) \\ f_a^m \in \mathbb{R}_+ & (m \in M, a \in F) \end{array}$$

such that

$$\begin{array}{llll} \displaystyle\sum_{Q\ni a} \alpha_Q^m & \leq & c_a d^m & (a \in C \setminus F, m \in M) \\ \displaystyle\sum_{Q\ni a} \alpha_Q^m & \leq & f_a^m & (a \in F \setminus C, m \in M) \\ \displaystyle\sum_{Q\ni a} \alpha_Q^m & \leq & f_a^m + c_a d^m & (a \in C \cup F, m \in M) \\ \displaystyle\sum_{m\in M} f_a & \leq & f_a & (a \in F) \\ K & \leq & \displaystyle\sum_{\substack{m\in M \\ Q\in\mathcal{S}^m}} \alpha_Q^m. & \end{array}$$

Proof of lemma 1

Let α_Q^m, f_a^m and β_a be the dual variables associated with constraints (6), (3) and (2) respectively. Using Farkas lemma and assuming without loss of generality that $\beta_a = c_a$ $(a \in C)$, we obtain the result.

$\square$

2.2 The set of constraints

Consider an instance G in $\mathcal{G}$, as defined in subsection 1.3.

In this section, a set of contraints $\mathcal{C}$ is presented. The validity of the constraints in $\mathcal{C}$ for the polyhedron P'_{xy}, and therefore for the polyhedron P_{xy} as well is proved using lemma 1.

We prove in the next section that this set of constraints is actually sufficient for the description of the polyhedron P_{xy}.

Let the constraints

$$\begin{array}{rcl} \sum_{a \in A^u} y_a + y^{u,0} & \geq & 1 \\ \sum_{a \in A^v} y_a + y^{v,0} & \geq & 1 \\ \sum_{a \in A^u} y_a + \sum_{a \in A^v} y_a & \geq & 1 \end{array}$$

be denoted by C^u, C^v, $C^{u,v}$ respectively.

We have the following lemma.

Lemma 2 *The constraints C^u, C^v, $C^{u,v}$ are valid for the polyhedron P_{xy}.*

Proof of lemma 2

Fix the dual variables α_Q^m, f_a^m as follows

$$\begin{array}{lcll} \alpha_Q^{u,0} & = & 1 & (Q = A^u \cup \{(v,u)\}) \\ f_a^{u,0} & = & 1 & (a \in A^u \cup \{(v,u)\}) \\ \alpha_Q^m, f_a^m & = & 0 & \text{otherwise.} \end{array}$$

Then the validity of constraint C^u follows immediately by lemma 1 and the relation (7).

Constraint C^v is treated symmetrically.

Fix now the dual variables α_Q^m, f_a^m as follows

$$\begin{array}{lcll} \alpha_Q^{u,0} & = & 1 & (Q = A^u \cup A^v) \\ f_a^{u,0} & = & 1 & (a \in A^u \cup A^v) \\ \alpha_Q^m, f_a^m & = & 0 & \text{otherwise.} \end{array}$$

Then the validity of constraint C^{uv} follows immediately by lemma 1 and the relation (7).

$\square$

For any set of arcs D^u in A^u, for any set of commodities N^u in M^u, let the constraint

$$\sum_{a\in A^u\setminus D^u} x_a + \sum_{a\in D^u} y_a d^{N^u} + \sum_{m\in N^u} (y^{u,0}+y^m)d^m \geq d^{N^u}$$

be denoted by C^{D^u,N^u}. For any set of arcs D^v in A^v, for any set of commodities N^v in M^v, define the constraint C^{D^v,N^v} symmetrically.

We have the following lemma.

Lemma 3 *The constraints C^{D^p,N^p} $(p\in\{u,v\}, D^p\subseteq A^p, N^p\subseteq M^p)$ are valid for the polyhedron P_{xy}.*

Proof of lemma 3

Consider any subsets D^u, N^u in A^u, M^u respectively.

Pose

$$\begin{array}{lcll} \alpha_Q^m & = & d^m & (m\in N^u, Q=A^u\cup\{(v,u),(v,s^m)\}) \\ f_a^m & = & d^m & (m\in N^u, a\in D^u\cup\{(v,u),(v,s^m)\}) \\ \alpha_Q^m, f_a^m & = & 0 & \text{otherwise.} \end{array}$$

Then the validity of constraint C^{D^u,N^u} follows immediately by lemma 1 and the relation (7).

The constraint C^{D^v,N^v} is treated analogously.

$\square$

For any subsets D^u, N^v, D^v, N^u in A^u, M^v, A^v, M^u respectively, pose

$$d^{D^u,N^v} = \sum_{a\in A^u\setminus D^u} x_a + \sum_{a\in D^u} y_a(d^M - d^{N^v}) + \sum_{m\in N^v}(y^{v,0}+y^m)d^m \quad (8)$$

$$d^{D^v,N^u} = \sum_{a\in A^v\setminus D^v} x_a + \sum_{a\in D^v} y_a(d^M - d^{N^u}) + \sum_{m\in N^u}(y^{u,0}+y^m)d^m \quad (9)$$

and consider the constraint defined by

$$d^{D^u,N^v} + d^{D^v,N^u} \geq d^M$$

and denoted by C^{D^u,N^v,D^v,N^u}.

We have the following lemma.

Lemma 4 *For any subsets D^u, N^v, D^v, N^u in A^u, M^v, A^v, M^u respectively, the constraints C^{D^u,N^v,D^v,N^u} is valid for the polyhedron P_{xy}.*

Proof of lemma 4

Pose

$$\begin{array}{lcll} \alpha_Q^m & = & d^m & (m \in N^u, Q = A^u \cup \{(v,u),(v,s^m)\}) \\ \alpha_Q^m & = & d^m & (m \in N^v, Q = A^v \cup \{(u,v),(u,s^m)\}) \\ \alpha_Q^m & = & d^m & (m \in M \setminus N^u \setminus N^v, Q = A^u \cup A^v) \\ f_a^m & = & d^m & (m \in N^u, a \in D^u \cup \{(v,u),(v,s^m)\}) \\ f_a^m & = & d^m & (m \in N^v, a \in D^v \cup \{(u,v),(u,s^m)\}) \\ f_a^m & = & d^m & (m \in M \setminus N^u \setminus N^v, a \in D^u \cup D^v) \\ \alpha_Q^m, f_a^m & = & 0 & \text{otherwise.} \end{array}$$

Then the validity of constraint C^{D^u,N^v,D^v,N^u} follows immediately by lemma 1 and the relation (7).

□

In the following, D^p, D_i^p and N^p, N_i^p $(p \in \{u,v\}, i \in \{1,2\})$ always denote subsets of A^p and M^p respectively, even if it is not specifically made explicit.

Finally, let C^{x_a} and C^a be the positivity constraints of the variables x_a $(a \in C)$ and y_a $(a \in F)$.

So we consider the set of constraints

$$\begin{array}{rcl} \mathcal{C} & = & \{C^u, C^v, C^{u,v}\} \\ & \cup & \{C^{D^p,N^p} : p \in \{u,v\}, D^p \subseteq A^p, N^p \subseteq M^p\} \\ & \cup & \{C^{D^u,N^v,D^v,N^u} : D^u \subseteq A^u, N^v \subseteq M^v, D^v \subseteq A^v, N^u \subseteq M^u\} \\ & \cup & \{C^{x_a} : a \in C\} \\ & \cup & \{C^a : a \in F\} \end{array}$$

and the polyhedron

$$P' = \{(x,y) \in I\!R^{|C|} \times I\!R^{|F|} : \forall (c,f,K) \in \mathcal{C} : cx + fy \geq K\}.$$

The results of this subsection are summed up in the following theorem.

Theorem 3 (Valid Constraints) *For any set of datas (N,D) defining an (SSNFD) instance as mentioned above, we have*

$$P_{xy} \subseteq P'.$$

Proof of theorem 3

This is a direct consequences of lemma 2, lemma 3 and lemma 4.

□

In the next section, we prove that the extreme points of P' are characteristic vectors of feasible solutions in $\mathcal{X}_G$, that is we prove $P' \subseteq P^I$. The latter completes the proof of the validity of the polyhedral characterization for the instances considered.

2.3 Comments

The classes of inequalities C^{D^u,N^u}, C^{N^v,D^v}, C^{D^u,N^v,D^v,N^u} is part of the class of dicut inequalities presented in [RW90].

In [RW90], a dicut inequality is called *simple* if there is at most one non-zero dual variable α_Q^m $(Q \in \mathcal{S}^m)$ with value d^m for every commodity m in M. The questions whether the collection of Dicut inequalities and the collection of simple Dicut inequalities are sufficient for the description of P^I are asked in [RW90]. By the result proved in the present paper, both questions are answered affirmatively for the instances in $\mathcal{G}$.

3 Proof of Polyhedral Characterization

So we are given a set of constraints $\mathcal{C}$ defining a polyhedron P' and we are given a set of combinatorial solutions defining the polyhedron P^I. In this section, we prove

$$P' \subseteq P^I. \tag{10}$$

3.1 Methodology

We first globally present the method we use for proving the polyhedral characterization.

As every solution (x, y) in P' can be expressed by a convex combination of extreme points plus a positive combination of extreme rays, in order to prove (10), it is sufficient to prove that every extreme point of P' belongs to P^I and that every extreme ray of P' belongs to the characteristic cone of P^I. The latter is trivial. Indeed, since the coefficients of the constraints in $\mathcal{C}$ are non-negative and since for any variable there is a constraint with positive coefficient for this variable, it follows that the characteristic cone of P' is the positive orthant, that is the characteristic cone of P^I.

So we only need to prove that every extreme point of P' belongs to P^I. For this, it is sufficient to prove that for every extreme point (x, y) of P', there is a solution X in $\mathcal{X}$ such that $(x, y) = (x^X, y^X)$.

Clearly, for the latter, since an extreme point is a face of dimension zero of the polyhedron, it is sufficient to prove that for every face O of P', there is a solution X in $\mathcal{X}$ such that (x^X, y^X) is in the face O.

Finally, as the extreme points are faces of dimension zero, and therefore bounded faces as well, we can consider only the faces of P' bounded in some sense, that is the faces containing no exreme rays or none of the rays in a given subset.

For any point (x, y) in P', let $\mathcal{C}^{(x,y)}$ denote the set of constraints in $\mathcal{C}$ tight for (x, y), that is the set

$$\mathcal{C}^{(x,y)} = \{(c, f, K) \in \mathcal{C} : cx + fy \leq K\}.$$

Let $O^{(x,y)}$ denote the face of P' defined by the point (x, y), that is the face defined by the set of tight constraints $\mathcal{C}^{(x,y)}$. The face $O^{(x,y)}$ is the smallest face containing the point (x, y). For every face O of P', there is a solution (x, y) in P' such that a constraint is tight for the face if and only if it is tight for (x, y) , i.e. is in $\mathcal{C}^{(x,y)}$. So for every face O of P', there is a solution (x, y) in P' such that the face O is $O^{(x,y)}$.

For any point (x, y) in P', a solution $(\tilde{x}, \tilde{y})$ in P' belongs to the face $O^{(x,y)}$ if and only if every constraint in $\mathcal{C}$ tight for (x, y) is tight for $(\tilde{x}, \tilde{y})$, that is if

$$\mathcal{C}^{(x,y)} \subseteq \mathcal{C}^{(\tilde{x},\tilde{y})}.$$

Let $\mathcal{C}^X$ denote the set of constraints tight for the characteristic vector (x^X, y^X), that is the set $\mathcal{C}^{(x^X,y^X)}$ for every solution X in $\mathcal{X}$.

The sufficient condition for proving (10) presented above, i.e. the condition that every face of P' contains a characteristic vector (x^X, Y^X), is then formulated as follows.

Criterion 1 (Sufficient condition) *If for any point (x, y) in P', there exists a solution X in $\mathcal{X}$ such that*

$$\mathcal{C}^{(x,y)} \subseteq \mathcal{C}^X$$

then we have

$$P' \subseteq P^I.$$

We may consider only the faces of P' bounded in some sense, that is the faces excluding rays in a given subset.

In the following, we consider only extreme rays associated with capacity variables y_a $(a \in F)$. For any arc a in F, let r^a denote the ray associated with variable y_a, that is the vector (x, y) in $I\!R^{|C|} \times I\!R^{|F|}$ where the only non-zero component is $y_a = 1$.

A constraint (c, f, K) of $\mathcal{C}$ is tight for the ray $r^a = (x, y)$ $(a \in F)$ if and only if we have

$$cx + fy = f_a \leq 0.$$

Let $\mathcal{C}^a$ be the set of constraints tight for the ray r^a. We have

$$\mathcal{C}^a = \{(c, f, K) \in \mathcal{C} : f_a \leq 0\}.$$

Finally, the ray r^a belongs to the characteristic cone of the face defined by the point (x, y), i.e. of the face $O^{(x,y)}$, if and only if every constraint of $\mathcal{C}$ tight for the point (x, y) is tight for the ray r^a, that is if

$$\mathcal{C}^{(x,y)} \subseteq \mathcal{C}^a.$$

Let F^0 be a set of arcs in F and let R^0 be the corresponding set of rays, that is the set $\{r^a : a \in F^0\}$. If we consider only the faces of P' containing none of the rays in R^0, then we obtain the following sufficient condition for proving (10).

Criterion 2 (Sufficient condition (Bounded faces)) *If for any point (x, y) in P', either there is an arc a in F^0 such that*

$$\mathcal{C}^{(x,y)} \subseteq \mathcal{C}^a$$

or there exists a solution X in $\mathcal{X}$ such that

$$\mathcal{C}^{(x,y)} \subseteq \mathcal{C}^X$$

then we have

$$P' \subseteq P^I.$$

Finally, we can prove the contraposition of criterion 2 as well. This gives us the following criterion.

Criterion 3 (Sufficient condition (Contraposition)) *If for any point (x, y) in P', we have that*

$$\forall a \in F^0 : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^a$$

and

$$\forall X \in \mathcal{X} : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^X$$

is a contradiction then we have

$$P' \subseteq P^I.$$

3.2 Scheme of the proof

By the last criterion 3, in order to prove (10), we need to prove the following steps.

3.2.1 Step 1

Consider any face O of the polyhedron P', and consider any point (x, y) in P' such that $O^{(x,y)} = O$. We first prove that if the face O contains none of the rays in the set

$$R^0 = \{r^a : a \in A^u \cup A^v \cup \{(u, v), (v, u)\}\}$$

then the constraints C^u, C^v are tight for the face O. That is, we prove the following lemmas.

Lemma 5 *For any point (x, y) in P', if we have*

$$\forall a \in A^u \cup \{(v, u)\} : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^a$$

then we have

$$C^u \in \mathcal{C}^{(x,y)}.$$

Lemma 6 *For any point (x, y) in P', if we have*

$$\forall a \in A^v \cup \{(u, v)\} : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^a$$

then we have

$$C^v \in \mathcal{C}^{(x,y)}.$$

3.2.2 Step 2

Consider now any face O of the polyhedron P' containing none of the rays in R^0, that is by lemmas 5, 6, a face O for which constraints C^u, C^v are tight, and consider a point (x, y) in P' such that $O^{(x,y)} = O$, that is a point (x, y) of P' for which constraints C^u, C^v are tight.

We prove that if the face O contains none of the characteristic vectors of the solutions in $\mathcal{X}^A$, $\mathcal{X}^{B^u}$, $\mathcal{X}^{B^v}$, then the constraints $C^{u,v}$, $C^{v,0}$, $C^{u,0}$ respectively are tight for the faces. That is, we prove the following lemmas.

Lemma 7 *For any point (x, y) in P', if we have*

$$C^u, C^v \in \mathcal{C}^{(x,y)}$$

and

$$\forall X \in \mathcal{X}^A : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^X$$

then we have

$$C^{u,v} \in \mathcal{C}^{(x,y)}.$$

Lemma 8 *For any point (x, y) in P', if we have*

$$C^u, C^v \in \mathcal{C}^{(x,y)}$$

and

$$\forall X \in \mathcal{X}^{B^u} : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^X$$

then we have

$$C^{v,0} \in \mathcal{C}^{(x,y)}.$$

Lemma 9 *For any point (x, y) in P', if we have*

$$C^u, C^v \in \mathcal{C}^{(x,y)}$$

and

$$\forall X \in \mathcal{X}^{B^v} : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^X$$

then we have

$$C^{u,0} \in \mathcal{C}^{(x,y)}.$$

3.2.3 Step 3

Finally, we prove that no point (x, y) in P' can satisfy at equality the constraints $C^{u,v}$, $C^{u,0}$, $C^{v,0}$ simultaneously.

So we prove the following lemma.

Lemma 10 *For any point (x, y) in P',*

$$C^{u,v}, C^{v,0}, C^{u,0} \in \mathcal{C}^{(x,y)}$$

is a contradiction.

Proof of lemma 10

Suppose, that such a point (x, y) exists. Since it satisfies at equality contraints $C^{u,v}$, $C^{u,0}$, $C^{v,0}$, we have

$$\begin{aligned} \sum_{a \in A^u} y_a + \sum_{a \in A^v} y_a &\leq 1 && (11) \\ y^{u,0} &\leq 0 && (12) \\ y^{v,0} &\leq 0. && (13) \end{aligned}$$

On the other hand, since (x, y) is a point in P', it satisfies the constraints C^u, C^v. This implies

$$\begin{aligned} \sum_{a \in A^u} y_a + y^{u,0} &\geq 1 && (14) \\ \sum_{a \in A^v} y_a + y^{v,0} &\geq 1 && (15) \end{aligned}$$

respectively.

Summing (11) to (15), we obtain

$$0 \geq 1$$

a contradiction.

$\square$

3.3 Structure of the faces

In this section, we consider a point (x, y) in P' and derive some structural properties of the set of constraints of $\mathcal{C}$ tight for (x, y), that is of the set $\mathcal{C}^{(x,y)}$.

Lemma 11 *Consider any point (x, y) in P' and any subsets D_i^p in A^p, N_i^p in M^p ($p \in \{u, v\}, i \in \{1, 2\}$). If we have*

$$C^{D_1^u, N_1^v, D_1^v, N_1^u}, C^{D_2^u, N_2^v, D_2^v, N_2^u} \in \mathcal{C}^{(x,y)}$$

then we have

$$C^{D_1^u, N_1^v, D_2^v, N_2^u}, C^{D_2^u, N_2^v, D_1^v, N_1^u} \in \mathcal{C}^{(x,y)}.$$

Proof of lemma 11

Let d^{D^u,N^v} and d^{D^v,N^u} be defined as in (8) and (9) of section 2 respectively. The fact that constraints $C^{D_1^u,N_1^v,D_1^v,N_1^u}$, $C^{D_2^u,N_2^v,D_2^v,N_2^u}$ are tight for (x, y) means that

$$\begin{aligned} d^{D_1^u,N_1^v} + d^{D_1^v,N_1^u} &\leq d^M && (16) \\ d^{D_2^u,N_2^v} + d^{D_2^v,N_2^u} &\leq d^M && (17) \end{aligned}$$

respectively.

On the other hand, the constraints $C^{D_1^u,N_1^v,D_2^v,N_2^u}$, $C^{D_2^u,N_2^v,D_1^v,N_1^u}$ are satisfied by (x, y), which gives

$$\begin{aligned} d^{D_1^u,N_1^v} + d^{D_2^v,N_2^u} &\geq d^M && (18) \\ d^{D_2^u,N_2^v} + d^{D_1^v,N_1^u} &\geq d^M && (19) \end{aligned}$$

respectively.

Summing (16) to (19), we obtain

$$0 \geq 0.$$

So equality holds everywhere. In particular, the constraints (18) and (19) are tight for (x, y).

□

Lemma 12 *Consider any point (x, y) in P' and let D_0^u be the set $\{a \in A^u : y_a \leq 0\}$. Consider any subsets D_i^u in A^u with $D_0^u \subseteq D_i^u$, N_i^u in M^u ($i \in \{1, 2\}$). If we have*

$$C^{D_1^u,N_1^u}, C^{D_2^u,N_2^u} \in \mathcal{C}^{(x,y)}$$

then we have either

$$D_1^u \subseteq D_2^u$$

or

$$D_1^u \supseteq D_2^u$$

or

$$N_1^u = N_2^u = N^u, C^{D_1^u \cup D_2^u,N^u} \in \mathcal{C}^{(x,y)}, C^{D_1^u \cap D_2^u,N^u} \in \mathcal{C}^{(x,y)}.$$

<u>Proof of lemma 12</u>

The constraints $(c_1, f_1, K_1) = C^{D_1^u,N_1^u}$ and $(c_2, f_2, K_2) = C^{D_2^u,N_2^u}$ are satisfied at equality by (x, y). On the other side, as (x, y) is in P', the constraints $(c_3, f_3, K_3) = C^{D_1^u \cup D_2^u,N_1^u \cap N_2^u}$ and $(c_4, f_4, K_4) = C^{D_1^u \cap D_2^u,N_1^u \cup N_2^u}$ are satisfied by (x, y). So the constraint $(c_3 + c_4 - c_1 - c_2, f_3 + f_4 - f_1 - f_2, K_3 + K_4 - K_1 - K_2)$ is satisfied by (x, y) as well. This gives

$$\sum_{a \in D_1^u \setminus D_2^u} y_a d^{N_1^u \setminus N_2^u} + \sum_{a \in D_2^u \setminus D_1^u} y_a d^{N_2^u \setminus N_1^u} \leq 0$$

as it is easy to check. Since the left hand side is non-negative, equality holds throughout and thus constraints (c_3, f_3, K_3) and (c_4, f_4, K_4) are tight for (x, y). On the other hand, we have that

$$\begin{aligned} \sum_{a \in D_1^u \setminus D_2^u} y_a d^{N_1^u \setminus N_2^u} &= 0 && (20) \\ \sum_{a \in D_2^u \setminus D_1^u} y_a d^{N_2^u \setminus N_1^u} &= 0. && (21) \end{aligned}$$

Now suppose that conclusions $D_1^u \subseteq D_2^u$ and $D_1^u \supseteq D_2^u$ do not hold, that is suppose that $D_1^u \setminus D_2^u$, $D_2^u \setminus D_1^u$ are non empty. By hypothesis, for every arc a in $D_1^u \setminus D_2^u$, $D_2^u \setminus D_1^u$, the variable y_a is positive, and by construction, the volumes of the commodities in M^u are positive. Then (20) and (21) imply $N_1^u \subseteq N_2^u$ and $N_2^u \subseteq N_1^u$ respectively. So we have $N_1^u = N_2^u = N^u$ and that the constraints $(c_3, f_3, K_3) = C^{D_1^u \cup D_2^u, N^u} \in \mathcal{C}^{(x,y)}, (c_4, f_4, K_4) = C^{D_1^u \cap D_2^u, N^u}$ are tight. This completes the proof.

□

Lemma 13 *Consider any point (x, y) in P' and let D_0^u be the set $\{a \in A^u : y_a \leq 0\}$. Consider any subsets D_i^u in A^u with $D_0^u \subseteq D_i^u$, N_i^v in M^v, D^v in A^v, N^u in M^u $(i \in \{1, 2\})$. If we have*

$$C^{D_1^u, N_1^v, D^v, N^u}, C^{D_2^u, N_2^v, D^v, N^u} \in \mathcal{C}^{(x,y)}$$

then we have either

$$D_1^u \subseteq D_2^u$$

or

$$D_1^u \supseteq D_2^u$$

or

$$N_1^v = N_2^v = N^v, C^{D_1^u \cup D_2^u, N^v, D^v, N^u} \in \mathcal{C}^{(x,y)}, C^{D_1^u \cap D_2^u, N^u, D^v, N^u} \in \mathcal{C}^{(x,y)}.$$

<u>Proof of lemma 13</u>

The proof of lemma 13 is entirely similar to the proof of lemma 12, using the valid constraints $C^{D_1^u \cup D_2^u, N_1^v \cup N_2^v, D^v, N^u}$ and $C^{D_1^u \cap D_2^u, N_1^v \cap N_2^v, D^v, N^u}$.

□

Lemma 14 *Consider any point (x, y) in P' and let D_0^u be the set $\{a \in A^u : y_a \leq 0\}$. Consider any subsets D_i^u in A^u with $D_0^u \subseteq D_i^u$, N_i^u in M^u, D_2^v in A^v, N_2^v in M^v $(i \in \{1, 2\})$. If we have*

$$C^{D_1^u, N_1^u}, C^{D_2^u, N_2^v, D_2^v, N_2^u} \in \mathcal{C}^{(x,y)}$$

then we have either

$$D_1^u \supseteq D_2^u$$

or

$$N_1^u = M^u, N_2^v = M^v, C^{D_1^u \cup D_2^u, M^u} \in \mathcal{C}^{(x,y)}, C^{D_1^u \cap D_2^u, M^v, D^v, N^u} \in \mathcal{C}^{(x,y)}.$$

<u>Proof of lemma 14</u>

The proof of lemma 14 is entirely similar to the proof of lemma 12, using the valid constraints $C^{D_1^u \cup D_2^u, N_1^u}$ and $C^{D_1^u \cap D_2^u, N_2^v, D_2^v, N_2^u}$.

□

3.4 Proof of step 1

In this section, we prove that for any face that contains none of the rays in $\{r^a : a \in A^u \cup \{(v,u)\}\}$, the constraint C^u is tight for the face, that is we prove lemma 5.

For this, we successively characterize the faces of P' not containing a given ray r^a ($a \in A^u$), then the faces containing none of the rays r^a ($a \in A^u$) and finally the faces not containing the ray $r^{(v,u)}$. Combining these results gives us the proof of lemma 5.

Lemma 6 being symmetrical can be proved analogously.

Consider a point (x,y) in P' and the corresponding face O. For any arc a in A^u, the following lemma characterizes the face O if it does not contain the ray r^a.

Lemma 15 *For any point (x,y) in P', and for any arc a in A^u, if we have*

$$\exists C \in \mathcal{C}^{(x,y)} \text{ s.t. } C \notin \mathcal{C}^a$$

then at least one of the following properties holds:

$$C^u \in \mathcal{C}^{(x,y)} \tag{22}$$

$$C^{u,v} \in \mathcal{C}^{(x,y)} \text{ and } D^M = 0 \tag{23}$$

$$C^a \in \mathcal{C}^{(x,y)} \tag{24}$$

$$\exists D^u, N^v, D^v, N^u \text{ s.t. } C^{D^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}, a \in D^u, d^M > d^{N^v} \tag{25}$$

$$\exists D^u, N^u \text{ s.t. } C^{D^u,N^u} \in \mathcal{C}^{(x,y)}, a \in D^u, d^{N^u} > 0. \tag{26}$$

Proof of lemma 15

A constraint $(cx + fy \geq K)$ is not tight for the ray r^a if and only if f_a is strictly positive and the face O does not contain the ray r^a if there is a constraint in $\mathcal{C}^{(x,y)}$ not tight for the ray. This gives us (22) or (23) with any value of d^M or (24) or (25) or (26). Finally, observe that if we have (23) with a positive value of d^M, then we have (25) by posing $D^u = A^u$, $D^v = A^v$, $N^u = \emptyset$, $N^v = \emptyset$. This closes the proof.

□

Consider a point (x,y) in P' and the corresponding face O. The following lemma characterizes the face O containing none of the rays r^a ($a \in A^u$).

Lemma 16 *For any point (x,y) in P', if we have*

$$\forall a \in A^u : \exists C \in \mathcal{C}^{(x,y)} \text{ s.t. } C \notin \mathcal{C}^a$$

then at least one of the following properties holds:

$$C^u \in \mathcal{C}^{(x,y)} \tag{27}$$

$$C^{u,v} \in \mathcal{C}^{(x,y)} \text{ and } D^M = 0 \tag{28}$$

$$\forall a \in A^u : C^a \in \mathcal{C}^{(x,y)} \tag{29}$$

$$\exists N^v, D^v, N^u \text{ s.t. } C^{A^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}, d^M > d^{N^v} \tag{30}$$

$$\exists N^u \text{ s.t. } C^{A^u,N^u} \in \mathcal{C}^{(x,y)}, d^{N^u} > 0. \tag{31}$$

Proof of lemma 16

Suppose that (27) and (28) do not hold.

Pose

$$D^0 = \{a \in A^u : C^a \in \mathcal{C}^{(x,y)}\}$$

and consider the collections of subsets of A^u

$$\begin{aligned}
\mathcal{D}^0 &= \{D^0\} \\
\mathcal{D}^1 &= \{D^u : \exists N^v, D^v, N^u \text{ s.t. } C^{D^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}, d^M > d^{N^v}\} \\
\mathcal{D}^2 &= \{D^u : \exists N^u \text{ s.t. } C^{D^u,N^u} \in \mathcal{C}^{(x,y)}, d^{N^u} > 0\} \\
\mathcal{D} &= \mathcal{D}^0 \cup \mathcal{D}^1 \cup \mathcal{D}^2.
\end{aligned}$$

We claim that the collections $\mathcal{D}$, $\mathcal{D}^1$, $\mathcal{D}^2$ have a maximum element for the inclusion order, that is an element that contains every other element in the collection considered. Note that the collections $\mathcal{D}^1$, $\mathcal{D}^2$ can be empty.

Consider the collection $\mathcal{D}^2$ and some element D^u maximally choosed in $\mathcal{D}^2$. We have thus $C^{D^u,N^u} \in \mathcal{C}^{(x,y)}$ for some non empty N^u in M^u. First we claim that $D^0 \subseteq D^u$. Indeed suppose it is not the case. It is not difficult to check that the constraint $C^{D^u \cup D^0,N^u}$ belongs to $\mathcal{C}^{(x,y)}$ as well. This implies $D^u \cup D^0$ is in $\mathcal{D}^2$. A contradiction with the choice of D^u. Second we claim that D^u is maximum in $\mathcal{D}^2$. Indeed suppose it is not the case. Then there are two maximally choosed elements D^u_1, D^u_2 in $\mathcal{D}^2$ that are not comparable. So we have $C^{D^u_1,N^u_1}, C^{D^u_2,N^u_2} \in \mathcal{C}^{(x,y)}$ for some non empty subsets $N^u_1, N^u_2 \in M^u$. Since we have $D^0 \subseteq D^u_1, D^0 \subseteq D^u_2$, $D^u_1 \not\subseteq D^u_2$, $D^u_1 \not\supseteq D^u_2$, lemma 12 implies $N^u_1 = N^u_2 = N^u$ and $C^{D^u_1 \cup D^u_2,N^u} \in \mathcal{C}^{(x,y)}$. But then $D^u_1 \cup D^u_2$ belongs to $\mathcal{D}^2$. A contradiction with the choices of D^u_1, D^u_2.

Consider now the collection $\mathcal{D}_1$. The fact that $\mathcal{D}^1$, if non empty, contains a maximum element incuding D^0 is proved entirely in a similar way, using lemma 13.

If at least one of the collection $\mathcal{D}^1, \mathcal{D}^2$ is empty, clearly the claim is proved, i.e. $\mathcal{D}$ contains a maximum element.

If it is not the case, let D^u_1, D^u_2 be the maximum elements in $\mathcal{D}^1, \mathcal{D}^2$ respectively. So we have $C^{D^u_1,N^v_1,D^v_1,N^u_1}, C^{D^u_2,N^u_2} \in \mathcal{C}^{(x,y)}$ for some subsets $N^v_1, D^v_1, N^u_1, N^u_2$ with $d^M > d^{N^v_1}, d^{N^u_2} > 0$. Suppose that the elements are not comparable. Then we have $D^0 \subseteq D^u_1, D^0 \subseteq D^u_2$, $D^u_1 \not\subseteq D^u_2$, $D^u_1 \not\supseteq D^u_2$, and lemma 14 implies $N^u_1 = M^u$ and $C^{D^u_1 \cup D^u_2,M^u} \in \mathcal{C}^{(x,y)}$. But then $D^u_1 \cup D^u_2$ belongs to $\mathcal{D}^2$. A contradiction with the choices of D^u_2.

So the collection $\mathcal{D}$ contains a maximal element, that we denote by D^*.

For any arc a in A^u, by the assumption made in the beginning of the proof, we have that conclusion (24), (25) or (26) of lemma 15 holds, which implies that a belongs to some element of $\mathcal{D}^0$, $\mathcal{D}^1$ or $\mathcal{D}^2$ respectively, and thus to the maximum element D^* as well. So we have $D^* = A^u$. But then the fact that the maximum element D^* belongs to $\mathcal{D}^0$, $\mathcal{D}^1$ or $\mathcal{D}^2$ gives us conclusion (29), (30) or (31) respectively. This completes the proof.

□

Consider a point (x, y) in P' and the corresponding face O. The following lemma characterizes the face O if it does not contain the ray $r^{(v,u)}$, that is the ray associated with the capacity variable $y^{u,0}$.

Lemma 17 *For any point (x, y) in P', if we have*

$$\exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^{(v,u)}$$

then at least one of the following properties holds:

$$C^u \in \mathcal{C}^{(x,y)} \tag{32}$$

$$C^{(v,u)} \in \mathcal{C}^{(x,y)} \tag{33}$$

$$\exists D^u, N^v, D^v, N^u \ s.t. \ C^{D^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}, d^{N^v} > 0 \tag{34}$$

$$\exists D^u, N^u \ s.t. \ C^{D^u,N^u} \in \mathcal{C}^{(x,y)}, d^{N^u} > 0. \tag{35}$$

Proof of lemma 17

First a constraint is not tight for the ray $r^{(v,u)}$ if and only if the coefficient of the variable $y^{u,0}$ is strictly positive and second the ray does not belong to the face if some constraint defining the face is not tight for the ray.

□

We are now ready to prove lemma 5, which states that if a face O of P' defined by a point (x, y) contains none the rays in $\{r^a : a \in A^u \text{ or } a = (v, u)\}$, then the constraint C^u, that is the constraint

$$\sum_{a \in A^u} y_a + y^{u,0} \geq 1$$

is tight.

Proof of lemma 5

Suppose that constraint C^u is not tight for (x, y), that is suppose

$$\sum_{a \in A^u} y_a + y^{u,0} > 1. \tag{36}$$

By lemma 16, we have (28), (29), (30) or (31). By lemma 17, we have (33), (34) or (35).

If (31) holds, then we have $C^{A^u,N^u} \in \mathcal{C}^{(x,y)}$ for some non empty subset N^u, which implies

$$(\sum_{a \in A^u} y_a + y^{u,0}) d^{N^u} \leq \sum_{a \in A^u} y_a d^{N^u} + \sum_{m \in N^u} (y^{u,0} + y^m) d^m \leq d^{N^u}.$$

As N^u is non empty and as the volumes of the commodities in M^u are positive, the latter expression is a contradiction with the absurd hypothesis (36).

If (28) holds, then we have that $d^M = 0$, which excludes (34) and (35). It follows that (33) holds. So we have

$$\sum_{a \in A^u} y_a + \sum_{a \in A^v} y_a \leq 1 \tag{37}$$

by (28) and

$$y^{u,0} \leq 0 \tag{38}$$

by (33). Summing (37) and (38), we obtain a contradiction with (36).

At this stage we have thus on one side that (29) or (30) holds and on the other side that (33), (34) or (35) holds.

We distinguish the following cases.

Case 1: property (33) holds

So we have

$$y^{u,0} \leq 0. \tag{39}$$

Observe that because of (39) and the fact that the constraint C^u is satisfied by (x, y), the property (29) does not hold. This implies that (30) holds, and thus that we have $(c_1, f_1, K_1) = C^{A^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}$ for some N^v, D^v, N^u with $d^M > dN^v$. On the other hand the constraint $(c_2, f_2, K_2) = C^{D^v,N^v}$ is satisfied by (x, y). So the constraint $(c_2 - c_1, f_2 - f_1, K_2 - K_1)$ is satisfied by (x, y) as well. This gives

$$\sum_{a\in A^u} y_a(d^M - d^{N^u}) + \sum_{a\in D^v} y_a(d^M - d^{N^u} - d^{N^v}) + \sum_{m\in N^u} (y^{u,0} + y^m)d^m \leq d^M - d^{N^v}$$

As $d^M > d^{N^v}$ and as the variables $y_a(a \in A^v), y^{u,0}, y^m(m \in M^u)$ are non-negative we have

$$\sum_{a\in A^u} y_a \leq 1$$

which, combined with (39), is a contradiction with (36).

Case 2: property (35) holds

So there is a constraint $(c_1, f_1, K_1) = C^{D_1^u,N_1^u} \in \mathcal{C}^{(x,y)}$ for some subsets D_1^u, N_1^u with $d^{N_1^u} > 0$.

If (29) holds, it is easy to check that the constraint C^{A^u,N_1^u} belongs to $\mathcal{C}^{(x,y)}$ as well, a contradiction with the absurd hypothesis (36), as shown in the beginning of the proof.

So we have (30), and thus that there is a constraint $(c_2, f_2, K_2) = C^{A^u,N_2^v,D_2^v,N_2^u}$ in $\mathcal{C}^{(x,y)}$ for some subsets N_2^v, D_2^v, N_2^u with $d^M > d^{N_2^v}$. On the other hand, the constraints $(c_3, f_3, K_3) = C^{D_1^u,N_2^v,D_2^v,N_2^u}$ and $(c_4, f_4, K_4) = C^{A^u,N_1^u}$ are satisfied by (x, y). It follows that the constraint $(c_3 + c_4 - c_1 - c_2, f_3 + f_4 - f_1 - f_2, K_3 + K_4 - K_1 - K_2)$ is satisfied by (x, y) as well. This gives

$$\sum_{a\in A^u} y_a(d^M - d^{N_1^u} - d^{N_2^v}) \leq \sum_{a\in D_1^u} y_a(d^M - d^{N_1^u} - d^{N_2^v})$$

As $D_1^u \subseteq A^u$, we have that equality holds throughout. In particular the constraint C^{A^u,N_1^u} is tight for (x, y), again a contradiction with (36).

Case 3: property (34) holds

So there is a constraint $C^{D_1^u,N_1^v,D_1^v,N_1^u} \in \mathcal{C}^{(x,y)}$ for some subsets $D_1^u, N_1^v, D_1^v, N_1^u$ with $d^{N_1^u} > 0$.

First we can assume $D_1^u = A^u$. Indeed. If (29) holds this is trivial. If (30) holds, there is a constraint $C^{A^u,N_2^v,D_2^v,N_2^u} \in \mathcal{C}^{(x,y)}$ for some subsets N_2^v, D_2^v, N_2^u. But then by lemma (11), the constraint $C^{A^u,N_2^v,D_1^v,N_1^u}$ belongs to $\mathcal{C}^{(x,y)}$ as well.

There is thus a constraint $(c_1, f_1, K_1) = C^{A^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}$ for some subsets N^v, D^v, N^u with $d^{N^u} > 0$.

On the other hand, the constraint $(c_2, f_2, K_2) = C^{A^u,N^v,D^v,\emptyset}$ is satisfied by (x, y) which implies that the constraint $(c_2 - c_1, f_2 - f_1, K_2 - K_1)$ is satisfied by (x, y) as well. This gives, using $d^{N^u} > 0$

$$\sum_{a \in D^v} y_a \geq y^{u,0}. \tag{40}$$

The constraint $(c_3, f_3, K_3) = C^{A^u,N^u}$ and $(c_4, f_4, K_4) = C^{N^v_1,D^v_1}$ are satisfied by (x, y), which implies that the constraint $(c_3 + c_4 - c_1, f_3 + f_4 - f_1, K_3 + K_4 - K_1)$ is satisfied by (x, y) as well. This gives

$$\sum_{a \in A^u} y_a(d^M - d^{N^u} - d^{N^v}) + \sum_{a \in D^v} y_a(d^M - d^{N^u} - d^{N^v}) \leq d^M - d^{N^u} - d^{N^v}.$$

If $d^M = d^{N^u} - d^{N^v}$, then we have that inequality holds throughout. In particular, the constraint C^{A^u,N^u} is tight with $d^{N^u} > 0$, a contradiction with (36).

If $d^M > d^{N^u} - d^{N^v}$, we have

$$\sum_{a \in A^u} y_a + \sum_{a \in D^v} y_a \leq 1. \tag{41}$$

But the combining (40) and (41), we obtain a contradiction with (36).

This close the proof.

□

3.5 Proof of step 2

In this section, we prove that for any face that contains none of the rays in R^0 and none of the characteristic vectors of the solutions in $\mathcal{X}^{B^u}$, $\mathcal{X}^{B^v}$, $\mathcal{X}^A$, the constraints $C^{v,0}$, $C^{u,0}$, $C^{u,v}$ respectively are tight for the face, that is we prove lemmas 8, 9, 7.

3.5.1 The solutions of $\mathcal{X}^{B^u}$

We first exclude the solutions in $\mathcal{X}^{B^u}$. For this, we successively characterize the faces of P' not containing a given solution $X^a \in \mathcal{X}^{B^u}$ ($a \in A^u$), then the faces containing none of the solutions in $\mathcal{X}^{B^u} = \{X^a : a \in A^u\}$. Using this last result, we prove lemma 8.

Lemma 9 being symmetrical can be proved analogously.

Consider a point (x, y) in P' and the corresponding face O. For any arc a in A^u, the following lemma characterizes the face O if it does not contain the solution $X^a \in \mathcal{X}^{B^u}$.

Lemma 18 *For any point (x, y) in P', and for any arc a in A^u, if we have*

$$\exists C \in \mathcal{C}^{(x,y)} \ s.t.\ C \notin \mathcal{C}^{X^a}$$

then at least one of the following properties holds:

$$C^{v,0} \in \mathcal{C}^{(x,y)} \tag{42}$$

$$C^{a} \in \mathcal{C}^{(x,y)} \tag{43}$$

$$C^{x_a} \in \mathcal{C}^{(x,y)} \text{ and } D^M > 0 \tag{44}$$

$$\exists D^u, N^u \text{ s.t. } C^{D^u,N^u} \in \mathcal{C}^{(x,y)}, a \notin D^u, d^M > d^{N^u} \tag{45}$$

$$\exists D^u, N^v, D^v, N^u \text{ s.t. } C^{D^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}, a \notin D^u, d^{N^v} > 0 \tag{46}$$

Proof of lemma 18

A solution $(\tilde{x},\tilde{y})$ in P' does not belong to the face O if there is some constraint in $\mathcal{C}^{(x,y)}$ not tight for $(\tilde{x},\tilde{y})$. Let $(\tilde{x},\tilde{y})$ be the characteristic vector of the solution X^a, i.e. $(\tilde{x},\tilde{y}) = (x^{X^a}, y^{X^a})$.

It is trivial to check that constraints $C^{v,0}$, C^a are not tight for $(\tilde{x},\tilde{y})$, that constraint C^{x_a} is not tight for $(\tilde{x},\tilde{y})$ if $d^M > 0$, that a constraint C^{D^u,N^u} is not tight for $(\tilde{x},\tilde{y})$ if $a \notin D^u, d^M > d^{N^u}$ and that a constraint C^{D^u,N^v,D^v,N^u} is not tight for $(\tilde{x},\tilde{y})$ if $a \notin D^u, d^{N^v} > 0$.

Finally the constraint mentioned above are the only constraints in $\mathcal{C}$ not tight for $(\tilde{x},\tilde{y})$.

□

Lemma 19 *For any point (x,y) in P', if we have*

$$\forall a \in A^u : \exists C \in \mathcal{C}^{(x,y)} \text{ s.t. } C \notin \mathcal{C}^{X^a}$$

and

$$C^{v,0} \notin \mathcal{C}^{(x,y)}$$

then there are subsets D^u_1, D^u_2 of A^u with $D^u_1 \subseteq D^u_2$ such that at least one of the following properties holds

$$D^u_1 = A^u \tag{47}$$

$$\forall a \in D^u_1 : C^{x_a} \notin \mathcal{C}^{(x,y)} \text{ or } C^a \in \mathcal{C}^{(x,y)}, d^M > 0 \tag{48}$$

$$\exists N^u_1 \text{ s.t. } C^{D^u_1,N^u_1} \in \mathcal{C}^{(x,y)}, d^M > d^{N^u_1} \tag{49}$$

$$\exists N^v_1, D^v_1, N^u_1 \text{ s.t. } C^{D^u_1,N^v_1,D^v_1,N^u_1} \in \mathcal{C}^{(x,y)}, d^{N^v_1} > 0 \tag{50}$$

and such that

$$\forall a \in D^u_2 : C^a \in \mathcal{C}^{(x,y)} \tag{51}$$

Proof of lemma 19

Pose

$$D^0 = \{a \in A^u : C^a \in \mathcal{C}^{(x,y)}\}$$

and consider the collections of subsets of A^u

$$\begin{aligned}
\mathcal{D}^0 &= \{\{a \in A^u : C^{x_a} \notin \mathcal{C}^{(x,y)} \text{ or } C^a \in \mathcal{C}^{(x,y)}\}\} \text{ if } d^M > 0, \emptyset \text{ otherwise} \\
\mathcal{D}^1 &= \{D^u : D^0 \subseteq D^u, \exists N^u \text{ s.t. } C^{D^u,N^u} \in \mathcal{C}^{(x,y)}, d^M > d^{N^u}\} \\
\mathcal{D}^2 &= \{D^u : D^0 \subseteq D^u, \exists N^v, D^v, N^u \text{ s.t. } C^{D^u,N^v,D^v,N^u} \in \mathcal{C}^{(x,y)}, d^{N^v} > 0\} \\
\mathcal{D} &= \mathcal{D}^0 \cup \mathcal{D}^1 \cup \mathcal{D}^2.
\end{aligned}$$

Observe that the collections $\mathcal{D}^0$, $\mathcal{D}^1$, $\mathcal{D}^2$, and thus possibly their union as well, can be empty.

In the same way as in the proof of lemma 16, using lemmas 12, 13, 14, it is not difficult to prove that the collections $\mathcal{D}^1$, $\mathcal{D}^2$, $\mathcal{D}$, if non empty, contain a mininum element for the inclusion order, that is an element contained by every other element in the collection considered. Let D_1^u denote the minimum element of $\mathcal{D}$ if $\mathcal{D}$ is non empty and let D_1^u be A^u otherwise.

Let D_2^u be D^0.

For any arc a in A^u, lemma 18 implies that if the characteristic vector of the solution X^a does not belong to the face, then either a belongs to D_2^u by (43), or, if not, it does not belong to some element of $\mathcal{D}^0$, $\mathcal{D}^1$, $\mathcal{D}^2$, by (44), (45), (46) respectively. Indeed. It is trivial if (44) holds. So Suppose that a is not in D_2^u and that (45) holds. Then there are subsets D^u, N^u such that the constraint C^{D^u,N^u} is tight with $a \notin D^u$, $d^M > d^{N^u}$. Pose $D_1^u = D^u \cup D^0$. Observe that the constraint $C^{D_1^u,N^u}$ is tight with $D^0 \subseteq D_1^u$, $a \notin D_1^u$, $d^M > d^{N^u}$. That is D_1^u is an element of $\mathcal{D}_1$ not containing a. The argument is the same if (46) instead of (45) holds for a .

We have thus shown that for arc a not in D_2^u is not contained in some element of $\mathcal{D}$, and is therefore not contained in the maximal element D_1^u either.

So we have

$$\forall \in A^u : a \in D_2^u \text{ or } a \notin D_1^u$$

which is equivalent to

$$D_1^u \subseteq D_2^u.$$

Finally, accordingly to whether $\mathcal{D}$ is empty or whether the maximum element D_1^u belongs to $\mathcal{D}^0$, $\mathcal{D}^1$, $\mathcal{D}^2$, we have (47), (48), (49), (50) respectively.

□

We are now ready to prove the main result of this section, i.e. to prove that if the face contains none of the characteristic vector of the solution in $\mathcal{X}^{B^u}$ and if the constraints C^u, C^v are tight, then the constraint $C^{v,0}$ is tight as well, that is to prove lemma 8.

Proof of lemma 8

Suppose that the constraint $C^{v,0}$ is not tight, that is suppose

$$y^{v,0} > 0. \tag{52}$$

By lemma 19, there are subsets D_1^u, D_2^u of A^u with $D_1^u \subseteq D_2^u$ such that D_1^u satisfies (47), (48), (49) or (50) and such that D_2^u satisfies (51).

Case 1: D_1^u satisfies (47)

By (47) and (51), we have

$$\sum_{a\in A^u} y_a \leq 0$$

which combined with the valid constraints C^u, gives a contradiction with (52).

Case 2: D_1^u satisfies (48)

Pose $N_1^u = \emptyset$. Observe that the constraint $C^{D_1^u,N_1^u}$ belongs to $\mathcal{C}^{(x,y)}$ and that $d^M > 0 = d^{N_1^u}$. So we have that D_1^u satisfies (49), which is the object of next case.

Case 3: D_1^u satisfies (49)

So there exists $N_1^u \subseteq M^u$ such that the constraint $(c_1, f_1, K_1) = C^{D_1^u,N_1^u}$ belongs to $\mathcal{C}^{(x,y)}$ with $d^M > d^{N_1^u}$. On the other hand, the constraint $(c_2, f_2, K_2) = C^{D_1^u,\emptyset,A^v,N_1^u}$ is satisfied by the solution (x, y). So the constraint $(c_2 - c_1, f_2 - f_1, K_2 - K_1)$ is satisfied by (x, y) as well. This gives, using $D^M > d^{N_1^u}$

$$\sum_{a\in D_1^u} y_a + \sum_{a\in A^v} y_a \geq 1. \tag{53}$$

Since $D_1^u \subseteq D_2^u$, (51) implies

$$\sum_{a\in D_1^u} y_a \leq 0. \tag{54}$$

On the other hand, by hypothesis, we have that the constraint C^v is tight, that is we have

$$\sum_{a\in A^v} y_a + y^{y,0} \leq 1. \tag{55}$$

Summing (53), (54), (55) we obtain a contradiction with (52).

Case 4: D_1^u satisfies (50)

So there are subsets N_1^v, D_1^v, N_1^u such that the constraint $(c_1, f_1, K_1) = C^{D_1^u,N_1^v,D_1^v,N_1^u}$ belongs to $\mathcal{C}^{(x,y)}$ with $d^{N_1^v} > 0$. On the other hand, the constraint $(c_2, f_2, K_2) = C^{D_1^u,\emptyset,D_1^v,N_1^u}$ is satisfied by (x, y). The constraint $(c_2 - c_1, f_2 - f_1, K_2 - K_1)$ is therefore satisfied by (x, y), which implies, using $d^{N_1^v} > 0$,

$$\sum_{a\in D_1^u} y_a \geq y^{v,0}. \tag{56}$$

On the other hand, as in case 3, we have (54). Summing (54) and (56), we directly obtain a contradiction with (52). This closes case 4 and thus the proof of the lemma.

□

3.5.2 The solutions of $\mathcal{X}^A$

We then exclude the solutions in $\mathcal{X}^A$. So we prove that if the face defined by a point (x, y) contains none of the characteristic vectors of the solutions in $\mathcal{X}^A$ and if the constraints C^u, C^v are tight for the face, then the constraint $C^{u,v}$ is tight as well, that is we prove lemma 7.

Consider a point (x, y) in P', and let

$$\begin{aligned} N^{u*} &= \{m \in M^u : y^m \leq 0\} \\ N^{v*} &= \{m \in M^v : y^m \leq 0\}. \end{aligned}$$

In the following, we characterize the faces containing none of the solutions $X^{a^u,a^v,N^{u*}}$, $X^{a^u,a^v,N^{v*}}$ ($a^u \in A^u, a^v \in A^v$) respectively. Then we show that it is sufficient to exclude these solutions to obtain the result.

Consider a point (x, y) in P' and the corresponding face O. For any arcs a^u, a^v, in A^u, A^v respectively, the following lemma characterizes the face O if it does not contain the solution $X^{a^u,a^v,N^{u*}}$ in $\mathcal{X}^{A^u}$.

Let $P_1^u(a)$ be the property satisfied by arc a in A^u if and only if at least one of the following conditions holds:

$$C^{x_a} \in \mathcal{C}^{(x,y)} \text{ and } d^{N^{u*}} > 0 \tag{57}$$

$$\exists D_1^u, N_1^u \text{ s.t. } C^{D_1^u,N_1^u} \in \mathcal{C}^{(x,y)}, a \notin D_1^u, N^{u*} \not\subseteq N_1^u. \tag{58}$$

Let $P_2^u(a)$ be the property satisfied by arc a in A^u if and only if at least one of the following conditions holds:

$$C^a \in \mathcal{C}^{(x,y)} \tag{59}$$

$$\exists D_2^u, N_2^v, D_2^v, N_2^u \text{ s.t. } C^{D_2^u,N_2^v,D_2^v,N_2^u} \in \mathcal{C}^{(x,y)}, a \in D_2^u, d^M > d^{N^{u*}} + d^{N_2^v} \tag{60}$$

$$\exists D_2^u, N_2^u \text{ s.t. } C^{D_2^u,N_2^u} \in \mathcal{C}^{(x,y)}, a \in D_2^u, N^{u*} \not\supseteq N_2^u. \tag{61}$$

Let $Q_1^u(a)$ be the property satisfied by arc a in A^v if and only if at least one of the following conditions holds:

$$C^{x_a} \in \mathcal{C}^{(x,y)} \text{ and } d^M > d^{N^{u*}} \tag{62}$$

$$\exists D_1^v, N_1^v \text{ s.t. } C^{D_1^v,N_1^v} \in \mathcal{C}^{(x,y)}, a \notin D_1^v, d^M > d^{N_1^v} + d^{N^{u*}}. \tag{63}$$

$$\exists D_1^u, N_1^v, D_1^v, N_1^u \text{ s.t. } C^{D_1^u,N_1^v,D_1^v,N_1^u} \in \mathcal{C}^{(x,y)}, a \notin D_1^v, N^{u*} \not\supseteq N_1^u. \tag{64}$$

Let $Q_2^u(a)$ be the property satisfied by arc a in A^v if and only if at least one of the following conditions holds:

$$C^a \in \mathcal{C}^{(x,y)} \tag{65}$$

$$\exists D_2^u, N_2^v, D_2^v, N_2^u \text{ s.t. } C^{D_2^u,N_2^v,D_2^v,N_2^u} \in \mathcal{C}^{(x,y)}, a \in D_2^v, N^{u*} \not\subseteq N_2^u. \tag{66}$$

Lemma 20 *For any point (x, y) in P', and for any arcs a^u, a^v, in A^u, A^v respectively, if we have*

$$\exists C \in \mathcal{C}^{(x,y)} \text{ s.t. } C \notin \mathcal{C}^{X^{a^u,a^v,N^{u*}}}$$

then, either the constraint $C^{u,v}$ is tight for (x, y), or at least one of the properties $P_1^u(a^u)$, $P_2^u(a^u)$, $Q_1^u(a^v)$, $Q_2^u(a^v)$ holds.

Proof of lemma 20

Let $(\tilde{x}, \tilde{y})$ be the characteristic vector of the solution $X^{a^u, a^v, N^{u*}}$, as defined in section 1.4. If the vector $(\tilde{x}, \tilde{y})$ does not belong to the face defined by (x, y), then some constraint in $\mathcal{C}^{(x,y)}$ is not tight for $(\tilde{x}, \tilde{y})$.

The constraint $C^{u,v}$ is never tight for $(\tilde{x}, \tilde{y})$.

A constraint C^{x_a} $(a \in A^u)$ is not tight for $(\tilde{x}, \tilde{y})$ only if we have $a = a^u$ and $d^{N^{u*}} > 0$.

A constraint C^{D^u, N^u} $(D^u \subseteq A^u, N^u \subseteq M^u)$ is violated by $(\tilde{x}, \tilde{y})$ if and only if, either we have $a^u \notin D^u$ and $N^{u*} \not\subseteq N^u$, or if we have $a^u \in D^u$ and $N^{u*} \not\supseteq N^u$. Indeed, writing the validity of the constraint C^{D^u, N^u} for the point $(\tilde{x}, \tilde{y})$, we obtain

$$\begin{array}{rcl} d^{N^{u*}} + d^{N^u \setminus N^{u*}} & \geq & d^{N^u} \\ d^{N^u} + d^{N^u \setminus N^{u*}} & \geq & d^{N^u} \end{array}$$

if $a^u \notin D^u$,$a^u \in D^u$ respectively. The latter inequalities are not tight if and only if we have $N^{u*} \not\subseteq N^u$, $N^{u*} \not\supseteq N^u$ respectively.

Simlarly a constraint C^{D^u, N^v, D^v, N^u} is not tight for the point $(\tilde{x}, \tilde{y})$ if and only if we have, either $a^v \notin D^v$ and $N^{u*} \not\supseteq N^u$, or $a^v \in D^v$ and $N^{u*} \not\subseteq N^u$, or $a^u \in D^u, d^M > d^{N^{u*}} + d^{N^v}$.

The characterization of the inequalities not tight for $(\tilde{x}, \tilde{y})$ in the other classes of constraints of $\mathcal{C}$ is left to the reader.

□

Lemma 21 *For any point (x, y) in P', if we have*

$$\forall a^u \in A^u, \forall a^v \in A^v : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^{X^{a^u, a^v, N^{u*}}}$$

then, either the constraint $C^{u,v}$ is tight for (x, y), or at least one of the following conditions holds

$$\forall a^u \in A^u : P_1^u(a^u) \ or \ P_2^u(a^u)$$
$$\forall a^v \in A^v : Q_1^u(a^v) \ or \ Q_2^u(a^v).$$

<u>Proof of lemma 21</u>

By lemma 20, we have

$$\forall a^u \in A^u, \forall a^v \in A^v : P_1^u(a^u) \text{ or } P_2^u(a^u) \text{ or } Q_1^u(a^v) \text{ or } Q_2^u(a^v)$$

which is equivalent to

$$(\forall a^u \in A^u : P_1^u(a^u) \text{ or } P_2^u(a^u)) \text{ or } (\forall a^v \in A^v : Q_1^u(a^v) \text{ or } Q_2^u(a^v).$$

□

We now show how the conclusions of lemmas 20 and 21 can be rewritten.

Lemma 22 *If we have*

$$\forall a^u \in A^u : P_1^u(a^u) \ or \ P_2^u(a^u)$$

then there is a set of arcs D_2^u in A^u such that

$$\sum_{a \in D_2^u} y_a + y^{u,0} \geq 1 \tag{67}$$

and such that at least one of the following conditions holds

$$\forall a \in D_2^u : C^a \in \mathcal{C}^{(x,y)} \tag{68}$$

$$\exists N_2^v, D_2^v, N_2^u \text{ s.t. } C^{D_2^u, N_2^v, D_2^v, N_2^u} \in \mathcal{C}^{(x,y)}, d^M > d^{N^{u*}} + d^{N_2^v} \tag{69}$$

Proof of lemma 22

Pose

$$D^0 = \{a \in A^u : C^a \in \mathcal{C}^{(x,y)}\}$$

and consider the collections of subsets of A^u

$$\begin{array}{rcl} \mathcal{D}_1^0 & = & \{\{a \in A^u : C^{x_a} \notin \mathcal{C}^{(x,y)} \text{ or } C^a \in \mathcal{C}^{(x,y)}\}\} \text{ if } d^{N^{u*}} > 0, \emptyset \text{ otherwise} \\ \mathcal{D}_1^1 & = & \{D_1^u : D^0 \subseteq D_1^u, \exists N_1^u \text{ s.t. } C^{D_1^u, N_1^u} \in \mathcal{C}^{(x,y)}, N^{u*} \not\subseteq N_1^u\} \\ \mathcal{D}_1 & = & \mathcal{D}_1^0 \cup \mathcal{D}_1^1. \end{array}$$

It is not difficult to see, by the same arguments as the ones used in the proof of lemma 16, using lemmas 12, that the collections $\mathcal{D}_1$, $\mathcal{D}_1^0$,$\mathcal{D}_1^1$, if not empty, contain a minimum element. Let D_1^u be the minimum element of $\mathcal{D}_1$ if $\mathcal{D}_1$ is non empty, and let D_1^u be A^u otherwise.

Pose

$$\begin{array}{rcl} \mathcal{D}_2^0 & = & \{D^0\} \\ \mathcal{D}_2^1 & = & \{D_2^u : \exists N_2^v, D_2^v, N_2^u \text{ s.t. } C^{D_2^u, N_2^v, D_2^v, N_2^u} \in \mathcal{C}^{(x,y)}, d^M > d^{N^{u*}} + d^{N_2^v}\} \\ \mathcal{D}_2^2 & = & \{D_2^u : \exists N_2^u \text{ s.t. } C^{D_2^u, N_2^u} \in \mathcal{C}^{(x,y)}, N^{u*} \not\supseteq N_2^u\} \\ \mathcal{D}_2 & = & \mathcal{D}_2^0 \cup \mathcal{D}_2^1 \cup \mathcal{D}_2^2. \end{array}$$

As above, we can obtain that the collections $\mathcal{D}_2$, $\mathcal{D}_2^0$, $\mathcal{D}_2^1$, $\mathcal{D}_2^2$ contain a maximum element. Let D_2^u be the maximum element of $\mathcal{D}_2$.

For any arc a^u in A^u, the fact that $P_2^u(a^u)$ holds means that a^u belongs to some element of $\mathcal{D}_2$, and thus to the maximum element D_2^u. Accordingly to wether the maximum element belongs to $\mathcal{D}_2^0$, $\mathcal{D}_2^1$, $\mathcal{D}_2^2$, we have thus that there is a subset D_2^u of A^u satisfying one of the following conditions

$$\forall a \in D_2^u : C^a \in \mathcal{C}^{(x,y)} \tag{70}$$

$$\exists N_2^v, D_2^v, N_2^u \text{ s.t. } C^{D_2^u, N_2^v, D_2^v, N_2^u} \in \mathcal{C}^{(x,y)}, d^M > d^{N^{u*}} + d^{N_2^v} \tag{71}$$

$$\exists N_2^u \text{ s.t. } C^{D_2^u, N_2^u} \in \mathcal{C}^{(x,y)}, N^{u*} \not\supseteq N_2^u \tag{72}$$

respectively, and such that D_2^u contains every arc a in A^u such that $P_2^u(a)$ holds.

For every arc a in A^u not in D_2^u, we have that $P_2^u(a)$ does not hold, and in particular, we have $a \notin D^0$. Further by the hypothesis, we have that $P_1^u(a)$ holds, which implies that a does not belong to some element of $\mathcal{D}_1$, and thus that is does not belong to the minimum element D_1^u either. So according to whether the minimum element belongs to the collection $\mathcal{D}_1^0$, $\mathcal{D}_1^1$ or whether $\mathcal{D}$ is empty, we have that there is a subset D_1^u of A^u satisfying one of the following conditions

$$\forall a \notin D_1^u : C^{x_a} \in \mathcal{C}^{(x,y)} \text{ and } d^{N^{u*}} > 0 \tag{73}$$

$$\exists N_1^u \text{ s.t. } C^{D_1^u, N_1^u} \in \mathcal{C}^{(x,y)}, N^{u*} \not\subseteq N_1^u \tag{74}$$

$$D_1^u = A^u \tag{75}$$

respectively such that D_1^u contains none of the arcs a of A^u for which $P_2^u(a)$ does not hold.

So we have proved the existence of a subset D_1^u satisfying (73), (74) or (75), the existence of a subset D_2^u satisfying (70), (71) or (72) such that for every arc a in A^u, we have either $a \in D_2^u$ or $a \notin D_1^u$, that is such that $D_1^u \subseteq D_2^u$.

This result can be simplified as follows.

First assume that (73) holds. Pose $N_1^u = \emptyset$ and observe that $C^{D_1^u, N_1^u}$ is a tight constraint with $N^{u*} \not\subseteq N_1^u$. So D_1^u satisfies (74) as well. We thus do not need to consider (73).

Suppose now that (74) holds. So there exists a subset N_1^u sucht that $(c_1, f_1, K_1) = C^{D_1^u, N_1^u}$ is a tight constraint with $N^{u*} \not\subseteq N_1^u$. On the other hand, the constraint $(c_2, f_2, K_2) = C^{D_1^u, N_1^u \cup N^{u*}}$ is satisfied by (x, y). It follows that the constraint $(c_2 - c_1, f_2 - f_1, K_2 - K_1)$ is satisfied by (x, y), which implies

$$\sum_{a \in D_1^u} y_a d^{N^{u*} \setminus N_1^u} + \sum_{m \in N^{u*} \setminus N_1^u} (y^m + y^{u,0}) d^m \geq d^{N^{u*} \setminus N_1^u}.$$

By $d^{N^{u*} \setminus N_1^u} > 0$ and as the variables $y^m (m \in N^{u*} \setminus N_1^u)$ are zero, this gives

$$\sum_{a \in D_1^u} y_a + y^{u,0} \geq 1. \tag{76}$$

Finally, observe that in the case that D_1^u satisfies (75), the equation (76) is trivially satisfied, since it is nothing else that the valid constraint C^u.

So (76) is satisfied in any case. Since we have $D_1^u \subseteq D_2^u$, we deduce

$$\sum_{a \in D_2^u} y_a + y^{u,0} \geq 1. \tag{77}$$

Suppose now that D_2^u satisfies (72). So there is a subset N_2^u such that $(c_3, f_3, K_3) = C^{D_2^u, N_2^u}$ is a tight constraint with $N^{u*} \not\supseteq N_2^u$. On the other hand, the valid constraint $(c_4, f_4, K_4) = C^{D_2^u, N_2^u \cap N^{u*}}$ is satisfied by (x, y). But the the fact that the constraint $(c_4 - c_3, f_4 - f_3, K_4 - K_3)$ is satisfied by (x, y) implies

$$\sum_{a \in D_2^u} y_a d^{N_2^u \setminus N^{u*}} + \sum_{m \in N_2^u \setminus N^{u*}} (y^m + y^{u,0}) d^m \leq d^{N_2^u \setminus N^{u*}}.$$

Since $N_2^u \setminus N^{u*}$ is non empty and since the variables y^m $(m \in N_2^u \setminus N^{u*})$ are strictly positive, the latter inequality implies

$$\sum_{a \in D_2^u} y_a + y^{u,0} < 1$$

a contradiction with (77).

So we have that D_2^u cannot satisfy (72). It follows that D_2^u satisfies either (70) or (71). But that is the thesis of the lemma.

□

Lemma 23 *If we have*

$$\forall a^v \in A^v : Q_1^u(a^v) \text{ or } Q_2^u(a^v)$$

then there is a set of arcs D_1^v in A^v such that

$$\sum_{a \in D_1^v} y_a \leq y^{u,0} \tag{78}$$

and such that at least one of the following conditions holds

$$\exists N_1^v \ s.t.\ C^{D_1^v, N_1^u} \in \mathcal{C}^{(x,y)}, d^M > d^{N^{u*}} + d^{N_1^v} \tag{79}$$

$$A^v = D_1^v \tag{80}$$

Proof of lemma 23

The proof of lemma 23, although slightly different from the proof of lemma 22, can nevertheless be obtained by entirely similar arguments and is therefore left to the reader.

□

The following lemma, summing up the results of the previous lemmas, characterize the faces of P' containing none of the rays in R^0 and none of the characteristic vectors of the solutions $X^{a^u, a^v, N^{u*}}$ ($a^u \in A^u, a^v \in A^v$).

Let P^u be the condition satisfied if and only if there exist subsets D_2^u, N_2^v, D_2^v, N_2^u such that the three following conditions are fulfilled:

$$C^{D_2^u, N_2^v, D_2^v, N_2^u} \in \mathcal{C}^{(x,y)} \tag{81}$$

$$d^M > d^{N^{u*}} + d^{N_2^v} \tag{82}$$

$$\sum_{a \in D_2^u} y_a + y^{u,0} \geq 1. \tag{83}$$

Let Q^u be the condition satisfied if and only if there exists subsets D_1^v, N_1^v such that the three following conditions are fulfilled:

$$C^{D_1^v, N_1^v} \in \mathcal{C}^{(x,y)} \tag{84}$$

$$d^M > d^{N^{u*}} + d^{N_1^v} \tag{85}$$

$$\sum_{a \in D_1^v} y_a \leq y^{u,0}. \tag{86}$$

Lemma 24 *For any point* (x,y) *in* P'*, if we have*

$$C^u, C^v \in \mathcal{C}^{(x,y)}$$

and

$$\forall a^u \in A^u, \forall a^v \in A^v : \exists C \in \mathcal{C}^{(x,y)} \ s.t. \ C \notin \mathcal{C}^{X^{a^u,a^v,N^{u*}}}$$

then we have that, either the constraint $C^{u,v}$ *is tight, or at least one of the conditions* P^u*,* Q^u *holds.*

Proof of lemma 24

By hypothesis, constraints C^u and C^v are tight for (x,y), while constraint $C^{u,v}$ is not. So we have

$$\sum_{a \in A^u} y_a + y^{u,0} \leq 1 \tag{87}$$

$$\sum_{a \in A^v} y_a + y^{v,0} \leq 1 \tag{88}$$

$$\sum_{a \in A^u} y_a + \sum_{a \in A^v} y_a > 1. \tag{89}$$

By lemmas 20, 21, 22, 23, we have either that the conclusion of lemma 22 holds, or that the conclusion of lemma 23 holds.

Suppose first that the conclusion of lemma 22 holds. So there is a set $D_2^u \subseteq A^u$ satisfying (67) and, either (68) or (69). If D_2^u satisfies (68), using (67), we obtain

$$y^{u,0} \geq 1. \tag{90}$$

Summing (87), (88), (90), we obtain a contradiction with (89).

So we have that D_2^u satisfies (67) and (69). But that is exactly the alternative P^u of the thesis.

Suppose now that the conclusion of lemma 23 holds. So there is a set $D_1^v \subseteq A^v$ satisfying (78) and, either (79) or (80). If D_1^v satisfies (80), then using (78), we obtain

$$\sum_{a \in A^v} y_a \leq y^{u,0}. \tag{91}$$

Summing (87), (91), we obtain a contradiction with (89).

So we have that D_1^v satisfies (78) and (79). But that is exactly the alternative Q^u of the thesis.

This closes the proof.

□

Symmetrically, we can obtain the following lemma, which characterize the faces of P' containing none of the rays in R^0 and none of the characteristic vectors of the solutions $X^{a^u,a^v,N^{v*}}$ $(a^u \in A^u, a^v \in A^v)$.

Let P^v be the condition satisfied if and only if there exists subsets D_2^u, N_2^v, D_2^v, N_2^u such that the three following conditions are fulfilled:

$$C^{D_2^u,N_2^v,D_2^v,N_2^u} \in \mathcal{C}^{(x,y)} \tag{92}$$

$$d^M > d^{N^{v*}} + d^{N_2^u} \tag{93}$$

$$\sum_{a \in D_2^v} y_a + y^{v,0} \geq 1. \tag{94}$$

Let Q^v be the condition satisfied if and only if there exists subsets D_1^u, N_1^u such that the three following conditions are fulfilled:

$$C^{D_1^u,N_1^u} \in \mathcal{C}^{(x,y)} \tag{95}$$

$$d^M > d^{N^{v*}} + d^{N_1^u} \tag{96}$$

$$\sum_{a \in D_1^u} y_a \leq y^{v,0}. \tag{97}$$

Lemma 25 *For any point (x, y) in P', if we have*

$$C^u, C^v \in \mathcal{C}^{(x,y)}$$

and

$$\forall a^u \in A^u, \forall a^v \in A^v : \exists C \in \mathcal{C}^{(x,y)} \text{ s.t. } C \notin \mathcal{C}^{X^{a^u,a^v,N^{v*}}}$$

then we have that, either the constraint $C^{u,v}$ is tight, or at least one of the conditions P^v, Q^v holds.

We are now ready for the main result of this part, that is lemma 7, that states that if a face O of P' defined by a point (x, y) contains none of the rays r^a $(a \in A^u \cup A^v \cup \{(u, v), (v, u)\})$ and none of the solutions of $\mathcal{X}^a$, then the constraint $C^{u,v}$ is tight for (x, y).

Proof of lemma 7

Suppose that the lemma is false, that is suppose $C^{u,v} \notin \mathcal{C}^{(x,y)}$. We have

$$\sum_{a \in A^u} y_a + \sum_{a \in A^v} y_a > 1. \tag{98}$$

By hypothesis, the constraints C^u and C^v are tight for (x, y). So we have

$$\sum_{a \in A^u} y_a + y^{u,0} \leq 1. \tag{99}$$

$$\sum_{a \in A^v} y_a + y^{v,0} \leq 1. \tag{100}$$

By lemma 24, we have that P^u or Q^u holds. By lemma 25, we have that P^v or Q^v holds. We therefore distinguish the following cases: case 1) P^u and P^v holds, case 2) Q^u and Q^v holds, case 3) P^u and Q^v holds, case 4) Q^u and P^v holds.

Case 1: P^u and P^v hold

By P^u, there are subsets D_2^u, N_2^v, D_2^v, N_2^u satisfying (81), (82), (83). By P^v there are subsets $\tilde{D}_2^u$, $\tilde{N}_2^v$, $\tilde{D}_2^v$, $\tilde{N}_2^u$ satisfying (92), (93), (94) By lemma 11, we can assume $D_2^u = \tilde{D}_2^u$, $N_2^v = \tilde{N}_2^v$, $D_2^v = \tilde{D}_2^v$, $N_2^u = \tilde{N}_2^u$.

So there are subsets D_2^u, N_2^v, D_2^v, N_2^u satisfying (83) and (94) such that the constraint $(c_1, f_1, K_1) = C^{D_2^u, N_2^v, D_2^v, N_2^u}$ is tight for (x, y). On the other hand the constraints $(c_2, f_2, K_2) = C^{D_2^u, N_2^u}$ and $(c_3, f_3, K_3) = C^{D_2^v, N_2^v}$ are satisfied for (x, y). It follows that the constraint $(c_3 + c_2 - c_1, f_3 + f_2 - f_1, K_3 + K_2 - K_1)$ is satisfied by (x, y) as well, which gives

$$\sum_{a \in D_2^u} y_a(d^M - d^{N_2^u} - d^{N_2^v}) + \sum_{a \in D_2^v} y_a(d^M - d^{N_2^u} - d^{N_2^v}) \leq (d^M - d^{N_2^u} - d^{N_2^v}).$$

Suppose first that $d^M > d^{N_2^u} + d^{N_2^v}$. We have then

$$\sum_{a \in D_2^u} y_a + \sum_{a \in D_2^v} y_a \leq 1. \tag{101}$$

Then summing (83), (94), (99), (100), (101) we obtain a contradiction with (98).

Suppose now that we have $d^M = d^{N_2^u} + d^{N_2^v}$, which can only occur in the case $N_2^u = M^u, N_2^v = M^v$. Then equality holds for every inequality afore mentioned. In particular, the constraint $(c_2, f_2, K_2) = C^{D_2^u, M^u}$ is tight. On the other hand, the valid constraint $(c_4, f_4, K_4) = C^{D_2^u, N^{u*}}$ is satisfied by (x, y). It follows that the constraint $(c_4 - c_2, f_4 - f_2, K_4 - K_2)$ is satisfied by (x, y) as well, which gives

$$\sum_{a \in D_2^u} y_a d^{M^u \setminus N^{u*}} + \sum_{m \in M^u \setminus N^{u*}} (y^m + y^{u,0}) d^m \leq d^{M^u \setminus N^{u*}}.$$

Finally, by (82), using $N_2^v = M^v$, we have that $M^u \setminus N^{u*}$ is non empty and by construction of N^{u*}, we have that the variables $y^m (m \in M^u \setminus N^{u*})$ are positive. This implies

$$\sum_{a \in D_2^u} y_a + y^{u,0} < 1$$

a contradiction with (83).

Case 2: Q^u and Q^v hold

By Q^u, there are subsets D_1^v, N_1^v satisfying (84), (85), (86). By Q^v there are subsets D_1^u, N_1^u satisfying (95), (96), (97). So the constraints $(c_1, f_1, K_1) = C^{D_1^u, N_1^u}$ and $(c_2, f_2, K_2) = C^{D_1^v, N_1^v}$ are tight for (x, y). On the other hand, the valid constraint $(c_3, f_3, K_3) = C^{D_1^u, N_1^v, D_1^v, N_1^u}$ is satisfied by (x, y). Then the constraint $(c_3 - c_2 - c_1, f_3 - f_2 - f_1, K_3 - K_2 - K_1)$ is satisfied by (x, y) as well, which gives

$$\sum_{a \in D_1^u} y_a(d^M - d^{N_1^u} - d^{N_1^v}) + \sum_{a \in D_1^v} y_a(d^M - d^{N_1^u} - d^{N_1^v}) \geq (d^M - d^{N_1^u} - d^{N_1^v}).$$

Suppose first that $d^M > d^{N_1^u} + d^{N_1^v}$. We have

$$\sum_{a \in D_1^u} y_a + \sum_{a \in D_1^v} y_a \geq 1. \tag{102}$$

Then summing (86), (97), (99), (100), (102) we obtain a contradiction with (98).

Suppose now that we have $d^M = d^{N_1^u} + d^{N_1^v}$, which can only occur in the case $N_1^u = M^u, N_1^v = M^v$. Then equality holds for every inequality afore mentioned. In particular, the constraint $(c_3, f_3, K_3) = C^{D_1^u, N_1^v, D_1^v, N_1^u}$ is tight. On the other hand, the valid constraint $(c_4, f_4, K_4) = C^{D_1^u, N^{v*}, D_1^v, N_1^u}$ is satisfied by (x, y). It follows that the constraint $(c_4 - c_3, f_4 - f_3, K_4 - K_3)$ is satisfied by (x, y) as well, which gives, using $M^v \setminus N^{v*} \neq \emptyset$ implied by (96)

$$\sum_{a \in D^u} y_a > y^{v,0}$$

a contradiction with (97).

Case 3: P^u and Q^v hold

By P^u, there are subsets D_2^u, N_2^v, D_2^v, N_2^u satisfying (81), (82), (83). By Q^v there are subsets D_1^u, N_1^u satisfying (95), (96), (97).

So the constraints $(c_1, f_1, K_1) = C^{D_2^u, N_2^v, D_2^v, N_2^u}$ and $(c_2, f_2, K_2) = C^{D_1^u, N_1^u}$ are tight for (x, y). On the other hand, the valid constraints $(c_3, f_3, K_3) = C^{D_2^u, N_1^u}$ and $(c_4, f_4, K_4) = C^{D_1^u, N_2^v, D_2^v, N_2^u}$ are satisfied by (x, y). It follows that the constraint $(c_4 + c_3 - c_2 - c_1, f_4 + f_3 - f_2 - f_1, K_4 + K_3 - K_2 - K_1)$ is satisfied by (x, y) as well, which gives

$$\sum_{a \in D_2^u} y_a (d^M - d^{N_1^u} - d^{N_2^v}) \leq \sum_{a \in D_1^u} y_a (d^M - d^{N_1^u} - d^{N_2^v}).$$

Suppose first that $d^M > d^{N_1^u} + d^{N_2^v}$. We have

$$\sum_{a \in D_2^u} y_a \leq \sum_{a \in D_1^u} y_a. \tag{103}$$

Then summing (83), (97), (99), (100), (103) we obtain a contradiction with (98).

Suppose now that we have $d^M = d^{N_1^u} + d^{N_2^v}$, which can only occur in the case $N_1^u = M^u, N_2^v = M^v$. But then, as in case 1, we have a tight constraint $C^{D_2^u, M^u}$, a contradiction with (83).

Case 4: Q^u and P^v hold

The proof of case 4 is obtained similarly as the proof of case 3, by permuting the roles played by u and v.

$\square$

3.6 Proof of Polyhedral Characterization

Finally, the proof that the multicommodity linear formulation is a polyhedral characterization for every instance in the class $\mathcal{G}$ considered is obtained as a consequence of the results proved previously.

Proof of theorem 1

As made explicit in section 1.2, the characteristic vector of every network belongs to P_{xy}. So we have

$$P^I \subseteq P_{xy}.$$

By the characterization of the valid constraints of P_{xy} of theorem 3, we have

$$P_{xy} \subseteq P'.$$

By the characterization of the faces of P' containing none of the rays of R^0 of lemmas 5 and 6 in step 1, by the characterization of the faces of P' containing none of the solutions in $\mathcal{X}^A$, $\mathcal{X}^{B^u}$, $\mathcal{X}^{B^v}$ of lemmas 7, 8, 9 in step 2, by the contradiction proved in lemma 10 in step 3, using criterion 3, we have

$$P' \subseteq P^I.$$

This close the proof.

□

4 Conclusions

In [Sch94], we prove that the multicommodity linear formulation is a polyhedral characterization for the SSNFD problem on Series-Parallel graphs if it is valid for a class of elementary instances $\mathcal{G}^{ele}$. The class of instances considered here $\mathcal{G}$ is a part of this class. This is the main motivation of this work.

Observe that as the instances in $\mathcal{G}$ are Series-Parallel, the validity of polyhedral characterization on $\mathcal{G}$ is necessary for our end objective. As there already seems to be no easy proof for this class of instances, there is probably no easy proof for the complete result as well.

We have no answer to the question whether it is easier to work in the space extended by the artificial variables rather than in the space of the natural variables. Observe nevertheless that, as the characterization of the valid constraints in the natural space is easy, to work in one space rather than in the other must be somehow equivalent.

In this paper, we completely describe the polyhedron associated with the SSNFD problem for the instances in $\mathcal{G}$. The question what is the set of inequalities describing the polyhedron for any Series-Parallel instances is open.

References

[EMV87] Ranel E. Erickson, Clyde L. Monma, and Arthur F. Veinott. Send and split methods for minimum concave cost flows. *Mathematics of Operations Research*, 12(4):634–664, 1987.

[RW90] Ronald L. Rardin and Laurence Wolsey. Valid inequalities and projecting the multicommodity extended formulation for uncapacitated fixed charge network flow problems. Core Discussion Paper 9024, Center for Operations Research and Econometrics, Voie du Roman Pays, 34, B-1348 Louvain-la-Neuve, Belgium, 1990.

[Sch90] Michel Schaffers. A polynomial algorithm for the single source network flow problem on series parallel graphs. Core Discussion Paper 9062, Center for Operations Research and Econometrics, Voie du Roman Pays, 34, B-1348 Louvain-la-Neuve, Belgium, 1990. Revised version 1993, submitted to Mathematical Programming.

[Sch94] Michel Schaffers. On links between graphs with bounded decomposability, existence of efficient algorithms, and existence of polyhedral characterizations. Doctoral Dissertation, Université Catholique de Louvain, 1994.

[VRW85] J. Van Roy and Laurence Wolsey. Valid inequalities and separations for uncapacitated charge networks. *Operations Research Letters*, 33(4):105–112, October 1985.

[Wol89] Laurence Wolsey. Strong formulation for mixed integer programming: a survey. *Mathematical Programming*, 45:173–191, January 1989.