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On the semidefinite representations of real functions applied to symmetric matrices

Guillaume Sagnol

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Abstract

We present a new semidefinite representation for the trace of a real function f applied to symmetric matrices, when a semidefinite representation of the convex function f is known. Our construction is intuitive, and yields a representation that is more compact than the previously known one. We also show with the help of matrix geometric means and a Riemannian metric over the set of positive definite matrices that for a rational exponent p in the interval $(0, 1]$, the matrix X raised to p is the largest element of a set represented by linear matrix inequalities. We give numerical results for a problem inspired from the theory of experimental designs, which show that the new semidefinite programming formulation can yield a speed-up factor in the order of 10.

Keywords semidefinite representability, optimal experimental designs, SDP, matrix geometric mean

1 Introduction

In this article we discuss semidefinite representations of scalar functions applied to symmetric matrices. We recall that it is possible to extend the definition of a function $f : I \mapsto \mathbb{R}, x \to f(x)$, where I is a real interval, to the set $\mathbb{S}_m^I$ of $m \times m-$symmetric matrices whose spectrum lies in I as follows: if $X = U \operatorname{Diag}(\lambda_1, \ldots, \lambda_m)U^T$ is an eigenvalue decomposition of X, then we define $f(X) := U \operatorname{Diag}\big(f(\lambda_1), \ldots, f(\lambda_m)\big)U^T$. Throughout this article we denote by $\mathbb{S}_m$ (resp. $\mathbb{S}_m^+, \mathbb{S}_m^{++}$) the set of $m \times m$ symmetric (resp. positive semidefinite, positive definite) matrices.

If the scalar function f is semidefinite representable, then a result of Ben-Tal and Nemirovski can be used to construct a semidefinite representation of $X \to \operatorname{trace} f(X)$. Indeed, $\operatorname{trace} f(X)$ can be rewritten as $\sum_i f(\lambda_i)$, which is a symmetric and semidefinite representable function of the eigenvalues of X, so that Proposition 4.2.1. in [BTN87] applies.

In this article, we show that the semidefinite representation of $x \to f(x)$ can be lifted to the matrix case $X \to \operatorname{trace} f(X)$ by an intuitive transformation which involves Kronecker products (Theorem 3.1). The resulting semidefinite representation of the epigraph

$$E = \{(t, X) \in \mathbb{R} \times \mathbb{S}_m : \operatorname{trace} f(X) \leq t\}$$

is more compact than the one obtained from the general construction of Ben-Tal and Nemirovski, in which the Ky-Fan k-norms of M must be bounded for $k = 1, \ldots, m$. Our numerical results of Section 5 moreover show that the semidefinite programs (SDP) based on the present representation are solved in a shorter time than the former SDP formulations, and that they are numerically more stable.

For the case where $f(x) = x^p : \mathbb{R}^+ \mapsto \mathbb{R}^+$, where p is a rational number in $(0, 1]$, we shall see that our construction yields a stronger result. Namely, we show in Theorem 4.2 that X^p has an extremal representation of the form

$$X^p = \max_{\preceq}\{T \in \mathbb{S}_m : \ T \in S\},$$

where the set S is semidefinite representable and $\max_{\preceq}$ denotes the largest element with respect to the Löwner ordering, which is defined over $\mathbb{S}_m$ as follows:

$$A \preceq B \Longleftrightarrow (B - A) \in \mathbb{S}_m^+.$$

The proof of this result uses the notion of *matrix geometric mean*, and the Banach fixed point theorem in the space $\mathbb{S}_m^{++}$ equipped with a Riemannian metric.

Our study is motivated by the *theory of optimal experimental designs*, where the general problem to solve takes the form

$$\begin{aligned} \max_{\boldsymbol{w} \in \mathbb{R}^s} \quad & \Phi_p \left(\sum_{i=1}^s w_i M_i \right), \\ \text{s.t.} \quad & \sum_{i=1}^s w_i = 1, \quad \boldsymbol{w} \geq \boldsymbol{0}, \end{aligned} \tag{1}$$

where $M_1, \ldots, M_s$ are given positive semidefinite matrices, and for $p \in [-\infty, 1]$ the Φ_p-criterion is defined over the set of positive definite matrices $M \in \mathbb{S}_m^{++}$ as

$$\Phi_p(M) = \begin{cases} \lambda_{min}(M) & \text{for } p = -\infty \ ; \\ (\frac{1}{m} \text{ trace } M^p)^{\frac{1}{p}} & \text{for } p \in \ (-\infty, 1], \ p \neq 0 \ ; \\ (\det(M))^{\frac{1}{m}} & \text{for } p = 0. \end{cases} \tag{2}$$

The definition of Φ_p is extended by continuity to singular matrices $M \in \mathbb{S}_m^+$, so that $\Phi_p(M) = 0$ if M is singular and $p \leq 0$. We refer the reader to Pukelsheim [Puk93] for more background on optimal experimental designs.

Note that any semidefinite representation of the function $M \to \operatorname{trace} M^p$ yields a semidefinite programming (SDP) formulation of Problem (1). The cases $p = -\infty$, $p = -1$, and $p = 0$, known as $E-$, $A-$ and $D-$optimal design problems have been extensively studied in the literature, and SDP formulations are known for these problems [BV04]. We also point out that lighter Second Order Cone Programming (SOCP) formulations exist for $p = -1$ and $p = 0$ [Sag11]. The general case ($p \in [-\infty, 1]$) deserved less attention. However, it was recently noticed by Papp [Pap12] that the a SDP formulation can be obtained by using Proposition 4.2.1. in [BTN87]. Our numerical results (cf. Section 5) show that the new SDP formulation from this paper can improve the computation time by a factor in the order of 10.

2 Preliminaries

In this section, we briefly recall some basic notion about semidefinite representability and matrix geometric means. We first recall the definition of a *Semidefinite Program* (SDP). The latter is an optimization problem in which a linear function $\boldsymbol{c}^T\boldsymbol{x}$ must be maximized, among the vectors $\boldsymbol{x}$ belonging to a set S defined by *linear matrix inequalities* (LMI):

$$S = \{\boldsymbol{x} \in \mathbb{R}^n : \; F_0 + \sum_i x_i F_i \succeq 0\}.$$

We now recall the definition of a semidefinite representable set, which was introduced by Ben-Tal and Nemirovski [BTN87]:

Definition 2.1 (Semidefinite representability)**.** A convex set $S \subset \mathbb{R}^n$ is said to be *semidefinite representable*, abbreviated *SDr*, if S is the projection of a set in a higher dimensional space which can be described by LMIs. In other words, S is *SDr* if and only if there exists symmetric matrices $F_0, \ldots, F_n, F'_1, \ldots, F'_{n'}$ such that

$$\boldsymbol{x} \in S \Longleftrightarrow \exists \boldsymbol{y} \in \mathbb{R}^{n'} : \quad F_0 + \sum_{i=1}^{n} x_i F_i + \sum_{i=1}^{n'} y_i F'_i \succeq 0.$$

Such an LMI is called a *semidefinite representation* (SDR) of the set S.

Definition 2.2 (SDR of a function)**.** A convex (resp. concave) function f is said *SDr* if and only if the epigraph of f, $\{(t,x) : f(x) \leq t\}$ (resp. the hypograph $\{(t,x) : t \leq f(x)\}$), is *SDr*.

It follows immediately from these two definitions that the problem of maximizing a concave *SDr* function (or minimizing a convex one) over a *SDr* set can be cast as an SDP.

We now give a short insight on the theory of matrix geometric means and the Riemannian metric of the set of positive definite matrices $\mathbb{S}_m^{++}$. We refer the reader to the book of Bhatia [Bha08] and the references therein for more details on this subject. The Geometric mean of two positive definite matrices $A, B \in \mathbb{S}_m^{++}$ was introduced by Ando [And79]:

$$A \,\sharp\, B := A^{1/2}\big(A^{-1/2} B A^{-1/2}\big)^{1/2} A^{1/2}.$$

In the latter paper, Ando shows that $A \,\sharp\, B$ satisfies the following extremal property:

$$A \,\sharp\, B = \max\nolimits_{\preceq} \left\{ X \in \mathbb{S}_m : \begin{pmatrix} A & X \\ X & B \end{pmatrix} \succeq 0 \right\} \tag{3}$$

The space of positive definite matrices is equipped with the Riemannian metric

$$\delta_2(A, B) = \| \log A^{-1/2} B A^{-1/2} \|_F,$$

where $\|M\|_F = \operatorname{trace}(M^T M)$ denotes the Frobenius norm of M. In this space, there exists a unique geodesic $[A, B]$ between two matrices A and B, which can be parametrized as follows:

$$\gamma(t) = A^{1/2}\big(A^{-1/2} B A^{-1/2}\big)^{t} A^{1/2}, \quad 0 \leq t \leq 1.$$

Note that $A \sharp B$ is the midpoint of this geodesic. The geometric mean of two matrices is commutative, i.e. $A \sharp B = B \sharp A$, and the map $X \to A \sharp X$ is *operator monotone*, i.e. $Y \succeq X \implies A \sharp Y \succeq A \sharp X$.

We also point out that the metric δ_2 enjoys an important convexity property, which will be useful in the proof of Theorem 4.2:

$$\forall A, B, C, D \in \mathbb{S}_m^{++}, \;\; \delta_2(A \sharp B, C \sharp D) \leq \frac{1}{2}\delta_2(A, C) + \frac{1}{2}\delta_2(B, D). \tag{4}$$

3 Lifting the SDR of a scalar function

In this section, we show that the *SDR* of a function $f : I \mapsto \mathbb{R}$ can be transformed in a simple way to a *SDR* of $\operatorname{trace} f : \mathbb{S}_m^I \to \mathbb{R}$:

Theorem 3.1. *Let $f : I \mapsto \mathbb{R}$ be a scalar function, where I is a real interval. Assume that f admits the following SDR: for all $x \in I$,*

$$f(x) \leq t \iff \exists \boldsymbol{y} \in \mathbb{R}^n : \;\; F_0 + xF_X + tF_T + \sum_{i=1}^{n} y_i F_i \succeq 0,$$

where the symmetric matrices $F_0, \ldots, F_n, F_X, F_T$ are given. Then, a SDR of the function $g : \mathbb{S}_m^I \mapsto \mathbb{R}, X \to \operatorname{trace} f(X)$ is given by: for all $X \in \mathbb{S}_m^I$,

$$\begin{aligned} \operatorname{trace} f(X) \leq t \iff \exists T, Y_1, \ldots, Y_n \in \mathbb{S}_m : \\ (i) \;\; & F_0 \otimes I_m + F_X \otimes X + F_T \otimes T + \sum_{i=1}^{n} F_i \otimes Y_i \succeq 0; \\ (ii) \;\; & \operatorname{trace} T \leq t, \end{aligned}$$

where I_m denotes the $m \times m$ identity matrix and $\otimes$ is the Kronecker product. In other words, the SDR is lifted from scalar to matrices by replacing each scalar by a corresponding matrix block of size $m \times m$.

Proof. Let X be an arbitrary matrix in $\mathbb{S}_m^I$, and $X = U \operatorname{Diag}(\boldsymbol{\lambda}) U^T$ be an eigenvalue decomposition of X. For $k = 1, \ldots, m$, define $t_k = f(\lambda_k)$. By assumption there exists a vector $\boldsymbol{y^{(k)}}$ such that

$$B_k := \;\; F_0 + \lambda_k F_X + t_k F_T + \sum_{i=1}^{n} y_i^{(k)} F_i \succeq 0.$$

Denote by $\mathcal{B}$ the block diagonal matrix with blocks $B_1, \ldots, B_m$ on the diagonal, and by $\boldsymbol{y_i}$ the vector of $\mathbb{R}^m$ with components $y_i^{(1)}, \ldots, y_i^{(m)}$. We may write

$$\mathcal{B} = I_m \otimes F_0 + \operatorname{Diag}(\boldsymbol{\lambda}) \otimes F_X + \operatorname{Diag}(\boldsymbol{t}) \otimes F_T + \sum_{i=1}^{n} \operatorname{Diag}(\boldsymbol{y_i}) \otimes F_i \succeq 0.$$

In the previous expression, we may commute the Kronecker products, which is equivalent to pre- and post-multiplying by a permutation matrix:

$$F_0 \otimes I_m + F_X \otimes \operatorname{Diag}(\boldsymbol{\lambda}) + F_T \otimes \operatorname{Diag}(\boldsymbol{t}) + \sum_{i=1}^{n} F_i \otimes \operatorname{Diag}(\boldsymbol{y_i}) \succeq 0.$$

Now, we multiply this expression to the left by the block diagonal matrix $\mathrm{Diag}(U,\dots,U) = I \otimes U$, and to the right by its transpose. This gives:

$$F_0 \otimes I_m + F_X \otimes X + F_T \otimes T + \sum_{i=1}^n F_i \otimes Y_i \succeq 0,$$

where we have set $T = U\,\mathrm{Diag}(\boldsymbol{t})U^T$ and $Y_i = U\,\mathrm{Diag}(\boldsymbol{y_i})U^T$. By construction, we have $T = f(X)$, and thus we have proved the "$\Rightarrow$" part of the theorem.

For the converse part, consider some matrices $T', Y_1', \dots Y_n' \in \mathbb{S}_m$ such that the LMI (i) of the theorem is satisfied. Define $H_T = T' - T$ and $H_i = Y_i' - Y_i$, where $T = f(X)$ and $Y_i = U\,\mathrm{Diag}(\boldsymbol{y_i})U^T$ are defined as in the first part of this proof. We will show that $\operatorname{trace} H_T \geq 0$, which implies $\operatorname{trace} T' \geq \operatorname{trace} f(X)$, and the proof will be complete.

So from (i) we have:

$$F_0 \otimes I_m + F_X \otimes X + F_T \otimes (T + H_T) + \sum_{i=1}^n F_i \otimes (Y_i + H_i) \succeq 0.$$

Again, we multiply this expression to the left by $I \otimes U^T$ and to the right by $I \otimes U$, and then we commute the Kronecker products. This gives:

$$\mathrm{Diag}(B_1,\dots,B_m) + U^T H_T U \otimes F_T + \sum_{i=1}^n U^T H_i U \otimes F_i \succeq 0.$$

For all $k = 1,\dots,m$, this implies that the kth diagonal block is positive semidefinite:

$$B_k + (U^T H_T U)_{k,k}\ F_T + \sum_{i=1}^n (U^T H_i U)_{k,k}\ F_i \succeq 0.$$

According to the SDR of the scalar function f, it means that

$$f(\lambda_k) \leq t_k + (U^T H_T U)_{k,k},$$

and since $f(\lambda_k) \leq t_k$ we obtain $(U^T H_T U)_{k,k} \geq 0$. From there, it is easy to conclude:

$$\operatorname{trace} H_T = \operatorname{trace} H_T U U^T = \operatorname{trace} U^T H_T U = \sum_{k=1}^m (U^T H_T U)_{k,k} \geq 0.$$

□

Example 3.2. A SDR of the function $x \to x^p$, where $p \in \mathbb{Q}$ is briefly sketched in [BTN87] and given with more details in [AG03] (note that this function is concave for $p \in [0,1]$ and convex for other values of p). For example, the epigraph of the convex function $x \to x^{-4/3}$ mapping $(0,\infty)$ onto itself, may be represented as follows: for all $t \geq 0$, $x > 0$:

$$\begin{aligned} x^{-4/3} \leq t &\iff 1 \leq x^4 t^3 \\ &\iff \exists u \geq 0, v \geq 0:\ 1 \leq xu,\ u^2 \leq tv,\ v^2 \leq t \\ &\iff \exists u \in \mathbb{R}, v \in \mathbb{R}: \begin{pmatrix} x & 1 \\ 1 & u \end{pmatrix} \succeq 0, \begin{pmatrix} t & u \\ u & v \end{pmatrix} \succeq 0, \begin{pmatrix} t & v \\ v & 1 \end{pmatrix} \succeq 0 \end{aligned}$$

By using Theorem 3.1, we obtain a SDR of the function $X \to \operatorname{trace} X^{-4/3}$:

$$\operatorname{trace} X^{-4/3} \le t \iff \exists U, V, T \in \mathbb{S}_m : \left\{ \begin{array}{l} \begin{pmatrix} X & I_m \\ I_m & U \end{pmatrix} \succeq 0 \\ \begin{pmatrix} T & U \\ U & V \end{pmatrix} \succeq 0, \\ \begin{pmatrix} T & V \\ V & I_m \end{pmatrix} \succeq 0, \\ \operatorname{trace} T \le t \end{array} \right.$$

□

Note however that LMI (i) of Theorem 3.1 does not imply the stronger property $f(X) \preceq T$. As a counter-example, consider the function $f(x) = x^4$, which admits the SDR

$$x^4 \le t \iff \exists u \in \mathbb{R} : \begin{pmatrix} u & x \\ x & 1 \end{pmatrix} \succeq 0, \ \begin{pmatrix} t & u \\ u & 1 \end{pmatrix} \succeq 0.$$

If we set $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $U = \begin{pmatrix} 8 & 8 \\ 8 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} 73 & 39 \\ 39 & 34 \end{pmatrix}$, the reader can check that the LMI (i) of Theorem 3.1 holds:

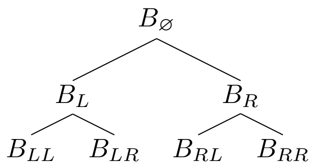
$$\begin{pmatrix} U & X \\ X & I_2 \end{pmatrix} \succeq 0, \ \begin{pmatrix} T & U \\ U & I_2 \end{pmatrix} \succeq 0,$$

but $X^4 \not\preceq T$. In the next section, we show that this stronger property holds for $f : x \to x^p$ when $p \in \mathbb{Q} \cap (0, 1]$.

4 Semidefinite representation of concave matrix powers

Throughout this section, p denotes a rational number in $(0, 1]$, and we assume that $p = \frac{\alpha}{\beta}$, with $0 < \alpha \le \beta$. We are going to show that the lifted SDR of the function f_p mapping $\mathbb{R}_+$ onto itself and defined by $f(x) = x^p$, also provides an extremal representation of X^p. In other words, there is a *SDr* set $S \in \mathbb{S}_m^+$ for which X^p is the largest element with respect to Löwner ordering.

To do this, we first present the construction of the SDR of f_p. As explained in [AG03], this SDR is based on binary trees whose nodes contain variables. Note that in a perfect binary tree, every node of depth k can be index by an element of $\Gamma_k := \{L, R\}^k$, which indicates the sequence of *left* or *right* turns needed to reach this node from the root of the tree. For example, a perfect binary tree B of depth 2 is index as follows:



We denote by $\mathcal{T}_n(m)$ the set of perfect binary trees of depth n, whose nodes are matrices of $\mathbb{S}_m$. The concatenation of tree indices is denoted by $\sqcup$, so that for example, $LR \sqcup L = LRL \in \Gamma_3$. We define n as the integer such that $2^{n-1} < \beta \leq 2^n$. Let $\sigma(X,T)$ denote a sequence of length 2^n that is a permutation of the sequence

$$\chi_{\alpha,\beta}(X,T) := (\underbrace{X,\dots,X}_{\alpha \text{ times}}, \underbrace{T,\dots,T}_{(2^n-\beta) \text{ times}}, \underbrace{I_m,\dots,I_m}_{(\beta-\alpha) \text{ times}}). \tag{5}$$

The elements of $\sigma(X,T)$ are indexed by $\gamma \in \Gamma_n$, in the order corresponding to the leaves of a tree of depth n from left to right. For example, if $\sigma(X,T) = (X, I_m, T, I_m)$, we have $\sigma(X,T)_{LL} = X$, $\sigma(X,T)_{RL} = T$, and $\sigma(X,T)_{LR} = \sigma(X,T)_{RR} = I_m$. We can now construct the SDR of f_p (already lifted to $\mathbb{S}_m^+$ by considering matrix blocks instead of scalar variables). It involves a tree whose root is T, leaves are defined by $\sigma(X,T)$, and a LMI related to the matrix geometric mean must be satisfied at each node:

$$\begin{aligned} \mathcal{S}(\sigma) = \{X,T \in \mathbb{S}_m^+ : \exists B \in \mathcal{T}_n(m) :& \\ & (i)\ B_\varnothing = T; \\ & (ii)\ \forall \gamma \in \Gamma_n,\ B_\gamma = \sigma(X,T)_\gamma; \\ & (iii)\ \forall k = 0,\dots,n-1,\ \forall \gamma \in \Gamma_k,\ \begin{pmatrix} B_{\gamma \sqcup L} & B_\gamma \\ B_\gamma & B_{\gamma \sqcup R} \end{pmatrix} \succeq 0\} \end{aligned}$$

Example 4.1. If $p = 1/3$, we have $\alpha = 1$, $\beta = 3$, $n = 2$, and $\sigma(X,T)$ must contain respectively $\alpha = 1$, $(2^n - \beta) = 1$ and $(\beta - \alpha) = 2$ copies of X, T, and I_m. If $\sigma(X,T) = (X, T, I_m, I_m)$, the set $\mathcal{S}(\sigma)$ is defined through a tree of the form

$$\begin{array}{ccccccc} & & & T & & & \\ & B_L & & & & B_R & \\ X & & T & & I_m & & I_m \end{array} \tag{6}$$

The property (iii) in the definition of $\mathcal{S}(\sigma)$ implies that B_R satisfies

$$\begin{pmatrix} I_m & B_R \\ B_R & I_m \end{pmatrix} \succeq 0.$$

So by Equation (3) we have $B_R \preceq I_m$, and the definition of $\mathcal{S}(\sigma)$ simplifies to:

$$(X,T) \in \mathcal{S}(\sigma) \iff \exists B_L \in \mathbb{S}_m : \begin{pmatrix} B_L & T \\ T & I_m \end{pmatrix} \succeq 0,\ \begin{pmatrix} X & B_L \\ B_L & T \end{pmatrix} \succeq 0.$$

□

Generally speaking, we point out that the order of the elements in the permutation σ can be chosen such that the definition of $\mathcal{S}(\sigma)$ involves no more than $2(n-1) = O(\log \beta)$ LMIs of size $2m \times 2m$.

Now, as a consequence of Equation (3), observe that property (iii) in the definition of $\mathcal{S}(\sigma)$ implies $B_\gamma \preceq B_{\gamma \sqcup L} \sharp B_{\gamma \sqcup R}$ (if the geometric mean is well defined, i.e. $B_{\gamma \sqcup L}, B_{\gamma \sqcup R} \in \mathbb{S}_m^{++}$). By operator monotonicity of the matrix geometric mean, we see that if the matrices X, T, B_L and B_R of Tree (6) are positive definite, then:

$$T \preceq B_L \sharp B_R \preceq (X \sharp T) \sharp (I_m \sharp I_m).$$

In the general case, a simple induction shows that for all positive definite matrices X, T,

$$(X,T) \in \mathcal{S}(\sigma) \Longrightarrow T \preceq \#_\sigma(X,T), \tag{7}$$

where $\#_\sigma(X,T)$ represents the expression with nested "$\sharp$-operations" in the binary tree whose leaves are defined through $\sigma(X,T)$. We can finally give the main result of this section:

Theorem 4.2 (Extremal representation of X^p)**.** *Let $p = \frac{\alpha}{\beta}$, $0 < \alpha \le \beta$, and let $\sigma(X,T)$ be a permutation of $\chi_{\alpha,\beta}(X,T)$. Then, X^p satisfies the following extremal property*

$$\forall X \in \mathbb{S}_m^+, \; X^p = \max\nolimits_{\preceq}\{T \in \mathbb{S}_m^+ : (X,T) \in \mathcal{S}(\sigma)\}.$$

Proof. Let $X \in \mathbb{S}_m^{++}$ be an arbitrary positive definite matrix. We are first going to show that $X^p = \max_{\preceq}\{T \in \mathbb{S}_m^{++} : (X,T) \in \mathcal{S}(\sigma)\}$. The general statement for all $X \in \mathbb{S}_m^+$ will be obtained at the end of this proof by continuity.

We first handle the case where $\beta = 2^n$. In this case, the matrix T does not appear in the sequence $\sigma(X,T)$, so every leaf of the tree B involved in the definition of $\mathcal{S}(\sigma)$ is either X or I_m. Define successively

$$\forall \gamma \in \Gamma_k, \; B_\gamma = B_{\gamma \sqcup L} \,\sharp\, B_{\gamma \sqcup R}$$

for $k = (n-1), (n-2), \ldots, 0$. By construction, we have $B_\varnothing = \#_\sigma(X,T)$, and a simple induction shows that $\#_\sigma(X,T) = X^{\frac{\alpha}{2^n}} = X^p$ (the geometric means are easy to compute because X and I_m commute). This shows that (X, X^p) belongs to $\mathcal{S}(\sigma)$. Conversely, if $T \in \mathbb{S}_m^{++}$, Equation (7) shows that $(X,T) \in \mathcal{S}(\sigma) \Rightarrow T \preceq X^p$.

The case $\beta < 2^n$ is more complicated. Let $T \in \mathbb{S}_m^{++}$ such that $(X,T) \in \mathcal{S}(\sigma)$, and let $B \in \mathcal{T}_n(m)$ be a tree satisfying properties $(i)-(iii)$. Define a new tree B' as follows:

$$\forall \gamma \in \Gamma_{n-1}, \; B'_\gamma := B_{\gamma \sqcup L} \,\sharp\, B_{\gamma \sqcup R} \succeq B_\gamma,$$

and

$$\forall \gamma \in \Gamma_k, \; B'_\gamma := B'_{\gamma \sqcup L} \,\sharp\, B'_{\gamma \sqcup R} \succeq B_\gamma$$

for $k = (n-2), \ldots, 0$. In particular, the root of B' is $T' := B'_\varnothing = \#_\sigma(X,T) \succeq T$. It remains to define the leaves of B', which we do according to $\sigma(X,T')$:

$$\forall \gamma \in \Gamma_n, \; B'_\gamma := \sigma(X,T')_\gamma \succeq B_\gamma.$$

By construction, it is clear that B' satisfies the property (iii) for the depth levels $k = 0, \ldots, n-2$. For a $\gamma \in \Gamma_{n-1}$, (iii) also holds, because

$$\begin{pmatrix} B'_{\gamma \sqcup L} & B'_\gamma \\ B'_\gamma & B'_{\gamma \sqcup R} \end{pmatrix} \succeq \begin{pmatrix} B_{\gamma \sqcup L} & B'_\gamma \\ B'_\gamma & B_{\gamma \sqcup R} \end{pmatrix} \succeq 0,$$

where the first inequality follows from $B'_{\gamma \sqcup L} \succeq B_{\gamma \sqcup L}$, $B'_{\gamma \sqcup R} \succeq B_{\gamma \sqcup R}$, and the second inequality is a consequence of $B'_\gamma = B_{\gamma \sqcup L} \,\sharp\, B_{\gamma \sqcup R}$. This shows that (X, T') belongs to $\mathcal{S}(\sigma)$.

Define $h : \mathbb{S}_m^{++} \mapsto \mathbb{S}_m^{++}$, $T \to \#_\sigma(X,T)$. So far, we have shown that $h(T) \succeq T$, and $(X,T) \in \mathcal{S}(\sigma) \Longrightarrow (X, h(T)) \in \mathcal{S}(\sigma)$. By using the convexity property of the Riemannian metric (Equation (4)), a simple induction shows that h is a contraction mapping with a contraction equal to the fraction of the number of leaves of B that take the value T:

$$\forall T, T' \in \mathbb{S}_m^{++},\ \delta_2(h(T), h(T')) \leq \frac{2^n - \beta}{2^n} \delta_2(T, T') < \delta_2(T, T').$$

Hence, the mapping $T \to h(T)$ is contractive in the space $\mathbb{S}_m^{++}$ equipped with the Riemannian metric δ_2. It is known that this space is complete (see e.g. [MZ11]), and hence we can apply the Banach fixed point theorem: the fixed point equation $T = h(T)$ has a unique solution $T^* \in \mathbb{S}_m^{++}$. Moreover for all $T \in \mathbb{S}_m^{++}$ the sequence defined by $T_0 = T$, $T_{i+1} = h(T_i)$ converges to T^*. In particular, if $(X, T) \in \mathcal{S}(\sigma)$, our previous discussion shows that $T \preceq T^*$ and $(X, T^*) \in \mathcal{S}(\sigma)$. This shows that T^* is the right candidate to be the largest element T such that $(X, T) \in \mathcal{S}(\sigma)$, and since X, X^p and I_m commute it is easy to verify that $X^p = h(X^p)$, i.e. $T^* = X^p$.

It remains to show that the statement of the theorem remains valid when the matrix $X \in \mathbb{S}_m^+$ is singular. To do this, chose a sequence $X_i \in \mathbb{S}_m^{++}$ such that $X_i \to X$ as $i \to \infty$, as well as a sequence $\epsilon_i > 0$ such that $\epsilon_i \to 0$. We know that $(X_i, X_i^p) \in \mathcal{S}(\sigma)$ for all i. Let $T \in \mathbb{S}_m^+$ such that $(X, T) \in \mathcal{S}(\sigma)$ and define $X_i' := (1-\epsilon_i)X + \epsilon_i X_i$, $T_i' := (1-\epsilon_i)T + \epsilon_i X_i^p$. By convexity of $\mathcal{S}(\sigma)$, we have $(X_i', T_i') \in \mathcal{S}(\sigma)$. Moreover, since the matrices T_i' and X_i' are positive definite, we know that $T_i' \preceq X_i'^p$. By taking the limit, we obtain $T \preceq X^p$. Finally, we must show that $(X, X^p) \in \mathcal{S}(\sigma)$. Consider the tree B with leaves $\sigma(X, X^p)$, and whose non-leaf nodes are defined by the relation: if $B_{\gamma \sqcup L} = X^{k_1}$ and $B_{\gamma \sqcup L} = X^{k_2}$, then $B_\gamma := X^{(k_1+k_2)/2}$. A simple induction shows that the root of this tree is $B_\varnothing = X^{\frac{n_X(\sigma)+n_T(\sigma)}{2^n}}$, where $n_X(\sigma)$ $n_T(\sigma)$ represent the number of times that X and T appear in $\sigma(X, T)$. Replacing $n_X(\sigma)$ by α and $n_T(\sigma)$ by $2^n - \beta$, we find $B_\varnothing = X^p$. Hence, $(X, X^p) \in \mathcal{S}(\sigma)$, and the proof is complete. □

Corollary 4.3. *Let $p \in \mathbb{Q} \cap (0, 1]$ and σ satisfy the assumptions of Theorem 4.2. If K is a $m \times r-$matrix, then the concave function $X \to \operatorname{trace} K^T X^p K$, which maps $\mathbb{S}_m^+$ to $\mathbb{R}^+$, has the following SDR representation: for all $X \in \mathbb{S}_m^+$,*

$$t \leq \operatorname{trace} K^T X^p K \iff \exists T \in \mathbb{S}_m^+ : (X, T) \in \mathcal{S}(\sigma),\ t \leq \operatorname{trace} K^T T K.$$

Proof. If $t \leq \operatorname{trace} K^T X^p K$, we set $T = X^p$, so that $t \leq \operatorname{trace} K^T T K$ and by Theorem 4.2 $(X, T) \in \mathcal{S}(\sigma)$. Conversely, assume that $(X, T) \in \mathcal{S}(\sigma)$. We know from previous theorem that $T \preceq X^p$. Hence, we have $\langle M, T\rangle \leq \langle M, X^p\rangle$ for all positive semidefinite matrix M. In particular,

$$\operatorname{trace} K^T T K = \langle K K^T, T\rangle \leq \langle K K^T, X^p\rangle = \operatorname{trace} K^T X^p K,$$

from which the conclusion follows. □

5 Numerical Results

In this section, we compare the CPU time required to solve problems of the form

$$\min_{\substack{\boldsymbol{w} \geq \mathbf{0} \\ \sum_i w_i = 1}} \quad \operatorname{trace} f\Big(\sum_{k=1}^{s} w_k M_k\Big), \tag{P_f}$$

by using the semidefinite representation of Theorem 3.1, and the one of Ben-Tal and Nemirovski [BTN87]. This problem is inspired from the application to optimal experimental design that is presented in the introduction. For the sake of variety, we do not limit ourselves to power functions $x \to x^p$ with $p < 1$. More precisely, assume that $f : I \to \mathbb{R}$ is a convex real valued function defined on the interval I, an SDR of f is known:

$$\forall x \in I,\ f(x) \leq t \Longleftrightarrow \exists \boldsymbol{y} \in \mathbb{R}^n :\ F_0 + xF_X + tF_T + \sum_{i=1}^{n} y_i F_i \succeq 0,$$

and the matrices $M_1, \ldots, M_s \in \mathbb{S}_m^I$ are given. We compare the efficiency of the following two SDP formulations of Problem (P_f): the one with block matrices resulting from Theorem 3.1,

$$\begin{aligned}
\min_{X,T,\{Y_i\},\boldsymbol{w}} \quad & \operatorname{trace} T & (SDP_f - 1)\\
\text{s. t.} \quad & F_0 \otimes I_m + F_X \otimes X + F_T \otimes T + \sum_{i=1}^{n} F_i \otimes Y_i \succeq 0;\\
& X = \sum_{k=1}^{s} w_k M_k, \quad \boldsymbol{w} \geq \boldsymbol{0}, \quad \sum_{k=1}^{s} w_k = 1,
\end{aligned}$$

and the SDP from [BTN87] that bounds each Ky-Fan Norm of X:

$$\begin{aligned}
\min_{X,\boldsymbol{t},\boldsymbol{x},\boldsymbol{y},\boldsymbol{\sigma},\{Z_j\}} \quad & \sum_{j=1}^{m} t_j & (SDP_f - 2)\\
\text{s. t.} \quad & F_0 + x_j F_X + t_j F_T + \sum_{i=1}^{n} y_i^{(j)} F_i \succeq 0, \quad (j = 1, \ldots, m);\\
& x_1 \geq x_2 \geq \ldots \geq x_m;\\
& \sum_{k=1}^{j} x_k - j\sigma_j - \operatorname{trace}(Z_j) \geq 0, \qquad (j = 1, \ldots, m-1);\\
& Z_j \succeq 0, \qquad (j = 1, \ldots, m-1);\\
& Z_j - X + \sigma_j I_m \succeq 0, \qquad (j = 1, \ldots, m-1);\\
& \operatorname{trace} X = \sum_{j=1}^{m} x_j;\\
& X = \sum_{k=1}^{s} w_k M_k, \quad \boldsymbol{w} \geq \boldsymbol{0}, \quad \sum_{k=1}^{s} w_k = 1.
\end{aligned}$$

Our computational results are summarized in Table 1. Besides rational power functions, we have also consider the function $f : (0,1) \mapsto \mathbb{R},\ x \to \frac{1}{x(x-1)}$, which has the SDR

$$\begin{aligned}
\forall x \in (0,1), \quad f(x) \leq t & \Longleftrightarrow \exists u \in \mathbb{R} : 1 \leq u(1-x),\ 1 \leq (t-u)x\\
& \Longleftrightarrow \exists u \in \mathbb{R} : \begin{pmatrix} u & 1 \\ 1 & 1-x \end{pmatrix} \succeq 0,\ \begin{pmatrix} t-u & 1 \\ 1 & x \end{pmatrix} \succeq 0,
\end{aligned}$$

$f(x)$	I	m	CPU time (s) ($SDP_f - 1$)	($SDP_f - 2$)
$-x^{\frac{1}{3}}$	$[0,\infty)$	10	0.40	0.80
		25	5.16	40.85
		40	59.19	*706.43*†
$-x^{\frac{2}{5}}$	$[0,\infty)$	10	0.58	1.28
		25	20.38	39.57
		40	298.90	*799.77*†
$x^{\frac{-8}{7}}$	$(0,\infty)$	10	0.49	0.90
		25	22.38	*40.07*†
		40	357.22	*691.75*†
$x^{\frac{7}{4}}$	$[0,\infty)$	10	0.41	1.23
		25	8.71	39.95
		40	120.16	*741.15*†
$\frac{1}{x(1-x)}$	$(0,1)$	10	0.30	0.76
		25	4.31	37.21
		40	51.79	*607.57*†
convex-env$(\frac{x^6}{6} - 3\frac{x^4}{2} + 4x^2 + x)$	$\mathbb{R}$	10	0.75	1.50
		25	63.62	*43.08*†
		40	1019.70	*903.55*†

Table 1: CPU time of two SDP formulations for Problem (P_f). The second column indicates the interval I where the function f is defined, and the third column specifies the size of the matrices $M_i \in \mathbb{S}_m^I$. †*The numbers displayed in italics indicate that the SDP solver stopped before reaching the optimality tolerance, because of numerical problems.*

as well as the convex envelope of a polynomial of degree 6. The fact that convex envelopes of univariate rational functions are SDr was proved by Laraki and Lasserre [LL08]. For the function $f : \mathbb{R} \mapsto \mathbb{R}$, $x \to \text{convex-env}(\frac{x^6}{6} - 3\frac{x^4}{2} + 4x^2 + x)$, the SDR of [LL08] is:

$$f(x) \leq t \iff \exists y_2, \dots, y_6 \in \mathbb{R} : \begin{pmatrix} 1 & x & y_2 & y_3 \\ x & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & y_6 \end{pmatrix} \succeq 0,$$
$$t \geq \frac{y_6}{6} - 3\frac{y_4}{2} + 4y_2 + x.$$

For all our instances, we have generated $s = 25$ random matrices $M_i \in \mathbb{S}_m^I$. We solved the SDPs by using SeDuMi [Stu99] on a PC with 8 processors at 2.2GHz. Our experiments show that the block matrix formulation ($SDP_f - 1$) improves the CPU time by a factor between 2 and 12, except for the case where f is the convex envelope of a polynomial of degree 6; but in this case, SeDuMi encountered numerical problems with ($SDP_f - 2$) and stopped the computation before reaching the optimality tolerance. Also note that the SDP solver was always able to compute an optimal solution with ($SDP_f - 1$), which suggests that the formulation from this paper is numerically more stable.

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