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Testing for S_n-Symmetry with a Recursive Detective

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Abstract

In the recent years symmetric chaos has been studied intensively. One knows which symmetries are admissible as the symmetry of an attractor and which transitions are possible. The numeric has been developed using equivariant functions for detection of symmetry and augmented systems for determination of transition points. In this paper we look at this from a sophisticated group theoretic point of view and from the view of scientific computing, i.e. efficient evaluation of detectives is an important point. The constructed detectives are based on Young's seminormal form for S_n. An application completes the paper.

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1 Introduction

In the recent years a lot of investigation of symmetry in chaos has been done. The book [10] shows that this is a fascinating topic. A great part of the theory and its numerics areunderstood. The symmetry of an attractor and its symmetry increasing bifurcation are detected with the help of some functions, called detectives, see [4], [7]. These functions recognize the symmetry of attractors and are thus essential for the detection of transition between attractors with different isotropy. In section 2 we will briefly recall part of those ideas. Also other aspects of the numeric have been investigated. From the theoretical side the admissible symmetry groups and the admissible symmetry breaking bifurcations have been understood, see e.g. [8] and [3].

In this paper we concentrate on a single aspect, namely the detectives. Since these functions are evaluated many times they are worth to be investigated further. There remain several open questions concerning the choice of such functions if they consist of polynomials. For the symmetric group S_n efficient evaluation turns out to be very important since the order of S_n grows rapidly with n.

Since polynomials and symmetry have been studied in algebra dating back to the last century, some consequences for detectives are discussed in section 3. The second question addressed in this section is how detectives are build from smaller functions. Concerning this construction we suggest criteria for the reliability of a detective.

For a reader not interested in theory section 4 is the main part of this paper. There the main result, the detective for S_n based on recursive evaluation is presented. The irreducible representations of S_n can easily be constructed from the Young tableaux. The representation matrices are called Young's seminormal form and their construction can be found in many text books on representation theory, i.e. [15], [16]. Implementation details have been taken from [5] where these irreducible representations are used for a S_n-version of FFT. The recursive evaluation makes the detective efficient in comparision to other choices. Besides this good numerical properties this function is also prefarable in terms of storage requirements. Results on the distances to the fixed point spaces complete the section.

Finally, this recursive detective is applied to Josephson junctions coupled in an array with S_n symmetry. In contrast to [21] where intensive numerical simulations for $n = 4$ and $n = 5$ have been done we give an example for $n = 10$.

2 The concept of detectives

In this section the detection of the symmetry of attractors is recalled.

We are interested in dynamical systems

$$\dot{x} = f(x, \lambda), \quad x \in R^n, \lambda \in R \tag{1}$$

or discrete dynamical systems

$$x^k = f(x^{k-1}, \lambda), \quad x \in R^n, \lambda \in R \tag{2}$$

which are equivariant with respect to a faithful, orthogonal representation $\vartheta : G \to Gl(R^n)$ of a finite group G. In the following ϑ always refers to this representation.

To make the notion of equivariance more precise we give the following definition.

Definition 2.1 *([13]) Let ϑ, ρ be two representations of G. A C^∞ function h is called* ϑ-ρ-equivariant, *if*

$$h(\vartheta(t)x) = \rho(t)h(x) \quad \forall\, t \in G, \forall\, x \in R^n.$$

The mapping f in (1) or (2) is assumed to be ϑ-ϑ-equivariant which usually is called G-equivariant.

In [4] attractors which do not lie completely in a fixed point space are thickened to open sets. So let $\mathcal{A}$ be the class of all open subsets A of R^n with piecewise smooth boundary that satisfy the dichotomy

$$\vartheta(t)A = A \text{ or } \vartheta(t)A \cap A = \emptyset \quad \forall\, t \in G,$$

where ϑ is the faithful representation in (1).

$$H(A) = \{t \in G | \vartheta(t)A = A\}$$

denotes the isotropy group of an attractor. Observables transform the symmetry of attractors into a *physical space* W.

Definition 2.2 *([4]) Let $\rho : G \to Gl(W)$ be a linear representation. A ϑ-ρ-equivariant C^∞ function $\phi : R^n \to W$ is called an* observable. *The vector*

$$K_\phi(A) := \int_A \phi d\mu$$

is an observation, *where μ is assumed to be the Lebesgue measure.*

The determination of $H(A)$ is thus shifted to determining the isotropy group of $K_\phi(A) \in W$ denoted by $H_\phi(A)$.
Checking isotropy may be done with *distances*:
Let $\mathrm{Fix}(H, W)$ be the fixed point space of a subgroup H of G within W and

$$P^{\rho,H}(y) = \frac{1}{|H|} \sum_{t \in H} \rho(t)(y), \tag{3}$$

the projection onto $\mathrm{Fix}(H, W)$. Then

$$d^H(y) = ||y - P^{\rho,H} y||_2^2 = ||(Id - P^{\rho,H})y||_2^2, \tag{4}$$

gives the distance to the fixed point space. Clearly, the isotropy of y is the maximal subgroup H with distance zero.

For the detection of symmetry of attractors it becomes important that ρ distinguishes all subgroups, i.e. all subgroups $H = G_y$ of G appear to be isotropy groups in W for one $y \in W$.

Definition 2.3 *([4] Def. 4.2): Two representations $\rho_1 : G \to Gl(W_1)$ and $\rho_2 : G \to Gl(W_2)$ are* lattice equivalent *if there exists a linear isomorphism $L : W_1 \to W_2$ such that*

$$L(Fix(H, W_1)) = Fix(H, W_2),$$

for every subgroup H of G.

Let $\vartheta^i, i = 1, \dots, h$ denote the inequivalent irreducible representations of G. ϑ^1 denotes the unit representation. For a linear representation $\rho : G \to Gl(R^n)$ let $m_i(\rho)$ be the multiplicity in the canonical decomposition $\rho = \sum_{i=1}^h m_i(\rho)\vartheta^i$. Let P_i denote the projection onto the isotypic component with respect to ϑ^i.

Lemma 2.4 *([4] Thm. 4.3): $\rho = \sum_{i=1}^h \vartheta^i$ distinguishes all subgroups of G, where ϑ^i denote the irreducible representations of G.*

A *detective* is an observable which generically determines all symmetries of sets in $\mathcal{A}$.

Definition 2.5 *([4] Def. 5.1) The observable ϕ is a detective for G if for each subset $A \in \mathcal{A}$ almost all near identity ϑ-ϑ-equivariant diffeomorphism ψ satisfy*

$$H_\phi(\psi(A)) = H(A).$$

Theorem 2.6 *([4] Thm. 5.2): Let $\phi^i, i = 1, \dots, h$ be ϑ-ϑ^i-equivariant observables which are polynomial and $\phi^i \not\equiv 0$. Then $\phi = (\phi^1, \dots, \phi^h)$ is a detective for G.*

It is clear that in Thm. 2.6 it is sufficient to consider all lattice inequivalent irreducible representations.

It turned out in case where the attractor is contained within a fixed point space of K one has to be more careful, see [12]. Then the symmetry of A may be one of the subgroups of $N_G(K)$. So the requirement is that $\phi_{|Fix(K)}$ is a detective for the group $N_G(K)$.

Before we discuss special detectives we shortly discuss the practical evaluation of the observation. Precise descriptions can be found in the literature. For discrete dynamical systems one uses

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N} \phi(f^k(x_0)),$$

provided the ergodic theorem is valid. For ordinary differential equations the attractor is $\{x(t) | t \geq 0\}$ and the observation becomes

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(x(t))dt.$$

There are two suggestions for detectives:
1.) In [4],[7] the observable $\phi(x) = xx^t$ was chosen as detective. ϕ is isomorphic to a ρ-ϑ-equivariant mapping where ρ is a subrepresentation of the *tensor product* $\vartheta \otimes \vartheta^*$ where $\vartheta^* : G \to Gl(R^n), \vartheta^*(t) = \vartheta(t^{-1})^t$ is the contragredient representation. In our case when ϑ is orthogonal then $\vartheta^* = \vartheta$. Obviously, range $\phi \neq R^{n,n}$.
If $\vartheta = \sum_{i=1}^h m_i(\vartheta)\vartheta^i$ is the canonical decomposition and

$$\vartheta^i \otimes \vartheta^j = \sum_{k=1}^h c_k^{ij} \vartheta^k$$

with the multiplicities c_k^{ij} being the Clebsch-Gordan coefficients, then

$$\vartheta \otimes \vartheta = \sum_{k=1}^h \left(\sum_{i,j=1}^h m_i(\vartheta) \cdot m_j(\vartheta) \cdot c_k^{ij} \right) \vartheta^k = \sum_{k=1}^h m_k(\vartheta \otimes \vartheta) \cdot \vartheta^k.$$

Assume ϑ has the following property:
For each $k = 1, \ldots, h$ there exists i and j such that

$$m_i(\vartheta) \neq 0, m_j(\vartheta) \neq 0 \text{ and } c_k^{ij} \neq 0.$$

Assuming this property we have $m_k(\vartheta \otimes \vartheta) > 0, k = 1, \ldots, h$. To prove that ϕ is a detective for G it remains to show that $P_k\phi \not\equiv 0, k = 1, \ldots, h$. Using the decomposition one needs to find for each k irreducible representations ϑ^i, ϑ^j with $c_k^{ij} > 0$ and $P_k x_i x_j^t \not\equiv 0$, where x_i, x_j behave like ϑ^i and ϑ^j, respectively. This can easily be implemented and checked in a Computer Algebra environment.

In [4] it is shown that $\vartheta \otimes \vartheta$ distinguishes all subgroups in the case of rings of coupled cells with $G = D_p$ symmetry. For this $p \geq 3$ and the number of equations per cell $m \geq 2$ is essential.
2.) A second detective was given by the left regular representation $L : G \to Gl(R^{|G|})$, see [7]. Its well-known decomposition is

$$L = \sum_{i=1}^{h} m_i(L)\vartheta^i = \sum_{i=1}^{h} dim(\vartheta^i)\vartheta^i.$$

So any ϑ-L-equivariant mapping ϕ with $P_i^L\phi \not\equiv 0, i = 1, \ldots, h$ is a detective. L and ϑ define a group action on the space of polynomial mappings

$$\begin{array}{rll} \delta: & G \to & Gl(R[x]^{|G|}) \\ & t \hookrightarrow & \delta(t) \\ \delta(t): & R[x]^{|G|} \to & R[x]^{|G|} \\ & \delta(t)(\phi(x)) & = L(t)\phi(\vartheta(t^{-1})x). \end{array}$$

This is a linear representation.

$$P^{\delta,G} = \frac{1}{|G|}\sum_{t \in G} \delta(t), \tag{5}$$

is a projection onto the trivial component wrt δ which consists of ϑ-L-equivariant mappings. In [21] it is suggested to choose a polynomial $p(x)$ and $q(x) = (p(x), 0, \ldots, 0)^t$. An observable is defined by

$$\phi(x) := P^{\delta,G}(q) = \frac{1}{|G|}\sum_{t \in G} \delta(t)(q) = \frac{1}{|G|}\sum_{t \in G} L(t^{-1})((p(\vartheta(t)x), 0, \ldots, 0)^t), \tag{6}$$

For the special choice $p(x) = x_1 x_2^2 \cdots x_{n-1}^{n-1}$ the mapping ϕ is a detective for S_n, see [21].

3 Poincaré series and irreducible observables

The aim of this section is to show the richness of polynomial detectives. Appliying some theory we derive results for the degree of detectives and its decomposition into smaller functions.

The polynomial, ρ-ϑ-equivariant mappings $\phi(x)$ form a module over the ring of ϑ-invariant polynomials. Then each component $\phi_i(x), i = 1, \ldots, dim(\rho)$ is a polynomial. ϕ_i

is said to be *homogeneous* of degree k if in the representation with monomials $\phi_i(x) = \sum_{j\in J} a_j x^j$ only monomials of degree k appear. The vector of polynomials $\phi(x)$ is said to be *homogeneous* of degree k if for each component either $\phi_i(x) \equiv 0$ or $\phi_i(x)$ is homogeneous of degree k. Let m_k denote the dimension of the vector space of ϑ-ρ-equivariant mappings which are homogeneous of degree k.

Theorem 3.1 *([22],[13]) Let ρ and ϑ be linear representations of G. Let m_k be the number of linear independent ρ-ϑ-equivariant polynomial mappings homogeneous of degree k. Then the Hilbert-Poincaré series is*

$$\sum_{k=0}^{\infty} m_k z^k = \frac{1}{|G|} \sum_{t\in G} \frac{tr(\rho(t^{-1}))}{det(Id - z\cdot\vartheta(t))}.$$

For $\rho = \vartheta$ this is exactly the series given by Sattinger [17]. For ρ being the trivial irreducible representation the series is the well-known Molien series for invariant polynomials which was proved at the end of the last century.

The righthand side is easily evaluated and thus the dimensions m_k for small k are cheaply determined with a Taylor expansion. This has been implemented in a Computer Algebra System.

The following theorem is a generalization of the results for the invariants, see e.g. [20] and for the isotypic components of $R[x]$.

Theorem 3.2 *([22],[13]) There exist n homogeneous invariants $\sigma_i, i = 1,\ldots,n$ such that the ϑ-ρ-equivariants form a free module over the subring $R[\sigma]$. (The module is Cohen-Macaulay.)*

Thm. 3.2 states that each equivariant has a unique representation $\sum A_i(\sigma_1,\ldots,\sigma_n)b_i$ with polynomials A_i. Note that if one considers the invariant ring instaed of the subring $R[\sigma]$ such representations are non-unique in general.

The free basis is determined with the help of projections and the series, see [13]. From Computer Algebra graded Gröbner bases with respect to weighted orderings are used for this purpose.

Lemma 3.3 *The minimal degree of a detective for a given group action ϑ is*

$$d = max_{i=1,\ldots,h}(kmin(\vartheta^i, \vartheta)),$$

where $\vartheta^i, i = 1,\ldots,h$ are the pairwise lattice inequivalent irreducible representations which are necessary to distinguish all subgroups of G. $kmin(\vartheta^i,\vartheta)$ is the minimal degree of a non-zero, homogeneous ϑ^i-ϑ-equivariant.

Proof: Thm. 2.6 means that for each detective $\phi(x)$ the restrictions $P_i^\rho\phi(x) \not\equiv 0, i = 1,\ldots,h$ hold. The mapping $P_i^\rho\phi(x)$ contains at least one ϑ^i-ϑ-equivariant mapping unequal zero which has minimal degree $kmin(\vartheta^i,\vartheta)$. □
The values $kmin(\vartheta^i,\vartheta)$ can easily be read off from the series in Theorem 3.1 with $\rho = \vartheta^i$.

Detectives may be build from smaller functions. In contrast to above where we used a decomposition on the image we will now study the consequences of a decomposition in the domain.

Definition 3.4 *A ϑ^i-ϑ^j-equivariant mapping $\phi^i_j(x) \not\equiv 0$ is called ϑ^i-ϑ^j-observable (*irreducible *observable).*

Remark: There are combinations ϑ^i, ϑ^j such that no ϑ^i-ϑ^j-observables exist.

Example: $D_6 = \{id, r, r^2, r^3, r^4, r^5, s, sr, sr^2, sr^3, sr^4, sr^5\}$,
$\vartheta^2(r) = 1, \vartheta^2(s) = -1$ and ϑ^5 the 2-dim. faithful representation. Then there exists no ϑ^2-ϑ^5-observable, but there are ϑ^5-ϑ^2-observables. $kmin(\vartheta^5, \vartheta^2) = 6$ can be read off from the series.

Tensor products of representations can be used to construct such small observables from known observables. Let ϕ be a η-δ-observable and ψ a η-ρ-observable where δ, η, ρ are irreducible representations. Then a $(\delta \otimes \rho)$-η-equivariant observable is defined by χ

$$\chi_{i+j}(x) = \phi_i(x) \cdot \psi_j(x) \quad i = 1, \ldots, dim(\delta), j = 1, \ldots, dim(\rho). \tag{7}$$

If $m_k(\delta \otimes \rho) = 1$ then $P_k\chi$ is a η-ϑ^k-observable.

Example: Let ϑ^2 be the non-trivial, 1-dim representation of S_4, ϑ^3 the 2-dim irreducible representation, and ϑ^4, ϑ^5 the 3-dim. irreducible representations of S_4. Let ϑ^4 be faithful. It is $\vartheta^5 = \vartheta^2 \otimes \vartheta^4$. Let $\phi(x)$ be a ϑ^3-ϑ^2-observable and ψ a ϑ^3-ϑ^4-observable. Then $\chi(x) = \phi(x) \cdot \psi(x)$ is a ϑ^3-ϑ^5-observable.

Since every representation ϑ has a decomposition $\vartheta = \sum_{i=1}^h m_i(\vartheta)\vartheta^i$ the small ϑ^i-ϑ^j-observables can help to build a detective, but in general more complicated functions are necessary.

Another composition of observables is the following case. Assume the representation ϑ in the domain decomposes into $\vartheta = \delta_1 + \delta_2$ and $R^n = V_1 + V_2$, respectively. Let $\phi_i : V_i \to R^m$ be δ_i-ρ-observables, $i = 1, 2$. Then $(\phi_1 + \phi_2) : R^n \to R^m$ is a ϑ-ρ-observable.

There may be detectives for G which do not depend on the full domain R^n. But the contrary seems to be appropriate. The following seems to be a reasonable demand for a detective $\phi : R^n \to R^m$

> For all ϑ-invariant subspaces W of R^n the function $\Psi : W \to R^n, \Psi(w) = \psi(w_1 + w)$ is not a constant function, where $w_1 \in W^\perp$ is a fixed value in the direct complement $W^\perp$ of W in R^n.

An even sharper demand is that $P_i^\rho \circ \Psi$ is not a constant function for $i = 2, \ldots, h$.

4 Detectives with recursive evaluation

Assume that H is a proper subgroup of G and that we already have a detective for H. Is it possible to use this information in order to construct a detective for G? We looked at theoretical results on induced representations such as the Frobenius reciprocity and the Theorem by Mackey (see [19] or [1]), but they are not suitable for practical considerations since no explicit formulas are obtained. However, for the symmetric group S_n explicit formulas for the relation between the irreducible representations σ^α of S_n and S_{n-1} are known, see [5]. The main advantage is that $Res_{S_{n-1}}\sigma^\alpha$ is already in block diagonal form. So no innerconnectivity matrices as in [11] are needed. Based on the Young tableaux we give a detective for S_n which is evaluated recursively. In order to clarify the group theoretic structure we recall a simple, but useful lemma.

Lemma 4.1 *Let $\eta : G \to Gl(V)$ be a linear represenation of G and H a proper subgroup of G. Furthermore let $t_i, i = 1, \ldots, [G : H]$ be representatives of left cosets and $w \in V$ a H-invariant vector with respect to $Res_H(\vartheta)$. Then*

$$v := \frac{1}{[G : H]} \sum_{i=1}^{[G:H]} \eta(t_i)w$$

is G-invariant.

Proof: The left cosets form a partition of G and each $g \in G$ corresponds to a permutation of the left cosets. □

The symmetric group S_n has irreducible representations σ^α, where $\alpha = (\alpha_1, \alpha_2, \ldots), \alpha_i \geq \alpha_{i+1}$ are the partitions of n, denoted by $\alpha \vdash n$. These representations can nicely be described with the Young diagrams. This is presented in a way suitable for applications in [5] from where the following recursive sheme for σ^α was taken. Also [15] and [16] are interesting references for the irreducible representations of the symmetric group.

Each partition α corresponds to an ordered collection of boxes, the Young diagrams. The numbers $1, \ldots, n$ are put into the diagram such that in each row and each column the numbers increase. The number of these so-called standard α-tableau equals the dimension of σ^α. The matrices $\sigma^\alpha(t)$ are given in a basis indexed by the standard α-tableau, which are ordered in the last letter sequence which successively compares the last entries in each row.

Now two facts are important:

a.) The restricted representation decomposes as $Res_{S_{n-1}}(\sigma^\alpha) = \sum_{\beta \vdash n-1, \beta \subset \alpha} \sigma^\beta$. Moreover one can work with the same coordinates. Thus $\sigma^\alpha(t) = \text{diag}(\sigma^\beta(t)), t \in S_{n-1}$.

b.) It is sufficient to give $\sigma^\alpha(t)$ for generators of the group, e.g. the neighboring transpositions $(i, i + 1), i = 1, \ldots, n - 1$.

The consequence is that the matrices $\sigma^\alpha(i, i + 1)$ are known from the matrices $\sigma^\beta(i, i + 1), \beta \vdash n - 1, i = 1, \ldots, n - 2$. For $i = n - 1$ a precise description of the sparse matrix $\sigma^\alpha(n - 1, n)$ is given in [5]. The sparse matrix $\sigma^\alpha(i, i + 1)$ can be stored in a vector of length $2 \cdot \dim(\sigma^\alpha)$, see [5].

Based on the two facts above we now develop a recursive detective for the representation ϑ which describes the permutation of variables. Let for all partitions $\beta \vdash (n - 1)$ the mappings $f^\beta : R^{n-1} \to R^{dim(\sigma^\beta)}$ be $Res_{S_{n-1}}(\vartheta)$-σ^β-equivariant. For each partition $\alpha \vdash n$ we have by condition a.) a mapping $f : R^{n-1} \to R^{dim(\sigma^\alpha)}, f = (f^{\beta_1}, \ldots, f^{\beta_r})$. All mappings f^β with $m_\beta(Res_{S_{n-1}}(\sigma^\alpha)) = 1$ or equivalently $\beta \subset \alpha$ are involved. The ordering of the f^β's in f is given by the last letter ordering of the standard α-tableaux. Let P be the projection $R^n \to R^{n-1} \quad (x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1})$. $f \circ P$ is $Res_{S_{n-1}}(\vartheta)$-η-equivariant with $\eta = \sum_{\beta \vdash n-1, \beta \subset \alpha} \sigma^\beta$.

As representatives of left cosets of S_n/S_{n-1} the cyclic permutations

$$t_i = (i, i + 1, \ldots, n) = (i, i + 1)(i + 1, i + 2) \cdots (n - 1, n), i = 1, \ldots, n - 1, t_n = id$$

are chosen.

Lemma 4.2 *Assume the above notations. The mappings* $F^\alpha : R^n \to R^{dim(\sigma^\alpha)}$

$$F^\alpha(x) = \sum_{i=1}^{n} \sigma^\alpha(t_i)[f \circ P](\vartheta(t_i^{-1})x) \tag{8}$$

are ϑ-σ^α*-equivariant for all partitions* $\alpha \vdash n$.

Proof: Apply Lemma 4.1 with $w = f \circ P$ and the representation $\eta(t)w = \sigma^\alpha(t)[f \circ P](\vartheta(t^{-1})x)$.

Remark 4.3 *i.) Evaluation of* $F^\alpha(x)$ *only needs evaluations at*

$$(x_1, \ldots, x_{n-1}, \hat{x}_n), (x_1, \ldots, \hat{x}_{n-1}, x_n), \ldots, (\hat{x}_1, x_2, \ldots, x_n),$$

where the symbol $\hat{x}_i$ *means that the variable* x_i *is dropped.*

ii.) Since the matrices $\sigma^\alpha(i, i+1)$ *are sparse the necessary matrix-vector operations can be performed cheaply, see [5, p. 131].*

Example 4.4 *For* $S_2 = Z_2 = \{id, (1,2)\}$ *the Young diagrams are*

$$\begin{array}{|c|c|}\hline & \\ \hline \end{array} \quad \textit{and} \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}.$$

The corresponding irreducible representations are given in the notation above by $\sigma^\beta(1,2) = 1$ *for* $\beta = 2$ *and* $\sigma^\beta(1,2) = -1$ *for* $\beta = (1,1)$. *For* $n = 3$ *and the partition* $\alpha = (1,2)$ *we have 2 standard Young tableaux*

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \quad < \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

By the branching theorem we have

$$\sigma^\alpha(1,2) = \begin{pmatrix} \sigma^{(1,1)}(1,2) & 0 \\ 0 & \sigma^2(1,2) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The last matrix is

$$\sigma^\alpha(2,3) = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} d^{-1} & 1 - d^{-2} \\ 1 & -d^{-1} \end{pmatrix},$$

where $d = |x - u| + |y - v| = 2$ *is the distance between the positions* (x, y) *and* (u, v) *for* $n = 3$ *in the standard* α*-tableaux. Representatives of left cosets are id, the transposition* $(1,2)$*, and the 3-cycle* $(1,2)\cdot(2,3)$. *Let* f^2 *and* $f^{(1,1)}$ *be invariant and equivariant functions, respectively,*

$$\begin{array}{rcccccl} f^2((1,2)(x_1,x_2)) & = & f^2(x_2,x_1) & = & \sigma^2(1,2)f^2(x_1,x_2) & = & f^2(x_1,x_2) \\ f^{(1,1)}((1,2)(x_1,x_2)) & = & f^{(1,1)}(x_2,x_1) & = & \sigma^{(1,1)}(1,2)f^{(1,1)}(x_1,x_2) & = & -f^{(1,1)}(x_1,x_2). \end{array}$$

Define $f(x_1,x_2) = (f^{(1,1)}(x_1,x_2), f^2(x_1,x_2))$ *and*

$$F^{(2,1)}(x_1,x_2,x_3) = f(x_1,x_2) + \sigma^{(2,1)}(2,3)f(x_1,x_3) + \sigma^{(2,1)}(1,2)\sigma^{(2,1)}(2,3)f(x_2,x_3).$$

The mapping $F^{(2,1)}$ *is* ϑ-$\sigma^{(2,1)}$*-equivariant, where* ϑ *is the permutation of variables.*

In order to construct a detective we need that F^α is not the zero-mapping. This is true if there exists $x_0 \in R^n$ with

$$F^\alpha(x_0) = \sum_{i=1}^{n} \sigma^\alpha(t_i)[f \circ P](\vartheta(t_i^{-1})x_0) \neq 0.$$

Lemma 4.5 *Let $f^\beta \not\equiv 0, \beta \vdash n-1$ be given $Res_{S_{n-1}}\vartheta$-σ^β-equivariant functions. Then for each partition $\alpha \vdash n$ there exists a S_{n-1}-invariant function $g : R^{n-1} \to R$ such that F^α as defined in (8) with $f = g \cdot (f^{\beta_1}, \ldots, f^{\beta_r}), \beta_i \subset \alpha$ is ϑ-σ^α-equivariant and not the zero mapping. $F = (F^{\alpha_1}, \ldots, F^{\alpha_s})$ is a detective for S_n where $\alpha_1, \ldots, \alpha_s$ denote the partitions of n.*

Proof: Choose $x_0 \in R^n$ such that it is not S_n-invariant and such that $f^\beta(Px_0) \neq 0$. Either we already have $F^\alpha(x_0) \neq 0$ with $f = (f^{\beta_1}, \ldots, f^{\beta_r})$ or we choose an S_{n-1}-invariant g. Since $f^\beta(x_0) \neq 0$ and x_0 is not invariant it is possible that the values $g(P\vartheta(t_i^{-1})x_0)$ are such that the vectors $g(P\vartheta(t_i^{-1})x_0)\sigma^\alpha(t_i)f(P(\vartheta(t_i^{-1})x_0))$ do not sum to zero. □

Of course the step of Lemma 4.2 can be repeated. Let s_j be representatives of the left cosets of S_{n-2} in S_{n-1}. For example choose $s_{n-1} = id$,

$$s_j = (j, j+1, \ldots, n-1) = (j, j+1) \cdots (n-2, n-1), j = 1, \ldots, n-2.$$

Then

$$\begin{aligned} F^\alpha(x) &= \sum_{i=1}^{n} \sigma^\alpha(t_i) \bigoplus_{\beta \vdash n-1, \beta \subset \alpha} f^\beta(P_{n-1}(\vartheta(t_i^{-1})x) \qquad (9) \\ &= \sum_{i=1}^{n} \sigma^\alpha(t_i) \bigoplus_{\beta \vdash n-1, \beta \subset \alpha} \sum_{j=1}^{n-1} \sigma^\beta(s_j) \bigoplus_{\gamma \vdash n-2, \gamma \subset \beta} f^\gamma(P_{n-2}(\vartheta(s_j^{-1})P_{n-1}\vartheta(t_i^{-1})x). \end{aligned}$$

We use P_k for the restriction to the first k coordinates. Secondly, let us introduce the notation $t_j^k = (j, j+1) \cdots (k-1, k)$ for cyclic permutations. For convenience $t_k^k = id$.

$$\begin{array}{lllllll}
k=1 & & f(x_1) & f(x_2) & f(x_3) & f(x_4) & \\
k=2 & f^2(x_1,x_2) & f^2(x_1,x_3) & f^2(x_1,x_4) & f^2(x_2,x_3) & f^2(x_2,x_4) & f^2(x_3,x_4) \\
 & f^{(1,1)}(x_1,x_2) & f^{(1,1)}(x_1,x_3) & f^{(1,1)}(x_1,x_4) & f^{(1,1)}(x_2,x_3) & f^{(1,1)}(x_2,x_4) & f^{(1,1)}(x_3,x_4)
\end{array}$$

$$\begin{array}{lllll}
k=3 & f^3(x_1,x_2,x_3) & f^3(x_1,x_2,x_4) & f^3(x_1,x_3,x_4) & f^3(x_2,x_3,x_4) \\
 & f^{(2,1)}(x_1,x_2,x_3) & f^{(2,1)}(x_1,x_2,x_4) & f^{(2,1)}(x_1,x_3,x_4) & f^{(2,1)}(x_2,x_3,x_4) \\
 & f^{(1,1,1)}(x_1,x_2,x_3) & f^{(1,1,1)}(x_1,x_2,x_4) & f^{(1,1,1)}(x_1,x_3,x_4) & f^{(1,1,1)}(x_2,x_3,x_4)
\end{array}$$

$$\begin{array}{lll}
k=4 & f^4(x_1,x_2,x_3,x_4) & f^{(2,1,1)}(x_1,x_2,x_3,x_4) \\
 & f^{(3,1)}(x_1,x_2,x_3,x_4) & f^{(1,1,1,1)}(x_1,x_2,x_3,x_4) \\
 & f^{(2,2)}(x_1,x_2,x_3,x_4) &
\end{array}$$

Figure 1: The quantities which are computed in Algorithm 4.6 for $n = 4$.

In these notations it is clear that (9) needs intermediate evaluations of type

$$f^\beta(P_{n-1}\vartheta(t_i^n)^{-1}x) = \sum_{j=1}^{n-1} \sigma^\beta(t_j^{n-1}) \bigoplus_{\gamma\subset n-2,\gamma\subset\beta} f^\gamma(P_{n-2}\vartheta(t_j^{n-1})^{-1}\vartheta(t_i^n)^{-1}x).$$

Observe that the arguments $P_{n-2}\vartheta(t_j^{n-1})^{-1}\vartheta(t_i^n)^{-1}x$ may be equal although i and j are different. We have $P_{n-2}\vartheta(t_j^{n-1})^{-1}\vartheta(t_i^n)^{-1}x = (x_{\nu_1},\ldots,x_{\nu_{n-2}}), \nu_i < \nu_{i+1}, i = 1,\ldots,n-2$, but $x_{\nu_{n-1}}, x_{\nu_n}$ with $\nu_{n-1},\nu_n \in \{1,\ldots,n\} \setminus \{\nu_1,\ldots,\nu_{n-2}\}$ have been deleted. Since the order of $x_{\nu_{n-1}}$ and x_{ν_n} does not matter this can be achieved in several ways.

Repeating the division process we have the following algorithm

Algorithm 4.6 *(Recursive evaluation of a detective for S_n)*
Given: $f : R \to R, f \not\equiv c, c \in R$.
Stored: $\sigma^\beta(k-1,k), \beta \vdash k, k = 2,\ldots,n$
Input: $x \in R^n$
Output: $F(x) \in R^m$, $m = \sum_{\alpha n} \dim \sigma^\alpha$
where F is ϑ-$(\sum_{\alpha\vdash n}\sigma^\alpha)$-equivariant and ϑ describes the permutation of variables.
Initialization: $k = 1$
Evaluate $f(x_1),\ldots,f(x_n)$.

for $k = 2$ to n do
for each $\beta \vdash k$ do
for all possible values of
$y = P_k\vartheta(t_{j_{k+1}}^{k+1})^{-1}\vartheta(t_{j_{k+2}}^{k+2})^{-1}\cdots\vartheta(t_{j_n}^n)^{-1}x, 1 \le j_i \le i, i = k+1,\ldots,n$
(equivalently $y = (x_{\nu_1},\ldots,x_{\nu_k}), \nu_i < \nu_{i+1}, i = 1,\ldots,k-1$)
evaluate
$f^\beta(y) = \frac{1}{k}\sum_{i=1}^k \sigma^\beta(t_i^k) \bigoplus_{\gamma\vdash k-1,\gamma\subset\beta} f^\gamma(P_{k-1}\vartheta((t_i^k)^{-1})y)$
if $\beta \neq k$ and $k \le n-1$ then $f^\beta(y) := g \cdot f^\beta(y)$ where $g = \sum_{j=0}^k (-f^k(y))^j$

Figure 1 shows the numbers which are computed in layers $k = 1,2,3,4$ for $n = 4$. The multiplication with g assures that no components are the zero mapping and gives a numerical balancing in the components.

Lemma 4.7 *Algorithm 4.6 evaluates a detective for S_n at $x \in R^n$. More precisely: Let the representation ϑ denote the permutation of variables and σ^α the irreducible representations of S_n corresponding to the partitions $\alpha \vdash n$ (Young's seminormal form). The values $f^\alpha(x)$ are computed for ϑ-σ^α-equivariant functions $f^\alpha \not\equiv 0$.*

Proof: From Lemma 4.2 and the multiple division it is clear that f^α is ϑ-σ^α-equivariant. It remains to show that $f^\alpha \not\equiv 0$. Using Lemma 4.5 we need to show that the S_k-invariant functions g have been chosen sufficiently generic. The main point for a polynomial f for this to happen is that the polynomial degree of f^β is sufficiently large.

The degree of a ϑ-$\sigma^{(1,\ldots,1)}$-equivariant polynomial $f^{(1,\ldots,1)} \not\equiv 0$ is $\ge \frac{n(n-1)}{2}$. (These equivariants form a module over the invariant ring generated by $\prod_{i=1,j=i+1}^n (x_i - x_j)$). Since $(1,\ldots,1)$ is the most sensitive case and the functions g are chosen sufficiently generic the statement follows by Lemma 4.5. $\square$

The proof includes already the proof of the following lemma.

n	$n!$	regular rep.	stored perms.	Young's seminormal form
5	120	0.01 sec	0.01 sec	0.01 sec
6	720	0.01 sec	0.02 sec	0.02 sec
7	5040	0.08 sec	0.04 sec	0.10 sec
8	40320	0.67 sec	0.29 sec	0.42 sec
9	362880	6.54 sec	6.60 sec	1.98 sec
10	3628800	77.55 sec	− sec	9.47 sec
11	39916800	− sec	− sec	48.66 sec

Table 1: Comparison between a detective based on regular representation (implementations with and without storage of permutations) and a detective based on Young's seminormal form (recursion): Computing times for one function evaluation on a Sun 4 implemented in C.

Lemma 4.8 *Let ϑ denote the representation of S_n which permutes n variables. Then the lowest degree $max_{\alpha\vdash n}kmin(\sigma^\alpha, \vartheta)$ of a polynomial detective is $\binom{n}{2}$.*

Recall the formula $\frac{1}{|G|}\sum \rho(t)q(\vartheta(t^{-1})x)$ for a ϑ-ρ-equivariant mapping. A simple choice is $\rho = L$ the regular representation and $q(x) = (p(x), 0, \ldots, 0)$ such that p is a monomial. Lemma 4.8 shows that $p(x) = x_1x_2^2 \cdots x_{n-1}^{n-1}$ is a monomial of lowest possible degree in order to give a detective. In [21] it is shown that this is indeed a detective.

The numerical properties of Algorithm 4.6 are the following. In each step k evaluation at $\binom{n}{k}$ values y are needed. The tupel $(f^k, \ldots, f^{(1,\ldots,1)})$ has dimension $\sum_{\alpha\vdash k} \dim \sigma^\alpha$. The determination of these $\binom{n}{k} \sum_{\alpha\vdash k} \dim \sigma^\alpha$ values necessitates $\#(\alpha \vdash k)\frac{k(k-1)}{2}$ matrix-vector multiplications which are performed cheaply. Here $\#(\alpha \vdash k)$ denotes the number of partitions. The number of additions at layer k is $\binom{n}{k}$ $(k-1)\sum_{\alpha\vdash k} \dim \sigma^\alpha$.

If we compare this detective $F = \oplus_{\alpha\vdash n} f^\alpha$ with the detective above based on the left regular representation we notice that the dimension is much smaller. We have $\sum_{\alpha\vdash n} \dim \sigma^\alpha$ with $\sum_{\alpha\vdash n}(\dim \sigma^\alpha)^2 = |S_n| = n!$ (the general formula for the dimensions of irreducible representations) in comparison to $n!$ itself. Table 1 shows the performance of one function evaluation as n increases. Various implementations for the detective with the left regular representation have been tested. The powers x_i^j are computed once and stored. The first alternative is to generate all permutations while the detective is evaluated, but this requires a lot of trivial computations. We have implemented the algorithm in [18] for this purpose. The second alternative is to generate the permutations in advance, which is necessary for the determination of the symmetry of the vector. For this one needs to know at which position which permutation is placed. But the storage needs $n!(n-1)$ integers for pointers. This leads to the effect that for n large a lot of system cpu is spent on administration of this storage. For $n = 10$ it even fails to allocate the storage.

The detective based on Young's seminormal form also necessitates some amount of storage shown in Table 2. Although the matrices $\sigma^\alpha(k-1, k), \alpha \vdash k$ and some pointers for $\sigma^\alpha(i, i+1), i < k-1$ need to be stored the computing time is smaller if all $\sigma^\alpha(i, i+1), i = 1, \ldots, k-1$ are stored. For $n = 10$ the representation matrices need space for 227376 integers and the intermediate function values are stored in 123108 reals. The experience shows that this detective is prefarable for larger n due to lower dimension and smaller storage requirements.

n	$n!$	$\#(\alpha \vdash n)$	$\sum_{\alpha\vdash n} \dim \sigma^\alpha$	rep.matrices	f's
5	120	7	26	288 int	141 real
6	720	11	76	1048 int	498 real
7	5040	15	232	3832 int	1849 real
8	40320	22	764	14528 int	7192 real
9	362880	30	2620	56448 int	29185 real
10	3628800	42	9496	227376 int	123108 real
11	39916800	56	35696	941296 int	538077 real

Table 2: Comparison of storage requirements between detectives based on regular representation and based on Young's seminormal form.

Finally, we like to mention that Algorithm 4.6 uses a principle known as Divide and Conquer. For other divide and conquer algorithms see [18].

Due to [4], [7] [6] detecting the symmetry of an attractor means computing $w := \frac{1}{N}\sum_{i=1}^{N} F(y_i)$, and distances $||(P^H - Id)w||_2$ for all subgroups H. Here we denote by P^H the projection on the fixed point space of H. The maximal subgroup H with $||(P^H - Id)w||_2 = 0$ is the symmetry group of the attractor. One still needs to think about how to perform $P^H - Id$.

For completeness we state:

Lemma 4.9 *If $\eta : H \to Gl(V)$ with $\eta(t) = diag(\vartheta^i(t)), \vartheta^i, i = 1, \ldots, h$ being the irreducible representations of H, ϑ^1 being the trivial irreducible representation then $(P^H - Id)w = (0, w_2, \ldots, w_m), m = \sum_{i=1}^{h} \dim(\vartheta^i)$.*

This suggests to collect within algorithm 4.6 all tupel $\oplus_{\beta\vdash k,\beta\neq k} f^\beta(y)$ for all $k = 2, \ldots, n$ and all possible y. But this needs too much storage.

Lemma 4.10 *Let S_n, generated by $(1,2), \ldots, (n-1,n)$ be represented by η acting as $diag(\sigma^\alpha)$, $\alpha \vdash n$ where σ^α are the irreducible representations given by the Young tableaux. Then for S_k generated by $(1,2), \ldots, (k-1,k), k = 2, \ldots, n$ there exists a set of indices I_k such that*

$$((P^{S_k} - Id)w)_i = \begin{cases} w_i & i \in I_k \\ 0 & i \notin I_k \end{cases} \qquad (10)$$

For conjugate subgroups sS_ks^{-1} we have the formula

$$P^{sS_ks^{-1}} - Id = \eta(s)(P^{S_k} - Id)\eta(s^{-1}).$$

Proof: The vector w decomposes as $w = (w^n, \ldots, w^\alpha, \ldots, w^{(1,\ldots,1)})$ into subvectors w^α, α a partition of n. Since $\sigma^\alpha = \sum_{\beta\vdash n-1,\beta\subset\alpha} \sigma^\beta$ each w^α decomposes into subvectors $w^{\alpha,\beta}, \beta \vdash n-1$. The ordering of $w^{\alpha,\beta}$ depends on the last letter ordering. Repeating this step we obtain $w^{\alpha_1,\ldots,\alpha_k}$ where α_n is a partition of n and α_i is a partition of i with $\alpha_i \subset \alpha_{i+1}, i = n-1, \ldots, k$. The trivial irreducible representation of S_k is denoted by σ^k. This yields

$$(P^{S_k} - Id)w^{\alpha_n,\ldots,\alpha_k} = \begin{cases} w^{\alpha_n,\ldots,\alpha_k}, & \text{if } \alpha_k \neq k \\ 0, & \text{if } \alpha_k = k \end{cases}$$

$\square$

Remark: For proper subgroups H of S_k it is more difficult to evaluate the distance $||(P^H - Id)w||$. But once the projections $P^{H,\beta}, \beta \vdash k$ in the coordinates of an irreducible representation σ^β of S_k are known, the distance can be evaluated using $P^{S_k} - Id$.

5 Example

Coupled arrays of Josephson junctions are a typical example of a dynamical system with S_n-symmetry given by permutations. These arrays have been discussed in various papers, e.g. [2]. In [21] a lot of numerical simulations of symmetric chaos are presented for the Josephson junctions for $n = 4$ and $n = 5$. Both, the pure capacitive and the pure resistive cases are treated in that article.
In contrast to [21] our aim is to perform calculations for larger n.
The equations for the pure capacitive load read

$$\begin{array}{rcl} \dot{\xi}_k & = & \psi_k \\ \dot{\psi}_k & = & \frac{1}{3+\beta} I - \frac{1}{\beta}(\psi_k + sin(\xi_k) - \frac{3}{n(3+\beta)} \sum_{j=1}^n (\psi_j + sin(\xi_j))) \end{array} \quad k = 1, \ldots, n.$$

We have done computations for $n = 10$ using the program code++ [14]. Figure 2 shows an S_{10}-invariant attractor, where the parameter values have been chosen to be $\beta = 0.2$ and $I = 1.05$. The solution seems to converge against a periodic orbit with S_{10}-symmetry. The triangular shape in the right picture is explained by the fact that $\{\xi_1 = \xi_2, \psi_1 = \psi_2\}$, $\{\xi_1 = \xi_3, \psi_1 = \psi_3\}$, and $\{\xi_2 = \xi_3, \psi_2 = \psi_3\}$ are fixed point spaces which are flow invariant.

The value of the distance $||Id - P^{S_{10}}v||$ to the fixed point space of S_{10} is $1.26185e-07$ where the approximate observation $v = \sum_{i=1}^N F(\xi^i)$ was used and the recursive detective F was evaluated at $N = 3000$ points. This small value clearly indicates that the type of symmetry is S_{10}. It is remarkable small since usually already a value of 0.05 is accepted to indicate a symmetry type.

In Figure 3 a more complicated attractor is presented. The parameter values are $\beta = 0.23$ and $I = 1.13$. The distances have been computed for $sS_ks^{-1}, k = 2, \ldots, n$ yielding a symmetry different from S_{10}.

In Algorithm 4.6, the recursive detective, the function f was chosen as $f(x_1) = \frac{x_1}{4} + \frac{1}{4x_1} + 1$.

These computations for S_{10} clearly demonstrated that one needs a sophisticated function for the detection of symmetry. The computing time depends on the detective since it is evaluated many times and secondly the distances are computed for a lot of subgroups of S_n. The recursive detective in Algorithm 4.6 was used successfully and is a typical example of modern algorithm technique.

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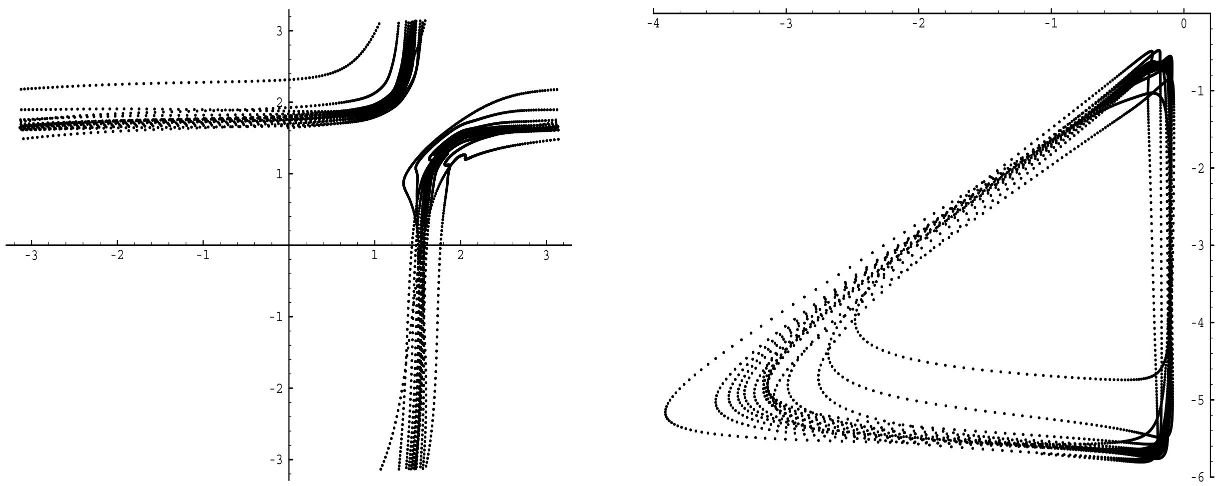

Figure 2: S_{10}-symmetric attractor in the coupled array of Josephson junctions (20 equations) for the parameter values $\beta = 0.2, I = 1.05$. The left picture shows ξ_1 versus ξ_2 plotted modulo 2π and the right picture shows $\xi_1 - \xi_3$ versus $\xi_2 - \xi_3$.

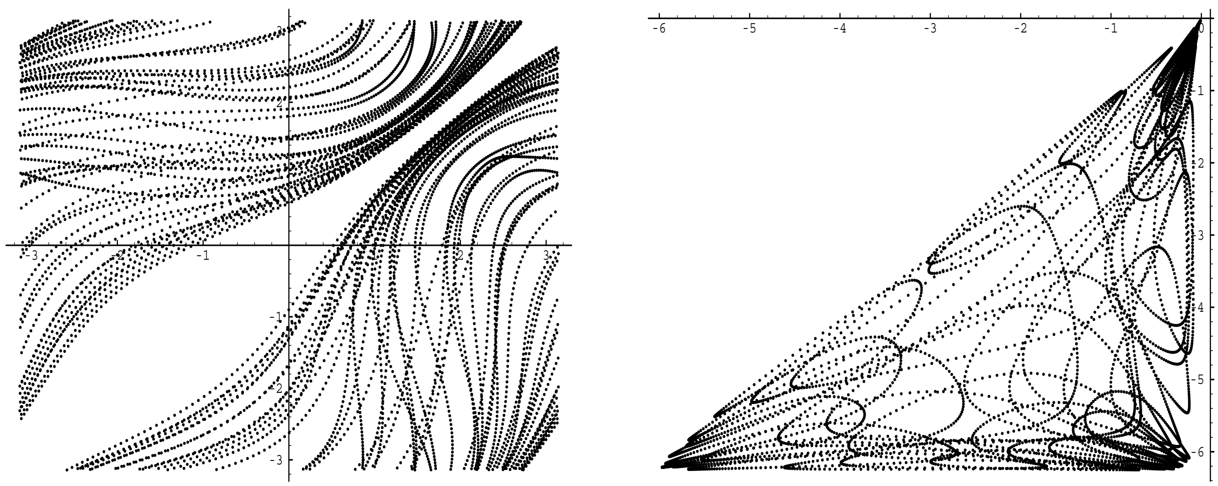

Figure 3: Symmetric attractor in the coupled array of Josephson junctions for the parameter values $\beta = 0.23, I = 1.13$. The left picture shows ξ_7 versus ξ_8 plotted modulo 2π and the right picture shows $\xi_1 - \xi_3$ versus $\xi_2 - \xi_3$.

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