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Restricted Singular Value Decomposition of Matrix Triplets

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Abstract

In this paper we introduce the concept of restricted singular values (RSV's) of matrix triplets. A theorem concerning the RSV's of a general matrix triplet (A, B, C), where $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$, which is called restricted singular value decomposition (RSVD) of matrix triplets, is derived. This result generalizes the wellknown SVD, GSVD and the recently proposed product induced SVD (PSVD). Connection of RSV's with the problem of determination of matrix rank under restricted perturbation is also discussed.

Keywords: Matrix rank, singular values, generalized singular values, product induced singular values, restricted singular values, matrix decompositions.

Subject Classification: AMS(MOS): 15A09, 15A12, 15A23, 65F20.

Inhalt

1. Introduction

Rank determination of matrices is an important problem in numerical linear algebra [7]. In applications, A_0 the matrix the rank of which is to be determined is always contaminated with errors, i.e. instead of knowing A_0 exactly we only have $A = A_0 + E$ an approximation of A_0, where E represents the error or perturbation matrix. Rank determination problem is how to estimate the rank of A_0, if A and some information of E are available. Usually only an upper bound on certain norms of E, e.g. 2-norm, is assumed to be known. In this case singular value decomposition is an useful tool for solving the problem [4,7].

In many situations, however, more informations about the error matrix E than the simple upper bound of its 2-norm are available, e.g. E has some special structure or in other words is restricted to a special class of matrices. SVD-based methods in these situations are likely to lead to conservative rank estimations.

In order to illustrate the situation, we give the following simple example. Consider matrix

$$A_0 = \begin{pmatrix} 0 & 1 \\ a_2 & a_1 \end{pmatrix}$$

if we assume that A_0 is resulted from the second order ordinary differential equation

$$\frac{d^2x}{dt^2} - a_2\frac{dx}{dt} - a_1 x = f \ ,$$

then only a_1 and a_2 are changeable, the "0" and "1" entry in A_0 are exact. Hence the error matrix E can only be of the following three forms:

i) only a_2 is changeable

$$E = \begin{pmatrix} 0 & 0 \\ e_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_{21}(1,0)$$

ii) only a_1 is changeable

$$E = \begin{pmatrix} 0 & 0 \\ 0 & e_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_{22}(0,1)$$

iii) both a_1 and a_2 are changeable

$$E = \begin{pmatrix} 0 & 0 \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (e_{21}, e_{22})$$

Observe that any E of the form in ii) can not change the rank of the original matrix A_0 while SVD-based method can not lead to such kind of conclusion. In this paper we consider the error matrix E which is restricted to a special class of matrices, i.e. $E = BDC$, where B and C are known matrices, and D is an arbitrary matrix with an upper bound on its 2-norm. In section 2 we introduce the concept of restricted singular values (RSV's) for the restricted error matrix $E = BDC$ and discuss the problem of rank determination of matrices. In section 3 we consider two special cases of RSV's, i.e. SV's and GVS's. In section 4 we derive the main result of this paper which we call the RSVD theorem of matrix triplets. Section 5 summarizes the paper and gives some comments concerning the further research of RSVD. Although only 2-norm is used in this paper, we note that the results of this paper can be extended to the case of unitarily invariant norms [5].

Notations. In this paper, only the complex matrices are considered while the case of real matrices can be considered similarly. Throughout the paper $\mathbb{C}^{m\times n}$ denotes the set of all $m \times n$ complex matrices. The matrix A^H is the complex conjugate transpose of A, $\|\cdot\|$ and $\|\cdot\|_F$ are the 2-norm and Frobenius norm respectively. I_s represents the identity matrix of order s, O with different subscripts and superscripts (e.g. $O_A^{(1)}$) denotes zero matrices of different dimensions. Sometimes we just use I and O to denote identity matrix or zero matrix of different dimensions when their dimension is not important or clear from the context.

Note. Originally we used the name "Structured Singular Values" for the concept introduced in this paper. Some people, especially B. De Moor, G. Golub and S. Van Huffel, brought to our attention that the name was already used in control theory under a different setting. Therefore we adopt here the name "Restricted Singular Values" which was suggested by B. De Moor and G. Golub.

2. Restricted Singular Values and Rank Determination of Matrices

Let $A \in \mathbb{C}^{m\times n}$ and the error matrix be of the form $E = BDC$, where $B \in \mathbb{C}^{m\times p}$, $D \in \mathbb{C}^{p\times q}$ and $C \in \mathbb{C}^{q\times n}$.

Definition 2.1. The restricted singular values (RSV's) of the matrix triplet (A, B, C) are defined as follows:

$$\sigma_k(A,B,C) = \min_{D\in\mathbb{C}^{p\times q}} \{\|D\|_2 | \text{ rank } (A+BDC) \leq k-1\} \qquad k=1,\ldots,n \tag{2.1}$$

Remark 2.2. If for some $k(1 \leq k \leq n)$, there is no $D \in \mathbb{C}^{p\times q}$ such that rank $(A+BDC) \leq k-1$ then $\sigma_k(A,B,C)$ is defined to be ∞.

Remark 2.3. For notational convenience, we simply define $\sigma_k(A,B,C) = 0$, for $k = n - \min(m,n),\ldots,n$.

Remark 2.4. It is easy to verify that the RSV's are arranged in nondecreasing order, i.e.

$$\sigma_k(A,B,C) \geq \sigma_{k+1}(A,B,C) \qquad k=1,\ldots,n-1 \tag{2.2}$$

We now briefly discuss the connection of RSV's and rank determination of matrices. The problem is to estimate the rank of

$$A_0 = A + BDC$$

where (A,B,C) is known and in addition $\| D \|_2 \leq \varepsilon$.
Assume further that the following inequalities for ε hold:

$$\sigma_1(A,B,C) \geq \ldots \geq \sigma_{k+2}(A,B,C) > \varepsilon \geq \sigma_{k+1}(A,B,C) \geq \ldots \geq \sigma_n(A,B,C)$$

then the best possible estimation of the rank of A_0 is k, in the following sense that there exists a matrix D_0, satisfying $\| D_0 \|_2 \leq \varepsilon$ such that

$$\text{rank } (A + BD_0C) = k$$

but there exists no D satisfying $\| D \|_2 \leq \varepsilon$ such that

$$\text{rank } (A + BDC) < k.$$

Such strategy of estimation is also used in the determination of numerical rank [4,7].

3. Singular Values and Generalized Singular Values

In this section we discuss two special cases of GSV's, i.e.

$$\begin{array}{lccc} 1. & B = I_m & \text{and} & C = I_n \\ 2. & B = I_m & \text{or} & C = I_n \end{array}$$

we will show that the RSV's of the matrix triplet (A, B, C) corresponding to these two special cases are just the wellknown SV's and GSV's respectively.

3.1 Singular Values of a Complex Matrix

We first cite the following result:

Theorem 3.1 [4,5] Let the SV's of A be

$$\sigma_1 \geq \ldots \geq \sigma_n \geq 0 \tag{3.1}$$

then

$$\sigma_k = \min_{E \in \mathbb{C}^{m \times n}} \{ \| E \|_2 | \text{ rank } (A + E) \leq k - 1 \} \qquad k = 1, \ldots, n \tag{3.2}$$

We note that remark 2.3 is also applicable here, i.e. we simply define $\sigma_k = 0$ for $k = n - \min(m, n), \ldots, n$. Using the notations of definition 2.1 we can rewrite theorem 3.1 as

Corollary 3.2.

$$\sigma_k(A, I_m, I_n) = \sigma_k \qquad k = 1, \ldots, n \tag{3.3}$$

It is also easy to establish the following inequalities.

Corollary 3.3. Assume that $B \neq 0$ and $C \neq 0$ then

$$\sigma_k \leq \| B \|_2 \| C \|_2 \sigma_k(A, B, C) \qquad k = 1, \ldots, n \tag{3.4}$$

3.2 Generalized Singular Values

We only consider the case that $B = I_m$ and C is a general complex matrix. The error matrix is now $E = DC$. The case that B is a general matrix and $C = I_n$ can be discussed similarly.

The concept of GSV's of matrix pencils was introduced by Van Loan [8] (where he used the name B-SV's). Paige and Saunders provided a slight generalization of Van Loan's result in order to treat all the possible cases [6]. Since GSV's have many applications in numerical linear algebra problems and thus are of their own interests, here we give an alternative derivation of the so called generalized singular value decomposition (GSVD) of matrix pencils, in which the two matrices have the same number of columns. Our approach here is different from Van Loan's, Paige's and Saunders'.

Theorem 3.4. [6,8]. Let $A \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{q\times n}$, then there exist unitary matrices U and V and nonsingular matrix Q such that

$$UAQ = \underset{\quad k \quad n-k}{(\Sigma_A, O)} \qquad VCQ = \underset{\quad k \quad n-k}{(\Sigma_C, O)} \tag{3.5}$$

$$\Sigma_A = \begin{pmatrix} I_r & & \\ & S_A & \\ & & O_A \end{pmatrix} \qquad \Sigma_C = \begin{pmatrix} O_C & & \\ & S_C & \\ & & I_{k-r-s} \end{pmatrix} \tag{3.6}$$

where

$$S_A = \mathrm{diag}\,(\alpha_{r+1}, \ \ldots, \alpha_{r+s}), S_C = \mathrm{diag}\,(\beta_{r+1}, \ldots, \beta_{r+s})$$

and

$$\begin{array}{ll} 1 > \alpha_{r+1} \geq \ldots \geq \alpha_{r+s} > 0, & 0 < \beta_{r+1} \leq \ldots \leq \beta_{r+s} < 1 \\ \alpha_i^2 + \beta_i^2 = 1 & i = r+1, \ldots, r+s \end{array} \tag{3.7}$$

Proof. The proof is constructive and consists of four steps. The transformations of each step are of the following form

$$A^{(k+1)} = U^{(k)}A^{(k)}Q^{(k)}, \ C^{(k+1)} = V^{(k)}C^{(k)}Q^{(k)}$$

where $U^{(k)}$ and $V^{(k)}$ are unitary matrices and $Q^{(k)}$ nonsingular. In each step we only specify the $U^{(k)}$, $V^{(k)}$ and $Q^{(k)}$ and the resulted matrices $A^{(k+1)}$ and $C^{(k+1)}$. Set $A^{(1)} = A$ and $C^{(1)} = C$.

Step 1. Let the SVD of C be $U_1 C V_1 = \text{diag}\ (O, \Sigma_C^{(1)})$, where $\Sigma_C^{(1)} = \text{diag}\ (s_1, \ldots, s_t)$ and $s_1 \geq \ldots \geq s_t > 0$. Set

$$U^{(1)} = I, \quad V^{(1)} = U_1$$
$$Q^{(1)} = V_1 \text{diag}\ (I, \Sigma_C^{-1})$$

then

$$A^{(2)} = \begin{array}{c} \begin{pmatrix} A_1^{(2)} & A_2^{(2)} \end{pmatrix} \\ \begin{array}{cc} n-t & t \end{array} \end{array}$$

$$C^{(2)} = \begin{pmatrix} O & O \\ O & I_t \end{pmatrix}$$

Step 2. Let the SVD of $A_1^{(2)}$ be $U_2 A_1^{(2)} V_2 = \text{diag}\ (\Sigma_A^{(2)}, O)$ where $\Sigma_A^{(2)} = \text{diag}\ (t_1, \ldots, t_r)$ and $t_1 \geq \ldots \geq t_r > 0$. Set

$$U^{(2)} = U_2, \quad V^{(2)} = I$$
$$Q^{(2)} = \text{diag}\ (V_2, I) \text{diag}\ ((\Sigma_A^{(2)})^{-1}, I)$$

then

$$A^{(3)} = \begin{array}{c} \begin{pmatrix} I_r & O & A_{13}^{(3)} \\ O & O & A_{23}^{(3)} \end{pmatrix} \\ \begin{array}{ccc} r & n-r-t & t \end{array} \end{array}$$

$$C^{(3)} = C^{(2)}$$

Step 3. Let the SVD of $A_{23}^{(3)}$ be $U_3 A_{23}^{(3)} V_3 = \text{diag}\ (\Sigma_A^{(3)}, O)$ where $\Sigma_A^{(3)} = \text{diag}\ (w_1, \ldots, w_s)$ and $w_1 \geq \ldots \geq w_s > 0$. Let $\alpha_i = w_i(1 + w_i^2)^{-\frac{1}{2}}$ and $\beta_i = (1 + w_i^2)^{-\frac{1}{2}}$, $i = r+1, \ldots, r+s$, and $S_A = \text{diag}\ (\alpha_{r+1}, \ldots, \alpha_{r+s}), S_C = \text{diag}\ (\beta_{r+1}, \ldots, \beta_{r+s})$. It is easy to check that α_i, β_i $(i = r+1, \ldots, r+s)$ satisfy (3.7). Set

$$U^{(3)} = \text{diag}\ (I, U_3), \quad V^{(3)} = \text{diag}\ (I, V_3^H)$$
$$Q^{(3)} = \begin{pmatrix} I & -A_{13}^{(3)} \\ O & I \end{pmatrix} \text{diag}\ (I, V_3) \text{diag}\ (I, S_C, I)$$

then

$$A^{(4)} = \begin{array}{c} \begin{pmatrix} I_r & O & & \\ & & S_A & \\ & & & O \end{pmatrix} \\ \begin{array}{cccc} r & n-r-t & s & t-s \end{array} \end{array}$$

$$C^{(4)} = \begin{array}{c} \begin{pmatrix} O & & \\ & S_C & \\ & & I_{k-r-s} \end{pmatrix} \\ \begin{array}{ccc} n-t & s & t-s \end{array} \end{array}$$

Step 4. After suitable permutations P_1 and P_2 and set $k = t + r$ we obtain

$$A^{(5)} = A^{(4)} P_1 = \left(\begin{array}{ccc|c} I_r & & & \\ & S_A & & O \\ & & O_A & \end{array} \right)$$

$$C^{(5)} = P_2 C^{(4)} P_1 = \left(\begin{array}{ccc|c} O_C & & & \\ & S_C & & O \\ & & I_{k-r-s} & \end{array} \right)$$

which completes the proof. ■

According to [6], corresponding to each column in (3.5) is ascribed a generalized singular pair (α_i, β_i). Following (3.6) we take for the first k of those as

$$\alpha_i = 1, \quad \beta_i = 0, \quad i = 1, \ldots, r \tag{3.8.a}$$

$$\alpha_i, \beta_i \text{ as in } S_A \text{ and } S_B \quad i = r+1, \ldots, r+s \tag{3.8.b}$$

$$\alpha_i = 0, \quad \beta_i = 1, \quad i = r+s+1, \ldots, k \tag{3.8.c}$$

and call them the nontrivial generalized singular pairs of (A, C); $\frac{\alpha_i}{\beta_i}(i = 1, \ldots, k)$ are called GSV's of (A, C). The other $n - k$ pairs corresponding to the zero columns in (3.5) are called trivial generalized singular pairs of (A, C).

The following result gives a new characterization of the GSV's of a general matrix pencil and states that GSV's are a special case of RSV's.

Theorem 3.5. With the notation as used in definition 2.1 and theorem 3.4 we have the following results

1.
$$\sigma_i(A, I_m, C) = \frac{\alpha_i}{\beta_i} \qquad i = 1, \ldots, k \tag{3.9.a}$$

and

$$\sigma_i(A, I_m, C) = 0 \qquad i = k+1, \ldots, n \tag{3.9.b}$$

2. let $l =$ rank $\begin{pmatrix} A \\ C \end{pmatrix}$ $-$ rank (C) and $u = \min\left(m, \text{ rank } \begin{pmatrix} A \\ C \end{pmatrix}\right)$ then $\forall D \in \mathbb{C}^{m\times q}$

$$l \leq \text{ rank } (A + DC) \leq u \tag{3.10}$$

and $\forall k$ integer satisfying $l \leq k \leq n$, there exists matrix $D_k \in \mathbb{C}^{m\times q}$ such that

$$\text{rank } (A + D_k C) = k.$$

Proof. Let the GSVD of (A, C) be as in theorem 3.4. For arbitrary $D \in \mathbb{C}^{m\times q}$, let $UDV^H = (D_{ij})_{i,j=1}^3$ be partioned conformally with that of Σ_A and Σ_C, then

$$\begin{aligned}
& \text{rank } (A + DC) \\
= \; & \text{rank } (UAQ + UDV^H VCQ). \\
= \; & \text{rank } \left\{ \begin{pmatrix} I_r & D_{12}S_C & D_{13} & O \\ O & S_A + D_{22}S_C & D_{23} & O \\ O & D_{32}S_C & D_{33} & O \end{pmatrix} \right\} \\
= \; & r + \text{ rank } \left(\begin{pmatrix} S_A S_C^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} D_{22} & D_{23} \\ D_{32} & D_{33} \end{pmatrix} \right)
\end{aligned}$$

using theorem 3.1, the proof of this part is completed.
One can verify that

$$k = \text{ rank } \begin{pmatrix} A \\ C \end{pmatrix}$$
$$r = \text{ rank } \begin{pmatrix} A \\ C \end{pmatrix} - \text{ rank } (C).$$

The proof of this part can be easily derived from these expressions. ∎

In the following we discuss the problem of uniqueness of GSVD. From the GSVD in theorem 3.4, let

$$U_i A Q_i = (\Sigma_A, O) \quad V_i C Q_i = (\Sigma_C, O) \ \ (i = 1, 2)$$

then

$$(U_2U_1^H)(\Sigma_A, O) = (\Sigma_A, O)(Q_2^{-1}Q_1) \tag{3.11.a}$$

$$(V_2V_1^H)(\Sigma_C, O) = (\Sigma_C, O)(Q_2^{-1}Q_1)\,. \tag{3.11.b}$$

Let

$$U_2U_1^H = (U_{ij})_{i,j=1}^3\,,\quad V_2V_1^H(V_{ij})_{i,j=1}^3$$

and

$$Q_2^{-1}Q_1 = (Q_{ij})_{i,j=1}^4$$

be block matrices partitioned conformally with the partitions of Σ_A and Σ_C. From (3.11.a)

$$\begin{pmatrix} U_{11} & U_{12}S_A & O & O \\ U_{21} & U_{22}S_A & O & O \\ U_{31} & U_{32}S_A & O & O \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ S_AQ_{21} & S_AQ_{22} & S_AQ_{23} & S_AQ_{24} \\ O & O & O & O \end{pmatrix}$$

hence

$$\begin{aligned} &U_{31} = O,\ U_{32} = O,\ Q_{13} = O,\ Q_{14} = O,\ Q_{23} = O,\ Q_{24} = O \\ &U_{11} = Q_{11},\ U_{12}S_A = Q_{12},\ U_{21} = S_AQ_{21},\ U_{22}S_A = S_AQ_{22} \end{aligned}$$

Since $U_2U_1^H$ is unitary, therefore $U_{13} = O$, $U_{23} = O$. From (3.11.b)

$$\begin{pmatrix} O & V_{12}S_C & V_{13} & O \\ O & V_{22}S_C & V_{23} & O \\ O & V_{32}S_C & V_{33} & O \end{pmatrix} = \begin{pmatrix} O & O & O & O \\ S_CQ_{21} & S_CQ_{22} & S_CQ_{23} & S_CQ_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \end{pmatrix}$$

hence

$$\begin{aligned} &V_{12} = O,\ V_{13} = O,\ Q_{21} = O,\ Q_{31} = O,\ Q_{34} = O \\ &V_{22}S_C = S_CQ_{22},\ V_{23} = S_CQ_{23},\ V_{32}S_C = Q_{32},\ Q_{33} = V_{33}. \end{aligned}$$

Since $V_2V_1^H$ is unitary, therefore $V_{21} = O$, $V_{31} = O$. Furthermore since $U_{21} = S_AQ_{21} = O$, hence $U_{12} = O$ and $Q_{12} = O$. Since $V_{32} = Q_{32}S_C^{-1} = O$, hence $V_{23} = O$ and $Q_{23} = O$. From $P_{22} = S_A^{-1}U_{22}S_A$ and $P_{22} = S_C^{-1}V_{22}S_C$, we obtain

$$(S_AS_C^{-1})\ V_{22} = U_{22}(S_AS_C^{-1}).$$

Let $\sigma_i = \dfrac{\alpha_{i+r}}{\beta_{i+r}}$ $i = 1, \ldots, s$ and $\Sigma := S_AS_C^{-1} = \operatorname{diag}(\sigma_{i_1}I_{s_1}, \ldots, \sigma_{i_l}I_{s_l})$ where $\sigma_{i_1} > \ldots > \sigma_{i_l}$ and $\sum_{t=1}^{l} s_t = s$. Since $\alpha_{i+r}^2 + \beta_{i+r}^2 = 1$ $i = 1, \ldots, s$. Hence S_A and S_C have the same partition as that of Σ, i.e.

$$S_A = \operatorname{diag}(\alpha_{i_1}I_{s_1}, \ldots, \alpha_{i_l}I_{s_l}),\ S_C = (\beta_{i_1}I_{s_1}, \ldots, \beta_{i_l}I_{s_l}).$$

From $\Sigma V_{22} = U_{22}\Sigma$, we can verify

$$\Sigma^2 V_{22} = V_{22}\Sigma^2 \ , \ \ \Sigma^2 U_{22} = U_{22}\Sigma^2$$

therefore

$$U_{22} = V_{22} = \text{diag}\ (\tilde{U}_1, \ldots, \tilde{U}_l)$$

where $\tilde{U}_i \ \ (i = 1, \ldots, l)$ is unitary matrix of order s_i.
Summarize the above we obtain

$$Q_1 = Q_2 \left(\begin{array}{ccc|c} U_{11} & & & \\ & U_{22} & & O \\ & & V_{33} & \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{array} \right) \qquad U_1^H = U_2^H \text{diag}\ (U_{11}, U_{22}, U_{33})$$

and

$$V_1^H = V_2^H \text{diag}\ (V_{11}, U_{22}, V_{33}) \tag{3.11.c}$$

where $U_{11}, U_{22}, U_{33}, V_{11}, V_{33}$ are unitary; Q_{44} is nonsingular and $U_{22} = \text{diag}\ (\tilde{U}_1, \ldots, \tilde{U}_l)$.
As pointed out in [6], the GSV's of (A, C) are just the SV's of AC^{-1}, if C is nonsingular. In the following we further discuss the case that C is a general matrix.

Corollary 3.6. Using the notations as in theorem 3.4 and let

$$C_A^+ = Q \begin{pmatrix} O_C^H & & \\ & S_C^{-1} & \\ & & I \end{pmatrix} V \ .$$

If rank $\begin{pmatrix} A \\ C \end{pmatrix} = n$, then C_A^+ is uniquely defined and the SV's of AC_A^+ contains the noninfinite GSV's of (A, C).

Proof. Since rank $\begin{pmatrix} A \\ C \end{pmatrix} = n$, any two sets of transformations in theorem 3.4 satisfy the following relations

$$Q_1 = Q_2 \text{diag}\ (U_{11}, U_{22}, V_{33}), \ U_1^H = U_2^H \text{diag}\ (U_{11}, U_{22}, U_{33})$$

and $V_1^H = V_2^H \text{diag}\,(V_{11}, U_{22}, V_{33})$, hence

$$
\begin{aligned}
& Q_1 \begin{pmatrix} O_C^H & & \\ & S_C^{-1} & \\ & & I \end{pmatrix} V_1 \\
= \; & Q_2 \begin{pmatrix} U_{11} & & \\ & U_{22} & \\ & & V_{33} \end{pmatrix} \begin{pmatrix} O_C^H & & \\ & S_C^{-1} & \\ & & I \end{pmatrix} \begin{pmatrix} V_{11}^H & & \\ & U_{22}^H & \\ & & V_{33}^H \end{pmatrix} V_2 \\
= \; & Q_2 \begin{pmatrix} O_C^H & & \\ & S_C^{-1} & \\ & & I \end{pmatrix} V_1
\end{aligned}
$$

Therefore we have proved that C_A^+ is well-defined. Furthermore observe that

$$U A C_A^+ V^H = \text{diag}\,(O, S_A S_C^{-1}, O)$$

and only the ∞ GSV's of (A, C) are changed to zero SV's of AC_A^+, the other GSV's are preserved in AC_A^+. ■

In the following we discuss some properties of C_A^+. It is easy to check that C_A^+ satisfies the following equations

$$C C_A^+ C = C \tag{3.12.a}$$

$$C_A^+ C C_A^+ = C_A^+ \tag{3.12.b}$$

$$(C C_A^+)^H = C C_A^+ \,. \tag{3.12.c}$$

Therefore in the notations of [1], C_A^+ is *a* {1,2,3}-inverse of C. It is interesting to know how one can uniquely characterize C_A^+ in the class of {1,2,3}-inverse of C. The following theorem answers this question under the condition that rank $\begin{pmatrix} A \\ C \end{pmatrix} = n$.

Theorem 3.7. If $\begin{pmatrix} A \\ C \end{pmatrix}$ is of full column rank, then C_A^+ is the unique solution of the following constrained minimization problem.

$$\min_{X \in \mathbb{C}^{n \times q}} \|AX\|_F \tag{3.13}$$

Subject to

$$CXC = C \tag{3.14.a}$$

$$XCX = X \tag{3.14.b}$$

$$(CX)^H = CX \tag{3.14.c}$$

The minimum value is $\sqrt{\sum_{i=r+1}^{r+s} (\frac{\alpha_i}{\beta_i})^2}$.

Proof. Let C have the decomposition as in (3.5)

$$C = V^H(\underset{k}{\Sigma_C}, \underset{n-k}{O})Q^{-1}$$

Since rank $\begin{pmatrix} A \\ C \end{pmatrix} = n$, so $k = n$ and $C = V^H \Sigma_C Q^{-1}$. Partition $Q^{-1}XV^H = (X_{ij})_{i,j=1}^3$ conformally with that of Σ_A and Σ_C. One can verify that X should be of the following form

$$X = Q \begin{pmatrix} O & X_{12} & X_{13} \\ O & S_C^{-1} & O \\ O & O & I_{n-r-s} \end{pmatrix} V$$

in order to satisfy (3.14.a) – (3.14.c).
Since

$$\begin{aligned}
& \|AX\|_F^2 \\
= \; & \|UAQ\, Q^{-1}XV^H\|_F^2 \\
= \; & \left\| \begin{pmatrix} I_r & & \\ & S_A & \\ & & O_A \end{pmatrix} \begin{pmatrix} O & X_{12} & X_{13} \\ O & S_C^{-1} & O \\ O & O & I_{n-r-s} \end{pmatrix} \right\|_F^2 \\
= \; & \|(X_{12}, X_{13})\|_F^2 + \|S_A S_C^{-1}\|_F^2 \\
\geq \; & \|S_A\, S_C^{-1}\|_F^2 \\
= \; & \sum_{i=r+1}^{r+s} (\frac{\alpha_i}{\beta_i})^2
\end{aligned}$$

The equality is satisfied if and only if $X_{12} = O$ and $X_{13} = O$, i.e. $X = C_A^+$. ■

Remark 3.9. Along the lines in the proof of theorem 3.7 we can also verify that C_A^+ is the unique solution of the following constrained minimization problem

$$\min_{X \in \mathbb{C}^{q \times n}} \|AX\|_F$$

subject to

$$\begin{aligned} &1. \quad CXC = C \\ &2. \quad (CX)^H = CX \end{aligned}$$

Remark 3.10. Exchange the rolls of A and C in (3.13) and (3.14), one can also show that

$$A_C^- := Q \begin{pmatrix} I & & \\ & S_C & \\ & & O \end{pmatrix} U$$

is the unique solution of the corresponding minimization problem.
Another way of uniquely characterizing C_A^+ is to generalize the Moore-Penrose conditions.

Theorem 3.11. If $\begin{pmatrix} A \\ C \end{pmatrix}$ has full column rank, then C_A^+ is the unique solution of the following four equations

$$CXC = C \tag{3.15.a}$$

$$XCX = X \tag{3.15.b}$$

$$(CX)^H = CX \tag{3.15.c}$$

$$(A^H AXC)^H = A^H AXC \tag{3.15.d}$$

Proof. As in the proof of theorem 3.7 X should be of the following form

$$X = Q \begin{pmatrix} O & X_{12} & X_{13} \\ O & S_C^{-1} & O \\ O & O & O \end{pmatrix} V$$

in order to satisfy (3.15.a) – (3.15.c). Since

$$
\begin{aligned}
A^H AXC &= Q^{-H} \begin{pmatrix} I & & \\ & S_A & \\ & & O \end{pmatrix} UU^H \begin{pmatrix} I & & \\ & S_A & \\ & & O \end{pmatrix} Q^{-1}Q \\
&\quad \begin{pmatrix} O & X_{12} & X_{13} \\ O & S_C^{-1} & O \\ O & O & I \end{pmatrix} VV^T \begin{pmatrix} O & & \\ & S_C^{-1} & \\ & & I \end{pmatrix} Q^{-1} \\
&= Q^{-H} \begin{pmatrix} O & X_{12} & X_{13} \\ O & S_A^2 S_C^{-2} & O \\ O & O & O \end{pmatrix} Q^{-1}
\end{aligned}
$$

hence $(A^H AXC)^H = A^H AXC$ if and only if $X_{12} = O$ and $X_{13} = O$ i.e. $X = C_A^+$. ∎

4. Restricted Singular Value Decomposition

In this section, B and C will be assumed to be general matrices. The key observation is the following

Lemma 4.1. Let $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ be nonsingular matrices, $U \in \mathbb{C}^{p\times p}$ and $V \in \mathbb{C}^{q\times q}$ be unitary matrices, then

$$\sigma_k(PAQ, PBU, VCQ) = \sigma_k(A, B, C) \quad k = 1, \ldots, n \tag{4.1}$$

This lemma specifies the class of transformations which preserves the RSV's of a matrix triplet.

Theorem 4.2. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$, then there exist nonsingular matrices $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$, unitary matrices $U \in \mathbb{C}^{p\times p}$ and $V \in \mathbb{C}^{q\times q}$ such that

$$PAQ = \begin{array}{c@{}c} \begin{array}{c} \\ s_2 \\ t_2 \end{array} & \begin{pmatrix} \Sigma_A & & \\ & O_A^{(1)} & \\ & & O_A^{(2)} \end{pmatrix} \\ & \begin{array}{ccc} & s_1 & t_1 \end{array} \end{array} \tag{4.2.a}$$

$$PBU = \begin{array}{c@{}c} \begin{array}{c} \\ t_2 \end{array} & \begin{pmatrix} \Sigma_B \\ O_B^{(2)} \end{pmatrix} \end{array} \tag{4.2.b}$$

$$VCQ = \begin{array}{c} \begin{pmatrix} \Sigma_C, & O_C^{(2)} \end{pmatrix} \\ \begin{array}{cc} & t_1 \end{array} \end{array} \tag{4.2.c}$$

$$\Sigma_A = \begin{array}{c} \begin{pmatrix} I_j & & & \\ & I_k & & \\ & & I_l & \\ & & & S_A \end{pmatrix} \\ \begin{array}{cccc} & & & r \end{array} \end{array} \tag{4.3.a}$$

$$\Sigma_B = \begin{array}{cc} k+l & \begin{pmatrix} I_j & & & \\ & O_B^{(1)} & & \\ & & S_B & \\ & & & I_{s_2} \end{pmatrix} \\ & \begin{array}{cc} p-j-r-s_2 & r \end{array} \end{array} \tag{4.3.b}$$

$$\Sigma_C = \begin{array}{cc} q-l-r-s_1 & \begin{pmatrix} O_C^{(1)} & & & \\ & I_l & & \\ & & S_C & \\ & & & I_{s_1} \end{pmatrix} \\ & \begin{array}{cc} j+k & r \end{array} \end{array} \tag{4.3.c}$$

where $S_A = \text{diag }(\alpha_i), \;\; S_B = \text{diag }(\beta_i), \;\; S_C = \text{diag }(\gamma_i)$

$$\alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1 \quad i = s+1, \ldots, s+r \tag{4.4.a}$$

where we denote $s = j + k + l$

$$1 > \alpha_i \geq \alpha_{i+1} > 0 \; ; \; 0 < \beta_i \leq \beta_{i+1} < 1 \; ; \; 1 > \gamma_i \geq \gamma_{i+1} > 0 \tag{4.4.b}$$

and

$$\frac{\alpha_i}{\beta_i \gamma_i} \geq \frac{\alpha_{i+1}}{\beta_{i+1}\gamma_{i+1}} \quad i = s+1, \ldots, s+r-1 \tag{4.4.c}$$

Proof. The proof is constructive and consists of four steps. The transformations of each step, according to lemma 4.1, are of the following form

$$A^{(k+1)} = P^{(k)} A^{(k)} Q^{(k)} \tag{4.5.a}$$

$$B^{(k+1)} = P^{(k)} B^{(k)} U^{(k)} \tag{4.5.b}$$

$$C^{(k+1)} = V^{(k)} C^{(k)} Q^{(k)} \tag{4.5.c}$$

where $P^{(k)}$ and $Q^{(k)}$ are nonsingular matrices, $U^{(k)}$ and $V^{(k)}$ are unitary matrices. In each step we only specify the $P^{(k)}$, $Q^{(k)}$, $U^{(k)}$ and $V^{(k)}$ and the resulted $A^{(k+1)}$, $B^{(k+1)}$ and $C^{(k+1)}$. Set

$$A^{(1)} = A, \;\; B^{(1)} = B, \;\; C^{(1)} = C$$

Step 1. Using theorem 3.4, let the GSVD of $(A^{(1)}, C^{(1)})$ be

$$U_1A^{(1)}Q_1 = \begin{array}{c} \left(\begin{array}{ccc|c} I_{j+k} & & & \\ & S_A^{(1)} & & O \\ & & O & \end{array}\right) \\ \begin{array}{ccc} l+r & s_1 & t_1 \end{array} \end{array}$$

$$V_1C^{(1)}Q_1 = \begin{array}{c} \left(\begin{array}{ccc|c} O & & & \\ & S_C^{(1)} & & O \\ & & I_{s_1} & \end{array}\right) \\ \begin{array}{cccc} j+k & l+r & s_1 & t_1 \end{array} \end{array}$$

Set

$$\begin{array}{ll} P^{(1)} = U_1 & Q^{(1)} = Q_1 \ \mathrm{diag}\,(I, (S_C^{(1)})^{-1}, I) \\ U^{(1)} = I & V^{(1)} = V_1 \end{array}$$

then

$$A^{(2)} = \begin{array}{c} \left(\begin{array}{ccc|c} I_{j+k} & & & \\ & S_A^{(1)}(S_C^{(1)})^{-1} & & O \\ & & O & \end{array}\right) \\ \begin{array}{ccc} l+r & s_1 & t_1 \end{array} \end{array}$$

$$B^{(2)} = \begin{array}{c} j+k \\ \\ \end{array} \begin{pmatrix} B_1^{(2)} \\ B_2^{(2)} \end{pmatrix}$$

$$C^{(2)} = \begin{array}{c} \left(\begin{array}{ccc|c} O & & & \\ & I_{l+r} & & O \\ & & I_{s_1} & \end{array}\right) \\ \begin{array}{cc} j+k & t_1 \end{array} \end{array}$$

Step 2. Using theorem 3.4 let the GSVD of $\left(\begin{pmatrix} S_A^{(1)}(S_C^{(1)})^{-1} \\ O \end{pmatrix}, B_2^{(2)}\right)$ be

$$P_2 \begin{pmatrix} S_A^{(1)}(S_C^{(1)})^{-1} \\ O \end{pmatrix} V_2 = \begin{array}{c} \\ \\ s_2+t_2 \\ \\ \end{array} \begin{array}{c} \left(\begin{array}{cc} I_l & \\ & S_4^{(2)} \\ \hline & O \end{array}\right) \\ \begin{array}{cc} l & r \end{array} \end{array}$$

$$
P_2 B_2^{(2)} U_2 = \begin{array}{c} \\ \\ \\ t_2 \end{array}
\left(\begin{array}{cc|c}
O & & \\
 & S_B^{(2)} & \\
 & & I_{s_2} \\
\hline
 & O &
\end{array}\right)
$$
$$
\qquad\qquad\qquad p-r-s_2 \quad r
$$

where $S_A^{(2)} = \text{diag}\ (s_1, \ldots, s_r)$, $S_B^{(2)} = \text{diag}\ (t_1, \ldots, t_r)$ and $s_i^2 + t_i^2 = 1$, $1 > s_i \geq \ldots \geq s_r > 0$ and $0 < t_i \leq \ldots \leq t_r < 1$. Set

$$
\begin{array}{ll}
P^{(2)} = \text{diag}\ (I, P_2) & Q^{(2)} = \text{diag}\ (I, V_2, I) \\
U^{(2)} = U_2 & V^{(2)} = \text{diag}\ (I, V_2^H, I)
\end{array}
$$

then

$$
A^{(3)} = \left(\begin{array}{cccc|c}
I_{j+k} & & & & \\
 & I_l & & & O \\
 & & S_A^{(2)} & & \\
 & & & O &
\end{array}\right)
$$
$$
\qquad\qquad\qquad s_1 \quad t_1
$$

$$
B^{(3)} = \left(\begin{array}{ccc}
B_1^{(3)} & B_2^{(3)} & B_3^{(3)} \\
O & & \\
 & S_B^{(2)} & \\
 & & I_{s_2} \\
\hline
 & O &
\end{array}\right)
$$
$$
\qquad\qquad p-r-s_2 \quad r
$$

and

$$
C^{(3)} = C^{(2)}
$$

Step 3. Set

$$
P^{(3)} = \begin{pmatrix}
I & & -B_2^{(3)}(S_B^{(2)})^{-1} & -B_3^{(3)} & \\
 & I & & & \\
 & & I & & \\
 & & & I & \\
 & & & & I
\end{pmatrix}
Q^{(3)} = \begin{pmatrix}
I & & -B_2^{(3)}(S_B^{(2)})^{-1} S_A^{(2)} & \\
 & I & & \\
 & & I & \\
 & & & I
\end{pmatrix}
$$

$U^{(3)} = I$ and $V^{(3)} = I$

then

$$\begin{aligned}
A^{(4)} &= A^{(3)} \\
B^{(4)} &= \left(\begin{array}{ccc} B_1^{(3)} & & \\ O & & \\ & S_B^{(2)} & \\ & & I_{s_2} \\ \hline & O & \end{array} \right) \\
C^{(4)} &= C^{(3)}
\end{aligned}$$

Step 4. Let the SVD of $B_1^{(3)}$ be

$$U_3 B_1^{(3)} V_3 = \begin{array}{c} j \\ k \end{array} \begin{pmatrix} \Sigma_B^{(2)} & O \\ O & O \end{pmatrix}$$

where $\Sigma_B^{(2)}$ is nonsingular. Let $s = j + k + l$ and

$$\begin{aligned}
\alpha_{s+i} &= \frac{s_i^2}{(1+s_i^2)^{\frac{1}{2}}} \\
\beta_{s+i} &= t_i \\
\gamma_{s+i} &= \frac{s_i}{(1+s_i^2)^{\frac{1}{2}}} \qquad i = 1, \ldots, r
\end{aligned}$$

It is easy to verify that $\{\alpha_{s+i}\}$, $\{\beta_{s+i}\}$ and $\{\gamma_{s+i}\}$ satisfy (4.4). Let $S_C =$ diag (γ_{s+i}), $S_A = S_A^{(2)} S_C$ and $S_B = S_B^{(2)}$, in addition set

$$\begin{aligned}
P^{(4)} &= \text{diag}\,((\Sigma_B^{(2)})^{-1}, I)\text{diag}\,(U_3, I) \\
Q^{(4)} &= \text{diag}\,(I, S_C, I)\text{diag}\,((\Sigma_B^{(2)})^{-1}, I)\text{diag}\,(U_3^H, I) \\
U^{(4)} &= \text{diag}\,(V_3, I) \text{ and} \\
V^{(4)} &= I
\end{aligned}$$

After some manipulation, we obtain the results as stated in (4.2) and (4.4). The proof is completed. ∎

Remark 4.3. We can also use D_1 and D_2 positive definite diagonal matrices to scale (S_A, S_B, S_c) to $(D_1 S_A D_2, D_s S_B, S_C D_2)$. For example, we can choose D_1 and D_2 such that $D_1 S_B$ and $S_C D_2$ are identity matrices.

Similar to (3.8) we define

$$\alpha_i = 1, \quad \beta_i = 1, \quad \gamma_i = 0 \qquad i = 1,\ldots,j \tag{4.6.a}$$

$$\alpha_i = 1, \quad \beta_i = 0, \quad \gamma_i = 0 \qquad i = j+1,\ldots,j+k \tag{4.6.b}$$

$$\alpha_i = 1, \quad \beta_i = 0, \quad \gamma_i = 1 \qquad i = j+k+1,\ldots,s \tag{4.6.c}$$

$$\alpha_i, \beta_i, \gamma_i \text{ as in } S_A, S_B \text{ and } S_C \quad i = s+1,\ldots,s+r \tag{4.6.d}$$

$$\alpha_i = 0, \quad \beta_i = 1, \quad \gamma_i = 1 \qquad i = s+r+1,\ldots,s+r+\min(s_1,s_2) \tag{4.6.e}$$

to be the nontrivial RSV triplets of (A, B, C).
The following theorem relates theorem 4.2 with the concept of RSV's and justifies the definition of (4.6) and calling theorem 4.2 the RSVD theorem.

Theorem 4.4. With the notations as in theorem 4.2 and (4.6) the following statements are true:

1.
$$\sigma_i(A,B,C) = \frac{\alpha_i}{\beta_i \gamma_i} \quad i = 1,\ldots,s+r+\min(s_1,s_2) \tag{4.7.a}$$

$$\sigma_i(A,B,C) = 0 \quad i = n-(s+r+\min(s_1,s_2))+1,\ldots,n \tag{4.7.b}$$

2. Let
$$l = \text{ rank } (A,B) + \text{ rank } \begin{pmatrix} A \\ B \end{pmatrix} - \text{ rank } \begin{pmatrix} A & B \\ C & O \end{pmatrix}$$

$$u = \min \left(\text{rank } (A,B), \text{ rank } \begin{pmatrix} A \\ C \end{pmatrix} \right)$$

then $\forall D \in \mathbb{C}^{p\times q}$
$$l \leq \text{ rank } (A + BDC) \leq u \tag{4.8}$$
and $\forall k$ integer satisfying $l \leq k \leq n$, there exists matrix $D_k \in \mathbb{C}^{p\times q}$ such that

$$\text{rank } (A + BD_kC) = k \ .$$

Proof. 1. Let $U^H D V^H = (D_{ij})_{i,j=1}^4$ be a block matrix partitioned conformally with that of Σ_B and Σ_C.

$$
\begin{aligned}
& \text{rank}\,(A + BDC) \\
= \ & \text{rank}\,(PAQ + PBUU^H DV^H VCQ) \\
= \ & \text{rank} \begin{pmatrix} I_j & O & D_{12} & D_{13}S_C & D_{14} & O \\ O & I_k & O & O & O & O \\ O & O & I_l & O & O & O \\ O & O & S_B D_{32} & S_A + S_B D_{33} S_C & S_B D_{34} & O \\ O & O & D_{42} & D_{44} S_C & D_{44} & O \\ O & O & O & O & O & O \end{pmatrix} \\
= \ & j + k + l + \text{rank} \left(\begin{pmatrix} S_B^{-1} S_A S_C^{-1} & O \\ O & O \end{pmatrix} + \begin{pmatrix} D_{33} & D_{34} \\ D_{43} & D_{44} \end{pmatrix} \right)
\end{aligned}
$$

using theorem 3.1., the proof of this part is completed.
2. For the upper bound, note that

$$A + BDC = (A, B) \begin{pmatrix} I \\ DC \end{pmatrix} = (I, BD) \begin{pmatrix} A \\ C \end{pmatrix}$$

hence

$$\text{rank}\,(A + BDC) \leq \text{rank}\,(A, B)$$

and

$$\text{rank}\,(A + BDC) \leq \text{rank} \begin{pmatrix} A \\ C \end{pmatrix}.$$

For the lower bound, we can verify that

$$
\begin{aligned}
\text{rank}\,(A, B) &= s + r + s_2 \\
\text{rank} \begin{pmatrix} A \\ C \end{pmatrix} &= s + r + s_1 \text{ and} \\
\text{rank} \begin{pmatrix} A & B \\ C & O \end{pmatrix} &= s + 2r + s_1 + s_2 \text{ hence} \\
s &= \text{rank}\,(A, B) + \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} - \text{rank} \begin{pmatrix} A & B \\ C & O \end{pmatrix}.
\end{aligned}
$$

∎

Remark 4.5 From the following linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} j \\ k \\ l \\ r \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \text{rank }(A) \\ \text{rank }(B) \\ \text{rank }(C) \\ \text{rank }(A,B) \\ \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} \\ \text{rank} \begin{pmatrix} A & B \\ C & O \end{pmatrix} \end{pmatrix}$$

we obtain the following expressions for the integer index:

$$\begin{aligned} j &= \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} + \text{rank }(B) - \text{rank} \begin{pmatrix} A & B \\ C & O \end{pmatrix} \\ k &= \text{rank} \begin{pmatrix} A & B \\ C & O \end{pmatrix} - \text{rank }(B) - \text{rank }(C) \\ l &= \text{rank }(A,B) + \text{rank }(C) - \text{rank} \begin{pmatrix} A & B \\ C & O \end{pmatrix} \\ r &= \text{rank} \begin{pmatrix} A & B \\ C & O \end{pmatrix} + \text{rank }(A) - \text{rank }(A,B) - \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} \\ s_1 &= \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} - \text{rank }(A) \\ s_2 &= \text{rank }(A,B) - \text{rank }(A) \end{aligned}$$

in addition, it is easy to see that

$$\begin{aligned} t_1 &= n - \text{rank} \begin{pmatrix} A \\ C \end{pmatrix} \\ t_2 &= m - \text{rank }(A,B) \end{aligned}$$

If we use $R_r(A)(R_C(A))$ and $N_r(A)(N_C(A))$ to denote the linear subspace spanned by the rows (or columns) of A and the row (column) null space of A respectively, furthermore $S \backslash T$ denotes the complement subspace of T in S, such that $S \backslash T \oplus T = S$ and $\dim(S)$ is the dimension of the subspace S, then we can express the above integer index using the following geometric terms:

$$
\begin{aligned}
j &= \dim\left(R_C\begin{pmatrix} A \\ C \end{pmatrix} \cap R_C\begin{pmatrix} B \\ O \end{pmatrix}\right) \\
k &= \dim\left(N_C(C)\backslash N_C\begin{pmatrix} A \\ C \end{pmatrix}\right) - \dim\left(R_C\begin{pmatrix} A \\ C \end{pmatrix} \cap R_C\begin{pmatrix} B \\ O \end{pmatrix}\right) \\
&= \dim(N_r(B)\backslash N_r(A,B)) - \dim(R_r(A,B) \cap R_r(C,O)) \\
l &= \dim(R_r(A,B) \cap R_r(C,O)) \\
r &= \dim(R_r(A) \cap R_r(C) - \dim(R_r(A,B) \cap R_r(C,O)) \\
&= \dim(R_C(A) \cap R_C(B)) - \dim\left(R_C\begin{pmatrix} A \\ C \end{pmatrix} \cap R_C\begin{pmatrix} B \\ O \end{pmatrix}\right) \\
s_1 &= \dim\left(N_C(A)\backslash N_C\begin{pmatrix} A \\ C \end{pmatrix}\right) \\
s_2 &= \dim(N_r(A)\backslash N_r(A,B)) \\
t_1 &= \dim\left(N_C\begin{pmatrix} A \\ C \end{pmatrix}\right) \\
t_2 &= \dim(N_r(A,B)).
\end{aligned}
$$

The above expressions can serve as a basis of a geometric derivation of RSVD. Before we discuss another two special cases of RSVD, we consider the uniqueness problem of the RSVD in theorem 4.2. In order to simplify the presentation, we only discuss the problem under some restrictions, while the general case can be discussed similarly.

Theorem 4.6. Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $C \in \mathbb{C}^{q\times n}$, in addition we assume that

$$
\begin{aligned}
&\text{i)} \quad \text{rank}\,(B) = p, \qquad \text{rank}\,(C) = q \\
&\text{ii)} \quad \text{rank}\,(A,B) = m, \qquad \text{rank}\begin{pmatrix} A \\ C \end{pmatrix} = n
\end{aligned}
$$

according to theorem 4.2 (here we use the scaled form as stated in remark 4.3) there exist nonsingular matrices P and Q, unitary matrices U and V such that

$$PAQ = \Sigma_A, \quad PBU = \Sigma_B \text{ and } VCQ = \Sigma_C \tag{4.9}$$

where with the restrictions of the ranks as in i) and ii) Σ_A, Σ_B and Σ_C are of the following form:

$$\Sigma_A = \text{diag}\,(I_j, I_k, I_l, \Sigma)$$

$$\Sigma_B = \begin{array}{c} \\ k \\ l \\ \\ \end{array} \begin{pmatrix} I_j & O \\ O & O \\ O & O \\ O & I_s \end{pmatrix} \qquad \Sigma_C = \begin{array}{c} \begin{pmatrix} O & O & I_l & O \\ O & O & O & I_t \end{pmatrix} \\ \begin{array}{cccc} & j & k & \end{array} \end{array}$$

where $\Sigma = \text{diag}\,(\sigma_i)$ $\sigma_i \geq 0$ $i = \min(s,t)$.
If there exist $\tilde{P}$, $\tilde{Q}$ and $\tilde{U}$, $\tilde{V}$ such that $\tilde{P}$ and $\tilde{Q}$ are nonsingular, $\tilde{U}$ and $\tilde{V}$ are unitary and

$$\tilde{P}A\tilde{Q} = \Sigma_A, \quad \tilde{P}B\tilde{U} = \Sigma_B \text{ and } \tilde{V}C\tilde{Q} = \Sigma_C \tag{4.10}$$

then

$$\tilde{P} = \begin{pmatrix} U_1 & S_1 & S_2 & O \\ O & S_3 & S_4 & O \\ O & O & V_1 & O \\ O & O & O & U_2 \end{pmatrix} P \quad Q = \tilde{Q} \begin{pmatrix} U_1 & S_1 & S_2 & O \\ O & S_3 & S_4 & O \\ O & O & V_1 & O \\ O & O & O & V_2 \end{pmatrix}$$

$$U = \tilde{U} \begin{pmatrix} U_1 & O \\ O & U_2 \end{pmatrix} \qquad V^H = \tilde{V}^H \begin{pmatrix} V_1 & O \\ O & V_2 \end{pmatrix}$$

where U_i, V_i, $(i = 1,2)$ are unitary, S_3 nonsingular and S_1, S_2 and S_4 are arbitrary. If in addition Σ hast the following form
1.

$$t > s \;\; \Sigma = (\text{diag}\,(\sigma_1 I_{i_1}, \ldots, \sigma_n I_{i_n}, O), O) \text{ and } s = \sum_{j=1}^{n+1} i_j \tag{4.11.a}$$

then

$$\begin{aligned} U_2 &= \text{diag}\,(U_{11}, \ldots, U_{nn}, U_{n+1,n+1}) \\ V_2 &= \text{diag}\,(U_{11}, \ldots, U_{nn}, V_2^{(2)}) \end{aligned} \tag{4.11.b}$$

where $U_{jj}(j = 1, \ldots, n+1)$ is $i_j \times i_j$ unitary matrix, and $V_2^{(2)}$ is unitary matrix of order $i_{n+1} + t - s$.
2.

$$\begin{aligned} &\text{if} \quad t \leq s, \;\; \Sigma = \begin{pmatrix} \text{diag}\,(\sigma, I_{i_1}, \ldots, \sigma_n I_{i_n}, O) \\ O \end{pmatrix} \\ &\text{and} \quad t = \sum_{j=1}^{n+1} i_j \end{aligned} \tag{4.12.a}$$

then

$$V_2 = \text{diag}\,(V_{11}, \ldots, V_{nn}, V_{n+1,n+1})$$
$$U_2 = \text{diag}\,(V_{11}, \ldots, V_{nn}, U_2^{(2)}) \tag{4.12.b}$$

where $V_{jj}(j = 1, \ldots, n+1)$ is $i_j \times i_j$ unitary matrix, $U_2^{(2)}$ is unitary matrix of order $i_{n+1} + s - t$.

Proof. From (4.9) and (4.10) we obtain

$$(\tilde{P}P^1)\Sigma_A = \Sigma_A(\tilde{Q}^{-1}Q), (\tilde{P}^{-1}P)\Sigma_B = \Sigma_B(\tilde{U}^H U) \text{ and}$$
$$(\tilde{V}V^H)\Sigma_C = \Sigma_C(\tilde{Q}^{-1}Q)\,.$$

Let

$$\bar{P} := \tilde{P}P^{-1} = (P_{ij})_{i,j=1}^4, \quad \bar{Q} := \tilde{Q}^{-1}Q = (Q_{ij})_{i,j=1}^4$$
$$\bar{U} := \tilde{U}^H U = (U_{ij})_{i,j=1}^2, \quad \bar{V} := \tilde{V}V^H = (V_{ij})_{i,j=1}^2$$

be partitioned conformally with those of Σ_A, Σ_B and Σ_C. From $\bar{P}\Sigma_B = \Sigma_B\bar{U}$ we obtain:

$$\begin{array}{llll} P_{11} = U_{11}, & P_{14} = U_{12}, & P_{21} = O, & P_{24} = O, \\ P_{31} = O, & P_{34} = O, & P_{41} = U_{21}, & P_{44} = U_{22} \end{array} \tag{4.13}$$

From $\bar{V}\Sigma_C = \Sigma_C\bar{Q}$ we obtain:

$$\begin{array}{llll} Q_{31} = O, & Q_{32} = O, & Q_{41} = O, & Q_{42} = O, \\ Q_{33} = V_{11}, & Q_{34} = V_{12}, & Q_{43} = V_{21}, & Q_{44} = V_{22} \end{array} \tag{4.14}$$

Substitute (4.13) and (4.14) into $\bar{P}\Sigma_A = \Sigma_A\bar{Q}$, we obtain:

$$\begin{pmatrix} U_{11} & P_{12} & P_{13} & U_{12}\Sigma \\ O & P_{22} & P_{23} & O \\ O & P_{32} & P_{33} & O \\ U_{21} & P_{42} & P_{43} & U_{22}\Sigma \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ O & O & V_{11} & V_{12} \\ O & O & \Sigma V_{21} & \Sigma V_{22} \end{pmatrix}$$

Since $U_{21} = O$, $V_{12} = O$ and $\bar{U}$, $\bar{V}$ are unitary, hence $U_{12} = O, V_{21} = O$. Furthermore:

$$\begin{array}{lllll} U_1 := Q_{11} = U_{11}, & Q_{21} = O, & Q_{14} = O, & Q_{24} = O, & U_2 := U_{22} \\ V_1 := P_{33} = V_{11}, & P_{32} = O, & P_{42} = O, & P_{43} = O, & V_2 := V_{22} \end{array}$$

and $U_2\Sigma = \Sigma V_2$.

In the following we only consider the case $t > s$, i.e.

$$\Sigma = (\hat{\Sigma}, O) \text{ where } \hat{\Sigma} = \text{diag}\,(\sigma_1 I_{i_1}, \ldots, \sigma_n I_{i_n}, O)\,.$$

Since

$$U_2 \hat{\Sigma}\hat{\Sigma}^H = \hat{\Sigma}\hat{\Sigma}^H U_2$$

U_2 must have the block diagonal form in (4.12.b). From $U_2\Sigma = \Sigma V_2$, we can obtain V_2 has the block diagonal form described in (4.12.b). The proof is completed. ∎

Corollary 4.7. Let A be nonsingular and the nonzero SV's of $CA^{-1}B$ be

$$\sigma_1 \geq \ldots \geq \sigma_r > 0$$

then (A,B,C) has $(n-r)$ ∞ RSV's and the r finite RSV's are

$$\frac{1}{\sigma_r} \geq \ldots \geq \frac{1}{\sigma_1} > 0\,.$$

Proof. Using the decomposition of theorem 4.2, one can show that

$$V(CA^{-1}B)U = \begin{pmatrix} O & & & \\ & O & & \\ & & O & \\ & & & S_C S_A^{-1} S_B \end{pmatrix}$$

∎

Corollary 4.8. (PSVD [3]) Let $B \in \mathbb{C}^{m \times p}$ and $C \in \mathbb{C}^{p \times n}$, then there exist unitary matrices U and V and nonsingular matrix T such that

$$UBT = \begin{pmatrix} I_j & & \\ & O_B & \\ & & \Sigma_B \end{pmatrix} \tag{4.15.a}$$

$$T^{-1}CV = \begin{pmatrix} O_C & & \\ & I_l & \\ & & \Sigma_C \end{pmatrix} \tag{4.15.b}$$

where

$$\Sigma_B = \text{diag}\,(s_i)\,, \qquad \Sigma_C = \text{diag}\,(t_i)$$

$$1 > s_1 \geq \ldots \geq s_r > O\,, \qquad 1 > t_i \geq \ldots \geq t_r > O$$

$$s_i^2 + t_i^{-2} = 1\,, \quad i = 1, \ldots, r\,.$$

Proof. Using theorem 3.2, let the RSVD of (I_p, B^H, C^H) be

$$\begin{aligned}
PI_pQ &= \text{diag}\,(I_j, I_k, I_l, S_A) \\
PB^H\tilde{U} &= \begin{pmatrix} I_j & & \\ & O_B^{(1)} & \\ & & S_B \end{pmatrix} \\
\tilde{V}C^HQ &= \begin{pmatrix} O_B^{(1)} & & \\ & I_l & \\ & & S_C \end{pmatrix} \\
\text{Set } \tilde{Q} &= Q\text{diag}\,(I, S_A^{-1}) \text{ then} \\
P\tilde{Q} &= I_p \\
PB^H\tilde{U} &= \begin{pmatrix} I_j & & \\ & O_B^{(1)} & \\ & & S_B \end{pmatrix} \\
\tilde{V}C^H\tilde{Q} &= \begin{pmatrix} O_C^{(1)} & & \\ & I_l & \\ & & S_CS_A^{-1} \end{pmatrix}
\end{aligned}$$

the proof is finished if we set $U = \tilde{U}^H$, $V = \tilde{V}^H$, $T = P^H$, $O_B = (O_B^{(1)})^H$, $O_C = (O_C^{(1)})^H$, $\Sigma_B = S_B$ and $\Sigma_C = S_CS_A^{-1}$. ■

Remark 4.9. Corollary 4.6 is a simplified version of the product induced SVD (PSVD) in [3]. We can also use the techniques established in proving theorem 3.4 and theorem 4.2 to give a direct proof of it.

In the following we discuss the relation between the RSVD of (A,B,C) and the eigenstructure problem of

$$\left(\begin{pmatrix} O & A \\ A^H & O \end{pmatrix}, \begin{pmatrix} BB^H & O \\ O & C^HC \end{pmatrix} \right).$$

From theorem 4.2 after suitable permutation Π we obtain

$$\begin{aligned}
&\Pi \begin{pmatrix} P & O \\ O & Q^H \end{pmatrix} \left(\begin{pmatrix} O & A \\ A^H & O \end{pmatrix} - \lambda \begin{pmatrix} BB^H & O \\ O & C^HC \end{pmatrix} \right) \begin{pmatrix} P^H & O \\ O & Q \end{pmatrix} \Pi^T \\
= \;& \text{diag} \left\{ \begin{pmatrix} -\lambda I_j & I_j \\ I_j & O \end{pmatrix}, \begin{pmatrix} O & I_k \\ I_k & O \end{pmatrix}, \begin{pmatrix} O & I_l \\ I_l & -\lambda I_l \end{pmatrix}, \right. \\
&\left. \begin{pmatrix} -\lambda S_B & S_A \\ S_A & -\lambda S_C \end{pmatrix}, \begin{pmatrix} -\lambda I_{s_1} & O \\ O & -\lambda I_{s_2} \end{pmatrix}, O \right\}
\end{aligned}$$

therefore the eigenstructure of the symmetric matrix pencil is the following

i) $2(j+l)$ ∞ eigenvalues corresponding to Jordan block of order 2 ($(j+l)$ 2×2 Jordan blocks).

ii) $2k$ ∞ eigenvalues corresponding to Jordan block of order 1.

iii) $2r$ nonzero finite eigenvalues $\pm \frac{\alpha_i}{\beta_i \gamma_i} i = s+1, \ldots, s+r$.

iv) $s_1 + s_2$ zero eigenvalues.

v) $(m+n) - 2(j+l+k+s) - s_1 - s_2$ Kronecker blocks of order 0.

5. Concluding Remarks

In this paper we introduce the concept of RSV's of matrix triplets. A main theorem called RSVD is proved for general matrix triplets. Three special cases of RSV's, i.e. the wellknown SV's GSV's and the recently proposed PSV's are also discussed. Numerical algorithms for computing the RSVD of a general matrix triplet and applications of RSVD to total least squares problem and regularization problem of general Gauss-Markov linear model will appear in separate papers. Perturbation analysis and further applications of RSVD will be the topics of future research. We hope that RSVD will be important not only as a useful theoretical tool for analysing problems in numerical linear algebra, statistics and control and system theory, but that its algorithmic aspects will also find applications in computer-based methods to solve realworld problems.

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