Konrad-Zuse-Zentrum für Informationstechnik Berlin
Heilbronner Str. 10, D-10711 Berlin-Wilmersdorf, Germany

Rekha Thomas
Robert Weismantel

Test sets and inequalities for integer programs: extended abstract

Preprint SC-95-36 (Dezember 1995)

Test sets and inequalities for integer programs: extended abstract

Rekha Thomas* Robert Weismantel†

Abstract

This paper presents some connections between test sets and valid inequalities of integer programs. The reason for establishing such relationships is the hope that information (even partial) on one of these objects can be used to get information on the other and vice versa. We approach this study from two directions: On the one hand we examine the geometric process by which the secondary polytope associated with a matrix A transforms to the state polytope as we pass from linear programs that have A as coefficient matrix to the associated integer programs. The second direction establishes the notion of classes of augmentation vectors parallel to the well known concept of classes of facet defining inequalities for integer programs. We show how certain inequalities for integer programs can be derived from test sets for these programs.

1 Introduction

Test sets and their algorithmic construction play a central role in several branches of Mathematics, like geometry of numbers, computational algebra and integer programming. We are concerned here with test sets for integer programming problems.
Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{R}^n$ be given. A *test set* T for the integer programming problem

$$(IP) \qquad \max cx : \ Ax \leq b, \ x \in \mathbb{N}^n$$

is a set of vectors such that any feasible point x of the integer program, i.e., $Ax \leq b$ and $x \in \mathbb{N}^n$, is not optimal if and only if there exists an element $t \in T$ such that $x + t$ is feasible and $ct > 0$. By a *universal test set* we mean a set of vectors that contains a test set for every $c \in \mathbb{R}^n$.

On the other hand, when one is concerned with the solution of integer programming problems, one often studies the geometry and combinatorics of the polyhedron

$$\text{conv}\{x \in \mathbb{N}^n : \ Ax \leq b\}.$$

In particular, one is interested in inequalities that describe faces of the polyhedron. Our goal in this paper is to present some connections between the elements of a minimal test set of an integer program and inequalities that are valid for the associated polyhedron.

*Dept. of Mathematics, Texas A & M University, College Station, TX 77843-3368, rekha@math.tamu.edu
†Konrad–Zuse–Zentrum für Informationstechnik Berlin, Heilbronner Straße 10, D–10711 Berlin, Germany, weismantel@zib–berlin.de

In Section 2 we present basic properties of irreducible elements in a test set and the normal vectors of facet-defining inequalities.

Section 3 examines the process by which the secondary polytope associated with the family of integer programs with coefficient matrix A transforms to the state polytope associated with the associated integer programs. The secondary polytope [GKZ94], [BFS90] of A is the Minkowski integral of all feasible regions of linear programs with coefficient matrix A whereas the state polytope of A [ST94] is the Minkowski integral of the convex hull of all feasible solutions of the associated integer programs. This study sheds light on how the LP optimum refines to the IP optimum for programs defined by a fixed matrix and cost function. The main tool used here are test sets while traditionally this process is carried out via cutting planes.

Finally we illustrate, in the case of 0/1-integer programming, further links between the augmentation problem and the separation problem.

2 Basic properties of test set elements

There are two elementary properties of normal vectors of facet-defining inequalities and so-called irreducible elements in a test set for an integer program that we introduce below. The following notation is needed.

From the objective function c of the integer programming problem (IP) we obtain a linear order on $\mathbb{Z}^n$ as follows: we choose an arbitrary term order $\prec_0$, (for example the lexicographic order $\prec_0 = \prec_{lex}$), and use it as a "tie breaker" on the points that have the same objective function value under c; that is, we define

$$x \ \prec_c \ y \qquad :\Longleftrightarrow \qquad \begin{cases} c^T x < c^T y, & \text{or} \\ c^T x = c^T y & \text{and } x \prec_0 y. \end{cases}$$

Definition 2.1 *A vector $t \in \mathbb{Z}^N$, $t \succ_c 0$ is called* reducible *by $w \in \mathbb{Z}^N$, $w \succ_c 0$ if $v^+ \le t^+$, $v^- \le t^-$ and $(Av)^+ \le (At)^+$. Otherwise, t is called* irreducible.

Note that, if $t \in \mathbb{Z}^N$ is reducible by $w \in \mathbb{Z}^N$, then, whenever $x + t$ is feasible for (IP), $x + w$ is feasible for (IP), too.

For $A \in \mathbb{Z}^n$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{R}^n$, let

$$(IP) \qquad \max\{cx : \ Ax \le b, \ x \in \mathbb{N}^n\}$$

denote an integer programming problem and

$$P_I = \text{conv}\{x \in \mathbb{N}^n : \ Ax \le b\}$$

the polyhedron associated with the program. With an integer program (IP) we associate a graph $G = (V, E)$. V is the set of nodes where node i corresponds to variable i in the integer programming formulation. Between nodes i and j we introduce an edge if there exists some row $k \in M := \{1, \ldots, m\}$ such that $a_{ki} = a_{kj} \neq 0$. Using this graph G we can derive a condition that must be satisfied by all normal vectors of facet-defining inequalities of P_I and by all irreducible elements in the test set.

Lemma 2.2 *Let t be an irreducible element in the test set for (IP). The subgraph $(S, E(S))$ with $S := \{i \in N := \{1, \ldots, n\} : t_i \neq 0\}$ is connected. Let $dx \leq d_0$ be a facet-defining inequality for P_I. The subgraph $(S, E(S))$ with $S := \{i \in N : d_i \neq 0\}$ is connected.*

There is another property of facet-defining inequalities that can be expressed in terms of irreducible elements in a test set. The linear subspace of the hyperplane defining the facet is composed of irreducible elements in the universal test set.

Theorem 2.3 *Let $dx \leq d_0$ be a facet-defining inequality for P_I and denote by p the dimension of P_I. There exists a basis $t_1, \ldots, t_{p-1}$ of the subspace $S := \{x \in \mathbb{R}^n : dx = 0\}$ such that $t_1, \ldots, t_{p-1}$ are irreducible elements in a universal test set of (IP).*

For illustration, consider the matroid $(E, \mathcal{M})$ defined on the ground set $E = \{1, \ldots, n\}$ with $\{i\} \in \mathcal{M}$ for all $i = 1, \ldots, n$ and $e_i := \chi^{\{i\}}$. The symbol $\succ_c$ denotes the order on the elements of E that we associate with the objective function c by using the lexicographic order in order to break ties. The convex hull of all incidence vectors of independent sets in the matroid is described by the non-negativity inequalities and the set of all rank inequalities of the type $x(F) \leq rank(F)$ for $F \subseteq E$ [Ed71]. For the rank inequality $x(F) \leq rank(F)$, a basis of $\{x \in \mathbb{R}^n : x(F) = 0\}$ is given by the vectors e_i, $i \in E \setminus F$ and by vectors of the type $e_i - e_j$ for $i, j \in F$. For an inequality $x_i \geq 0$, a basis of $\{x \in \mathbb{R}^n : x_i = 0\}$ is given by the vectors e_j, $j \in E \setminus \{i\}$. This fact is reflected in the structure of the test set for the matroid optimization problem.

Lemma 2.4 *For a matroid $(E, \mathcal{M})$ and an objective function $c : E \to \mathbb{R}$, the set*

$$T := \{e_i,\ i \in E \text{ and } i \succ_c 0\} \cup \{-e_i,\ i \in E \text{ and } i \prec_c 0\} \cup \{e_i - e_j :\ i, j \in E,\ i \neq j,\ i \succ_c j \succ_c 0\}$$

defines a test set for the matroid optimization problem $\max\{c\chi^M :\ M \in \mathcal{M}\}$.

3 From the secondary polyhedron to the state polyhedron

For a fixed matrix $A \in \mathbf{Z}^{m \times n}$ of rank m and $b \in pos_{\mathbf{Z}}(A) = \{Ap : p \in \mathbf{N}^n\}$, let $P_b = \{x \in \mathbf{R}^n_+ : Ax = b\}$ and P^I_b = convex hull $\{x \in \mathbf{N}^n : Ax = b\}$. We will assume that $\{x \in \mathbf{R}^n_+ : Ax = 0\} = \{0\}$ which guarantees that both P_b and P^I_b are polytopes for all $b \in pos_{\mathbf{Z}}(A)$. For a fixed cost vector c and right hand side vector $b \in pos_{\mathbf{Z}}(A)$, let $IP_{A,c}(b)$ denote the integer program minimize $cx : Ax = b, x \in \mathbf{N}^n$ and $LP_{A,c}(b)$ denote its linear relaxation minimize $cx :\ Ax = b, x \in \mathbf{R}^n_+$. A number of recent papers (see [CT91], [Th94], [ST94]) have established the existence and construction of the reduced Gröbner basis $\mathcal{G}_c \subset kernel(A) \cap \mathbf{Z}^n = ker_{\mathbf{Z}}(A)$ which is a unique minimal test set for the family of integer programs $IP_{A,c}(\cdot)$. The union of all reduced Gröbner bases associated with a fixed matrix A is a finite set UGB_A, called the *universal Gröbner basis* of A and is a universal test set for the family $IP_A = \{IP_{A,c}(b), \forall b \in pos_{\mathbf{Z}}(A), c \in \mathbf{R}^n\}$. It can be established from the mechanics of the simplex method that the *circuits* of A which are the primitive vectors of minimal support in $ker_{\mathbf{Z}}(A)$, constitute a minimal universal test set for LP_A. Further, the circuits of A are contained in UGB_A. Associated with A, one can construct two $n - m$-dimensional polytopes : the *secondary*

polytope $\Sigma(A)$ which is the *Minkowski integral* $\int_b P_b db$ ([GKZ94], [BiS92], [BFS90]) and the *state polytope* $St(A) = \int_b P_b^I db$ (see [ST94]) where db is a suitable probability measure in both cases. In fact, any polytope *normally equivalent* (having the same normal fan) as $\Sigma(A)$ (respectively, $St(A)$) is called a secondary polytope of A (respectively, state polytope of A). Both $\Sigma(A)$ and $St(A)$ can be constructed by taking a finite Minkowski sum of polytopes of the form P_b and P_b^I respectively. The inner normal fan of $\Sigma(A)$ is an n-dimensional complete fan called the *secondary fan* of A, denoted $\mathcal{N}(\Sigma(A))$ and the inner normal fan of $St(A)$ is also a complete polyhedral fan in $\mathbf{R}^n$, known as the *Gröbner fan* of A, denoted $\mathcal{N}(St(A))$. Construction methods for both fans can be found in [BFS90] and [ST94]. A number of relationships between the secondary and state polytopes have been established in [St91] and [ST94]. We state a few below:
1. The state polytope $St(A)$ is a *Minkowski summand* of the secondary polytope $\Sigma(A)$. Therefore the Gröbner fan $\mathcal{N}(St(A))$ is a *refinement* of the secondary fan $\mathcal{N}(\Sigma(A))$.
2. The edge directions of $\Sigma(A)$ and hence of all polytopes $P_b, b \in pos_A$ are the circuits of A while the edge directions of $St(A)$ and hence of all polytopes $P_b^I, b \in pos_{\mathbf{Z}}(A)$ are the elements in UGB_A.
3. Let $c \in \mathbf{R}^n$ be a *generic* cost function for IP_A, namely, c is optimized at a unique vertex in each polytope P_b^I, $b \in pos_{\mathbf{Z}}(A)$. Then c lies in the interior of an n-dimensional cell S_c in $\mathcal{N}(\Sigma(A))$ and in the interior of an n-dimensional cell K_c in $\mathcal{N}(\Sigma(A))$. By property 1, K_c is contained in S_c. We say that two cost functions c_1 and c_2 are *equivalent* with respect to IP_A (respectively LP_A) if $IP_{A,c_1}(b)$ and $IP_{A,c_2}(b)$ (respectively $LP_{A,c_1}(b)$ and $LP_{A,c_2}(b)$) have the same set of optimal solutions for all $b \in pos_{\mathbf{Z}}(A)$ (respectively, for all $b \in pos(A)$). Then the interior of K_c is precisely the equivalence class of c with respect to IP_A and the interior of S_c is the equivalence class of c with respect to LP_A. In particular, every full dimensional cone in $\mathcal{N}(\Sigma(A))$ either remains the same or partitions (refines) into subcones when we pass from LP_A to IP_A. This refinement reinforces the fact that integer programming is an arithmetic refinement of linear programming.

In this section we establish results that provide insight into how a fixed secondary cell partitions (if it does) into its associated Gröbner cells. For a fixed right hand side vector b, the passage from the optimal vertex of P_b with respect to a fixed cost vector c to the optimal vertex of P_b^I with respect to c is achieved by introducing cutting planes or "local" facets for P_b^I. On the level of $\Sigma(A)$ and $St(A)$, this study is done via Gröbner bases. Similarly, the Chvatal procedure (Chapter 23 in [Schr86]) shows how to iteratively use cutting planes to obtain P_b^I from P_b. This procedure requires the concept of TDI representations and its correctness can be proved via test sets. All these point toward interconnections between valid inequalities for an integer program and test sets for the program. Facets of P_b^I are the faces of codimension one while the faces of dimension one are elements in UGB_A. All these point toward interconnections between valid inequalities for an integer program and test sets for the program. Below we study these relationships from the point of view of the secondary and state polytopes associated with A. Understanding how a fixed secondary cell refines to its associated Gröbner cells completely explains how the secondary fan refines to the Gröbner fan. We first examine some conditions under which the state and secondary polytopes of a matrix coincide.

Definition 3.1 *A matrix A of full row rank is called c-unimodular if each of its maximal minors is one of $-c$, 0 or c, where c is a positive integral constant.*

It is known that if A is c-unimodular, then the circuits of A constitute UGB_A and that $St(A) = \Sigma(A)$. The latter fact follows from the following theorem and some additional arguments.

Theorem 3.2 *A matrix A is c-unimodular if and only if $P_b = P_b^I$ for all $b \in pos_{\mathbf{Z}}(A)$.*

The above theorem is well known in the case of 1-unimodular matrices. When A is 1-unimodular, $pos_{\mathbf{Z}}(A) = pos(A) \cap \mathbf{Z}^m$ and hence the above theorem would say that A is 1-unimodular if and only if $P_b = P_b^I$ for all integral $b \in pos(A)$. This is precisely Theorem 19.2 in [Schr86] and the above theorem is a generalization of this.

Remarks 3.3 *If A is c-unimodular where $c > 1$, then $pos(A) \cap c \cdot \mathbf{Z}^m \subseteq pos_{\mathbf{Z}}(A)$. This containment is often strict. For example, the incidence matrix A_4 of the complete graph K_4 is 2-unimodular. However, $A_4 \cdot (1,1,1,1,0,0)^t = (3,2,2,1) \notin 2 \cdot \mathbf{Z}^4$.*

However, there are matrices that are not c-unimodular for which the state and secondary polytopes coincide. For such matrices, it is no longer true that $P_b = P_b^I$ for all $b \in pos_{\mathbf{Z}}(A)$ and yet $\int_b P_b db = \int_b P_b^I db$. We establish a sufficient condition for the state and secondary polytopes to coincide and a family of non-unimodular matrices that has this property.

A vector $u \in ker_{\mathbf{Z}}(A)$ can be written uniquely as $u = u^+ - u^-$ where $u^+, u^- \in \mathbf{N}^n$. The polytope $P_{Au^+} = P_{Au^-}$ is called the *LP-fiber of* u and $P_{Au^+}^I = P_{Au^-}^I$ is called the *IP-fiber of* u. In particular, if u is a circuit of A, then P_{Au^+} and $P_{Au^+}^I$ are called a LP- and IP-*circuit fiber* respectively, of A and if $u \in UGB_A$ they are called the LP- and IP- *Gröbner fiber* respectively, of A. It is shown in [ST94] that $\Sigma(A) = \sum\{P_{Au^+} : u$ is a circuit of A $\}$ and $St(A) = \sum\{P_{Au^+}^I : u \in UGB_A$ $\}$. These facts imply the following sufficient condition for when the state and secondary polytopes of a matrix coincide.

Lemma 3.4 *For a matrix A, if the circuits of A constitute UGB_A and all LP-circuit fibers of A are integral, then $\Sigma(A) = St(A)$.*

Clearly, unimodular matrices satisfy the condition in the above lemma. The *Lawrence lifting* of a given matrix $A \in \mathbf{Z}^{m \times n}$, is defined to be the enlarged matrix $\Lambda(A) = \begin{pmatrix} A & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$ where $\mathbf{0}$ is a m by n matrix of zeroes and $\mathbf{1}$ denotes an identity matrix of size n. Lawrence matrices have many special properties and can be used to compute the universal Gröbner basis of IP_A. It was shown in [ST94] that the IP Gröbner fibers of $\Lambda(A)$ are one dimensional. We establish a similar result below.

Proposition 3.5 *Every LP circuit fibers of $\Lambda(A)$ is one dimensional and integral.*

Note that A and $\Lambda(A)$ have isomorphic kernels: $ker_{\mathbf{Z}}(\Lambda(A)) = \{(u, -u) : u \in ker_{\mathbf{Z}}(A)\}$. In particular, a vector u is a circuit of A if and only if the vector $(u, -u)$ is a circuit of $\Lambda(A)$. If x is a point in the LP circuit fiber of $\Lambda(A)$ corresponding to the circuit $(u, -u)$, then it can be shown that x is in the affine span of the line segment $[u^+, u^-]$ and hence the fiber is one dimensional. Integrality of the fiber follows from the fact that the line segment $[u^+, u^-]$ is an edge in this fiber. This establishes Proposition 3.5. Using Lemma 3.4 and Proposition 3.5 we have the following.

Proposition 3.6 *If the set of all circuits of $\Lambda(A)$ equals $UGB_{\Lambda(A)}$ then $\Sigma(\Lambda(A)) = St(\Lambda(A))$.*

The above result can also be infered from [ST94] using a different proof based on the notions of *circuit and Gröbner arrangements.*

For a general matrix A, it is not true that if circuits$(A) = UGB_A$ then $\Sigma(A) = St(A)$. An example of such a matrix is $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{bmatrix}$.

We saw above that for certain classes of matrices A, $St(A) = \Sigma(A)$ which is equivalent to saying that no cell in the secondary fan partitions into subcones while passing to the Gröbner fan. For an arbitrary matrix A, some cells in the secondary fan partition into subcones while passing to the Gröbner fan while other cells do not. We examine below properties of those secondary cells that do not subdivide. This is related to the classical concept of TDI-ness.

Definition 3.7 *[Schr86] A rational system $yA \leq c$ is TDI if the minimum in the LP duality equation $max\{yb : yA \leq c\} = min\{cx : Ax = b, x \geq 0\}$ has an integral optimal solution for every $b \in \mathbf{Z}^m$ for which the minimum is finite.*

In the terminology used so far, $yA \leq c$ is TDI if $LP_{A,c}(b)$ has an integral optimal solution for all $b \in pos_{\mathbf{Z}}(A)$. This optimal solution is then also optimal for $IP_{A,c}(b)$. We say that a full dimensional, simplicial, polyhedral, convex cone $K \subseteq \mathbf{R}^n$ with extreme rays generated by the primitive vectors $p_1, \ldots, p_n$ is *unimodular* if the square matrix with columns $p_1, \ldots, p_n$ has determinant 1 or -1. The monomial ideal generated by the unique leading terms of the elements in a reduced Gröbner basis $\mathcal{G}_c$ is called the initial ideal with respect to c, denoted $in_c(I_A)$ where I_A is the underlying ideal that depends on A used in the algebraic version of this theory. Also, associated with a generic cost vector c we have a *regular triangulation* of the columns of A denoted Δ_c. See [GKZ94], [BFS90] for details on regular triangulations. A regular triangulation Δ_c is said to be *unimodular* if all its maximal simplices have unit normalised volume. The following algebraic result can be infered from Theorem 5.3 in [KSZ92].

Proposition 3.8 *The initial ideal $in_c(I_A)$ is square-free if and only if the regular triangulation Δ_c is unimodular.*

We now relate certain properties discussed so far. Recall that the secondary cell of a generic cost vector c is denoted S_c and the Gröbner cell is denoted K_c.

Theorem 3.9 *Consider the following properties of $A \in \mathbf{Z}^{m\times n}$ and generic cost vector $c \in \mathbf{Q}^n$.*
(i) The secondary cell S_c coincides with the Gröbner cell K_c.

(ii) The system $yA \leq c$ is TDI.
(iii) The optimal solution of $LP_{A,c}(b)$ is integral for all $b \in pos_{\mathbf{Z}}(A)$.
(iv) The initial ideal $in_c(I_A)$ is square free.
(v) The secondary cell S_c modulo its lineality space is unimodular. Then

a) $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$,
b) $(ii) \Rightarrow (i)$ but $(i) \not\Rightarrow (ii)$,
c) $(i) \not\Rightarrow (v)$ and $(v) \not\Rightarrow (i)$,
d) $(ii) \not\Rightarrow (v)$ and $(v) \not\Rightarrow (ii)$.

Therefore the TDI-ness of $yA \leq c$ is a sufficient condition for the secondary cell S_c to not subdivide while passing to the Gröbner fan although the converse is false. Geometrically, the interior of S_c is the set of all cost functions that select the same LP optimum as c in the polytopes P_b for all $b \in pos_{\mathbf{Z}}(A)$ while the interior of K_c is the set of all cost vectors that select the same optimal vertex as c in the polytopes P_b^I for all $b \in pos_{\mathbf{Z}}(A)$. However since $(i) \not\Rightarrow (ii)$, it follows that all cost vectors in $S_c = K_c$ can pick the same LP optimum for all $b \in pos(A)$ and the same IP optimum for all $b \in pos_{\mathbf{Z}}(A)$, although for a given b, the LP and IP optima may be different - the LP optimum being fractional. This supports the fact that matrices that are not c-unimodular can have their state and secondary polytopes coincide.

4 Augmentation vectors

Throughout this section we assume that the integer programming problem is a 0/1-program that is given in the form

$$(IP) \qquad \max cx : \; Ax \leq b, \; x \in \{0,1\}^n$$

with $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^n$ and $c \in \mathbb{R}^n$. By a^j we mean column $j \in M = \{1, \ldots, m\}$.

In polyhedral combinatorics one often tries to find classes of inequalities that are valid for the integer polyhedron. Once such a class of inequalities is given, one derives a separation algorithm that, for a given fractional point, finds an inequality that is violated by the fractional point, or proves that no inequality in this class is violated by the fractional point. This dual approach is justified as separation and optimization are equivalent in terms of computational complexity [GLS88].

The primal counterpart of these questions is to ask for a set of vectors and an algorithm that either finds a vector (augmentation vector) in this set such that a current feasible point can be improved, or asserts that there is no vector in this set that yields an improvement of the current feasible solution. These questions are reasonable to ask, because for 0/1 integer programs the augmentation problem, the irreducible augmentation problem and the optimization problem are strongly polynomial time equivalent, see [SWZ95].

Below we present two examples. One shows that for a certain knapsack problem the augmentation problem can be solved in time being polynomial in the size of the input data. The other example illustrates how certain elements in a test set can be used to derive inequalities that are satisfied by all optimal solutions (objective based cutting planes).

4.1 Example: knapsack augmentation vectors

For a fixed number $K \in \mathbb{N}$ and natural numbers $d_1 < \ldots < d_K$, we consider the knapsack problem

$$P(d_1, \ldots, d_K) \qquad \max \sum_{i \in N} c_i x_i : \sum_{i \in N} a_i x_i \leq a_0, \ x \in \{0,1\}^n,$$

with $a_i \in \mathbb{N}$, $c_i \in \{d_1, \ldots, d_K\}$ for all $i \in N$ and $a_0 \in \mathbb{N}$. Let $N_1, \ldots, N_K$ be the subsets of items in N of objective function coefficients $d_1, \ldots, d_K$, respectively, and set $n_j := |N_j|$ for all $j \in \{1, \ldots, K\}$.

Lemma 4.1 *There exists a test set of $P(d_1, \ldots, d_K)$ of cardinality at most $(\prod_{j=1}^K n_j)3^K + \frac{n(n-1)}{2}$.*

A test set T of $P(d_1, \ldots, d_K)$ with respect to $\succ_c$ that consists of irreducible elements can be computed by the algorithm described in [UWZ94]. This set T can then be used as a set of candidates to improve a current feasible point x of the given integer program (IP) via the following scheme.

1 Sum up a subset S of the set of rows of the matrix A. This yields a problem of the type $\max cx : \sum_{i \in N}(\sum_{j \in S} a_{ji})x_i \leq \sum_{j \in S} b_j, \ x \in \{0,1\}^n$. By complementing a variable i if $(\sum_{j \in S} a_{ji}) < 0$, we obtain a knapsack problem.

2 Choose $K \in \mathbb{N}$ and natural numbers $d_1, \ldots, d_K$ and assign to every item $i \in N$ a coefficient $c'_i \in \{d_1, \ldots, d_K\}$, for instance: choose $d_1 < \ldots < d_K \in [\min_{i \in N} c_i, \ldots, \max_{i \in N} c_i]$. For every $i \in N$ set $c'_i := \max_{j=1 \ldots K}\{d_j : d_j \leq c_i\}$.

3 Compute a test set T of $P(d_1, \ldots, d_K)$.

4 For all $v \in T$ check whether $ct > 0$ and $x + v$ is feasible for (IP). In this case set $x := x + t$.

Note that the knapsack relaxation that we derive in Step 1 of the scheme above can simultaneously be used to derive valid inequalities for the integer programming problem (IP), see [CJP83].

4.2 Example: Minimal cover vectors

There is a one subclass of irreducible elements in every test set of (IP) that we want to introduce now. With each such element - as we will see - we can associate an inequality that is satisfied by all optimal points of the given instance. We call these elements *minimal cover vectors* and define them as follows.

Definition 4.2 *Let $K, K' \subseteq N$ be disjoint sets such that $\sum_{j \in K} a^j \leq \sum_{j \in K'} a^j$, but $\sum_{j \in K \cup \{i\}} a^j \not\leq \sum_{j \in K'} a^j$ for every $i \notin K$. If $\sum_{j \in K} e_j \succ_c \sum_{j \in K'} e_j$, then the vector $\sum_{j \in K} e_j - \sum_{j \in K'} e_j$ is called minimal cover vector.*

The name "minimal cover" reflects the fact that K' is a minimal subset with respect to inclusion such that $\sum_{j \in K'} a^j$ covers $\sum_{j \in K} a^j$. From the definition follows that if a minimum cover vector is reducible with respect to an objective function $c \in \mathbb{N}^n$, then the reduction vector is also a minimum cover vector. Hence, for every objective function $c \in \mathbb{N}^n$ there exists a subset of the set of all minimal cover vectors that must be contained in every test set of (IP). We did not succeed in designing a polynomial time algorithm that solves the augmentation problem for the set of minimal cover vectors. There is, however, a property of minimal cover vectors: they give rise to inequalities that must be satisfied by every optimum solution of the given integer programming problem.

Lemma 4.3 *If $v = \sum_{j \in K} e_j - \sum_{j \in K'} e_j$ is a minimum cover vector, then every optimum solution x satisfies the constraint*

$$\sum_{j \in K} x_j \geq \sum_{j \in K'} x_j - (|K'| - 1).$$

This example shows that a test set contains elements that can be used to derive inequalities satisfied by all optimal solutions. It can be further shown that the property of a minimum cover vector being reducible is reflected by the property that the associated inequality is dominated by an inequality that we derive from an irreducible vector. More precesily, if the minimal cover vector $v := \sum_{j \in K} e_j - \sum_{j \in K'} e_j$ is reducible, then there exists $\overline{K} \subseteq K$, $\overline{K'} \subseteq K'$ such that $w := \sum_{j \in \overline{K}} e_j - \sum_{j \in \overline{K'}} e_j \succ_c 0$ is a minimal cover vector. Under these assumptions the inequality

$$\sum_{j \in K} x_j \geq \sum_{j \in K'} x_j - (|K'| - 1)$$

is weaker than the inequality

$$\sum_{j \in \overline{K}} x_j \geq \sum_{j \in \overline{K'}} x_j - (|\overline{K'}| - 1).$$

In this sense, reducibility of augmentation vectors and domination of inequalities are at least in special cases related subjects.

References

[BiS92] L.J. Billera and B. Sturmfels, Fiber polytopes, Annals of Mathematics, 135, 1992, 527–549.

[BFS90] L.J. Billera, P. Filliman and B. Sturmfels, Constructions and complexity of secondary polytopes, Advances in Mathematics, 83, 1990, 155–179.

[CT91] P. Conti and C. Traverso, Buchberger algorithm and integer programming, Proceedings AAECC-9 (New Orleans), Springer Verlag LNCS 539, 1991, 130-139.

[CJP83] H. Crowder, E. L. Johnson and M. W. Padberg, Solving large-scale zero-one linear programming problems, Operations Research 31, No. 5, 1983, 803 - 834.

[Ed71] J. Edmonds, Matroids and the greedy algorithm, Mathematical Programming, 1, 1971, 127–136.

[GKZ94] I.M. Gel'fand, M. Kapranov and A. Zelevinsky, Multidimensional Determinants, Discriminants and Resultants, Birkhäuser, Boston, 1994.

[GLS88] M. Grötschel, L. Lovász, and A. Schrijver: *Geometric Algorithms and Combinatorial Optimization*, Algorithms and Combinatorics 2, Springer, Berlin, 1988; Second edition 1993.

[KSZ92] M.M. Kapranov, B. Sturmfels and A.V.Zelevinsky, Chow polytopes and general resultants, *Duke Math. Journal*, Vol 67, 1992, 189-218.

[Schr86] A. Schrijver, Theory of Linear and Integer Programming, Wiley-Interscience Series in Discrete Mathematics and Optimization, New York, 1986.

[St91] B. Sturmfels, Gröbner bases of toric varieties, Tôhoku Math. Journal, 43, 1991, 249–261.

[ST94] B.Sturmfels and R.Thomas, Variation of cost functions in integer programming, Technical Report, 1994, Cornell University.

[Th94] R.Thomas, A geometric Buchberger algorithm for integer programming, Mathematics of Operations Research, to appear.

[SWZ95] A. Schulz, R. Weismantel and G. Ziegler, 0/1 integer programming: optimization and augmentation are equivalent, to appear in Proceedings of the European Symposium on Algorithms 95, "Lecture Notes in Computer Science", Springer-Verlag, 1995.

[UWZ94] R. Urbaniak, R. Weismantel and G. Ziegler, A variant of Buchberger's algorithm for integer programming, *Preprint* SC 94-29, 1994, Konrad-Zuse-Zentrum Berlin.