Konrad-Zuse-Zentrum für Informationstechnik Berlin
Takustraße 7, D-14195 Berlin

Georg Ch. Pflug Andrzej Ruszczyński
Rüdiger Schultz

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Georg Ch. Pflug

Institut für Statistik und Operations Research
Universität Wien
Universitätsstraße 5, A-1090 Wien, Austria

Andrzej Ruszczyński

Department of Industrial Engineering
University of Wisconsin-Madison, Madison, WI 53706, U.S.A.

Rüdiger Schultz

Konrad-Zuse-Zentrum für Informationstechnik
Takustraße 7, D-14195 Berlin, Germany

Abstract

Expected recourse functions in linear two-stage stochastic programs with mixed-integer second stage are approximated by estimating the underlying probability distribution via empirical measures. Under mild conditions, almost sure uniform convergence of the empirical means to the original expected recourse function is established.

Key words: Stochastic Programming, Empirical Measures, Uniform Convergence, Value Functions of Mixed-Integer Linear Programs.

1 Introduction

Mathematical modelling of phenomena in nature, technology and economics typically involves some level of uncertainty. Depending on the modelling environment and the availability of (statistical) information on the random data stochastic programming offers models for finding optimal decisions under uncertainty. The present paper is devoted to linear two-stage stochastic programs with integer requirements in the second stage. Problems to be considered are formulated as follows

$$\min_{x \in X} \left[c^T x + \int f(x, \omega) P(d\omega) \right], \tag{1.1}$$

where $X \subset \mathbb{R}^{n_x}$ is the first stage feasible set and $f : \mathbb{R}^{n_x} \times \Omega \mapsto \mathbb{R}$ denotes the recourse function dependent on x and on an elementary event in some probability space (Ω, Σ, P). The recourse function is defined as the optimal value of the mixed-integer linear program

$$f(x, \omega) = \min \left\{ q^T y + q'^T y' \mid W y + W' y' = b(x, \omega),\ y' \in \mathbb{R}_+^{n'_y},\ y \in \mathbb{Z}_+^{n_y} \right\}. \tag{1.2}$$

Here, $b : \mathbb{R}^{n_x} \times \Omega \mapsto \mathbb{R}^{m_y}$ is a measurable function, q, q', W, W' are vectors and matrices of proper dimensions, and $\mathbb{Z}_+$ denotes the set of nonnegative integers.

Behind the model (1.1)-(1.2) there is a two-stage decision process under uncertainty. The first-stage decision x and the elementary event ω, which is observed only after x has been taken, determine the quantity $b(x, \omega)$ entering a second-stage (or recourse) optimization procedure. Optimization in (1.1) aims at finding a feasible first-stage (here-and-now) decision such that its direct costs plus the expectation of costs arising from the recourse procedure are minimal. The above scheme, although seemingly abstract, turns out to be quite powerful when optimizing decision making under uncertainty in many practical situations. For further details we refer to [8, 14].

Mathematical analysis of (1.1) focuses on the expectation $\int f(x, \omega) P(d\omega)$. If, for example, b is linear and (1.2) contains no integer requirements then f is convex piecewise linear, and the techniques of convex analysis can be applied. In the present setting, however, the integrality conditions in (1.2) lead to discontinuities in f. This is one of the reasons why integer recourse stochastic programming is, up to now, far less developed than its continuous counterpart. Some first contributions to theory and algorithms in stochastic integer programming can be found in [1, 2, 4, 7, 9, 10, 15, 16, 17].

We are interested in approximations of the expected value function

$$F(x) = \int f(x, \omega) P(d\omega) \tag{1.3}$$

obtained by replacing the original probability measure P by empirical measures. Given an independent identically distributed sample $s = \{s_i\}_{i=1}^{\infty} \in \Omega^\infty = \Omega^{\mathbb{N}}$, the empirical measures $P_n(s), n \in \mathbb{N}$ are defined by

$$P_n(s) = \frac{1}{n} \sum_{i=1}^{n} \delta_{s_i}, \tag{1.4}$$

where δ_{s_i} denotes point mass at s_i. This leads to the empirical mean

$$F_n(x) = \int f(x,\omega) P_n(s)(d\omega) = \frac{1}{n}\sum_{i=1}^{n} f(x,s_i). \tag{1.5}$$

As in [12], our main interest will be in deciding right from the data in (1.1)-(1.2) whether uniform convergence of F_n to F takes place for almost all s (with respect to the product probability P^∞ on Ω^∞). Again, a necessary and sufficient condition for uniform convergence of empirical means developed by Talagrand [19] will lead us to classes of right-hand sides $b(x,\omega)$ such that the mentioned uniform convergence takes place. In [12] we showed persistence of Talagrand's condition under transformations which, in particular, allow to form the value functions of linear programs with continuous variables. For building value functions of mixed-integer linear programs, however, using the pointwise integer round-up operation is indispensable in general [6]. Since Talagrand's condition does not persist under this transformation, mixed-integer recourse stochastic programs do not fall within the scope of [12]. The key issue of the present paper, therefore, is to elaborate an alternative verification of Talagrand's condition tailored to value functions of mixed-integer linear programs.

Let us add a few bibliographical comments. Our paper addresses a specific topic in stability of stochastic programs under perturbations of the integrating probability measure. Basically, there are two motivations for this line of research: incomplete information on the underlying measure and numerical difficulties in computing integrals like (1.3) for complicated (e.g., multivariate continuous) P. The majority of papers on stability of stochastic programs deals with continuous variables. With a certain accent on estimation via empirical measures, the references in [12] reflect developments in this field and are not repeated here. The papers [1, 15, 16] study continuity properties of the mapping assigning to the underlying measure P the expectation in (1.3); in [1], also the integrand in (1.3) may belong to the argument space of the mentioned mapping. A further joint feature of [1, 15, 16] is that integrands are discontinuous, namely in [1] general lower semicontinuous functions and in [15, 16] value functions of mixed-integer linear programs as in the present paper. Moreover, variations of P in these papers are more general than here. The measure P varies in a space endowed with weak convergence of probability measures. Results in [1, 15, 16] assert lower semicontinuity, continuity and Hölder estimates of the mentioned mappings, respectively. In addition, extensions towards stability of optimal values and sets of optimal solutions of the accompanying optimization problems are discussed. In [2], the authors study convergence of empirical means involving general lower semicontinuous integrands. Their setting comprehends the one adopted here since, as will be made precise in the next section, value functions of mixed-integer linear programs are lower semicontinuous under natural conditions. As a central result, epi-convergence of empirical means is established in [2] which can be seen as a one-sided version of the uniform convergence addressed in the present paper.

The rest of the paper is organized as follows. In Section 2 we collect the prerequisites on empirical measures and value functions that are needed for our analysis. Section 3 contains the main results on uniform convergence. In the final section, some concluding remarks are added.

2 Prerequisites on empirical measures and value functions

Our analysis of uniform convergence for expected recourse functions will combine known results on value functions of mixed-integer linear programs and on uniform convergence of abstract empirical means. To have a self-contained exposition and to introduce some notation, we collect these results in the present section.

Definition 2.1. A class of integrable functions $\varphi_x : \Omega \mapsto \mathbb{R}$, $x \in X$, is called a *P-uniformity class* if

$$\lim_{n\to\infty} \sup_{x\in X} \left| \int \varphi_x(\omega) P(d\omega) - \int \varphi_x(\omega) P_n(s)(d\omega) \right| = 0$$

for P^∞-almost all s.

In these terms, uniform convergence of (1.5) to (1.3) means that the family of functions $\omega \mapsto f(x,\omega)$, $x \in X$, is a P-uniformity class. Having in mind application to stochastic programming, we restrict our attention to functions which are measurable with respect to both arguments (x,ω). This avoids technical difficulties associated with non-measurability of sets defined with the use of the existence quantifier.

Of course, with $\varphi_x(\omega)$ taken as indicator functions of lower left orthants in a Euclidean space X, validity of the property in Definition 2.1 coincides with the well-known Glivenko-Cantelli Theorem on almost sure uniform convergence of distribution functions. This leads to the notion Glivenko-Cantelli problem for deciding whether an abstract family of integrands forms a P-uniformity class. In [19], a practicable necessary and sufficient criterion for detecting P-uniformity classes is given. As in [12], it will turn out to be most useful for the special case we have in mind. It is based on the following notion of P-stability that we reformulate with the mentioned simplification concerning measurability.

Definition 2.2. Let $\varphi : X \times \Omega \mapsto \mathbb{R}$ be measurable in both arguments. The class of functions $\omega \mapsto \varphi(x,\omega)$, $x \in X$, is called *P-stable* if for each $\alpha < \beta$ and each set $A \in \Sigma$ with $P(A) > 0$ there exists $n > 0$ such that

$$\begin{aligned} P^{2n} \Big\{ & (s_1,\ldots,s_n,t_1,\ldots,t_n) \in A^{2n} : (\exists x \in X) \\ & \varphi(x,s_i) < \alpha,\ \varphi(x,t_i) > \beta,\ i = 1,\ldots,n \ \Big\} < (P(A))^{2n}, \end{aligned}$$

where P^{2n} is the product probability on Ω^{2n}.

The criterion in [19] reads.

Theorem 2.3. ([19], Theorem 2). *Assume that the function $\varphi(x,\omega) : X \times \Omega \mapsto \mathbb{R}$ is measurable in both arguments. Then the following statements are equivalent:*

(a) the class of functions $\omega \mapsto \varphi(x,\omega)$, $x \in X$, is a P-uniformity class and $\int \varphi(x,\omega) P(d\omega)$, $x \in X$, is bounded;

(b) the class of functions $\omega \mapsto \varphi(x,\omega)$, $x \in X$, is P-stable and there exists v with $\int v(\omega)P(d\omega) < \infty$ such that, for all $x \in X$, $|\varphi(x,\omega)| \leq v(\omega)$ a.s.

Since we shall use this result arguing by contradiction, it is convenient to restate the definition of stability.

Remark 2.4. ([19], Proposition 4). *Let $\varphi : X \times \Omega \mapsto \mathbb{R}$ be measurable in both arguments. The class of functions $\omega \mapsto \varphi(x,\omega)$, $x \in X$, fails to be P-stable if and only if there exist $\alpha < \beta$ and $A \in \Sigma$ with $P(A) > 0$ such that for each $n \in \mathbb{N}$ and almost each $(s_1, \ldots, s_n) \in A^n$, for each subset I of $\{1, \ldots, n\}$ there is $x \in X$ with*

$$\varphi(x, s_i) < \alpha \quad \text{for} \quad i \in I$$

and

$$\varphi(x, s_i) > \beta \quad \text{for} \quad i \notin I.$$

The second basic ingredient of our subsequent analysis, the value function of a mixed-integer linear program, belongs to the well-studied objects in optimization theory. The properties displayed below can be looked up in [3, 5], for instance. We also refer to [16] where this collection of results is discussed in more detail than here.

Let

$$\Phi(t) = \min \left\{ q^T y + q'^T y' \mid Wy + W'y' = t,\ y' \in \mathbb{R}_+^{n'_y},\ y \in \mathbb{Z}_+^{n_y} \right\}$$

where W and W' are rational matrices. Assume further that

$$\text{for any} \quad t \in \mathbb{R}^{m_y} \quad \text{there exist} \quad y' \in \mathbb{R}_+^{n'_y}, y \in \mathbb{Z}_+^{n_y} \quad \text{such that} \quad Wy + W'y' = t, \tag{2.1}$$

$$\text{there exists} \quad u \in \mathbb{R}^{m_y} \quad \text{such that} \quad W^T u \leq q \quad \text{and} \quad W'^T u \leq q'. \tag{2.2}$$

Standard existence theorems from mixed-integer linear programming (cf., e.g., [11] Proposition I.6.7.) then yield $\Phi(t) \in \mathbb{R}$ for all $t \in \mathbb{R}^{m_y}$.

Proposition 2.5. *Suppose that* (2.1), (2.2) *are satisfied, then the following holds*

(i) *Φ is lower semicontinuous on $\mathbb{R}^{m_y}$ and there exists a countable partition $\mathbb{R}^{m_y} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$ such that Φ is Lipschitz continuous with uniform constant $L > 0$ on each of the $\mathcal{B}_i$; moreover, each of the $\mathcal{B}_i$ admits a representation $\mathcal{B}_i = \{t_i + \mathcal{K}\} \setminus \bigcup_{j=1}^{N_o} \{t_{ij} + \mathcal{K}\}$ where $\mathcal{K} := W'(\mathbb{R}_+^{n'_y})$, $t_i, t_{ij} \in \mathbb{R}^{m_y}$ $(i \in \mathbb{N},\ j = 1, \ldots, N_o)$, and N_o does not depend on i,*

(ii) *there exist constants $\gamma_1 > 0, \gamma_2 > 0$ such that for all $t_1, t_2 \in \mathbb{R}^{m_y}$ it holds*

$$|\Phi(t_1) - \Phi(t_2)| \leq \gamma_1 \|t_1 - t_2\| + \gamma_2,$$

(iii) *there exists a number $N_1 \in \mathbb{N}$ such that for any $t \in \mathbb{R}^{m_y}$ the ball around t with radius 1 intersects at most N_1 different subsets of the partition $\bigcup_{i=1}^{\infty} \mathcal{B}_i$.*

3 Uniform convergence

We now pass on to (1.3) and its approximation (1.5). Uniform convergence will be established by showing that, under suitable assumptions, the family of functions $\omega \mapsto \Phi(b(x,\omega)), x \in X$, is P-stable.

Before doing so, we again consider the partition $\bigcup_{i=1}^{\infty} \mathcal{B}_i$ arising in Proposition 2.5. By Proposition 2.5 (i), the boundary of each cell $\mathcal{B}_i$ may be decomposed into finitely many pieces such that each of them parallels a facet of $\mathcal{K}$ (by (2.1), $\mathcal{K}$ has to be full-dimensional). Introducing a sufficient number of hyperplanes that are parallel to facets of $\mathcal{K}$, each cell $\mathcal{B}_i$ can be subdivided into sets whose closures are polyhedra. Enriching, if necessary, the family of hyperplanes by members parallel to facets of the non-negative orthant, boundedness of the cells is achieved, and each $\mathcal{B}_i$ may be subdivided into sets of the mentioned type with arbitrarily small diameter. By Proposition 2.5 (iii), these subdivisions can be carried out in such a way that the number of subsets hit by an arbitrary ball of radius 1 is bounded above by a uniform constant. Of course, there are infinitely many subsets, but only finitely many "facet slopes" occur.

Altogether, for any $\delta > 0$, there exists a partition $\mathbb{R}^{m_y} = \bigcup_{i=1}^{\infty} \Pi_i$ such that, for each $i \in \mathbb{N}$, the closure of Π_i is a polyhedron with diameter less than δ, and the function Φ is Lipschitz continuous with modulus $L > 0$ on Π_i. Moreover, up to location in (infinitely many) parallel hyperplanes, there arise only finitely many different facets in the $\Pi_i, i \in \mathbb{N}$, and the number of cells hit by an arbitrary ball is bounded above by a constant depending only on the radius of the ball.

Proposition 3.1. *Suppose* (2.1) , (2.2) *to hold and assume further that*

(i) *for any fixed $a \in \mathbb{R}^{m_y}$, the family $\mathcal{I}$ of indicator functions*

$$\omega \mapsto \blacktriangleright_{x,a_o}(\omega), \quad x \in X, a_o \in \mathbb{R}$$

of the sets $\{\omega \in \Omega : a^T b(x,\omega) \leq a_o\}$ is P-stable,

(ii) *there exists v with $\int v(\omega)P(d\omega) < \infty$ such that $\|b(x,\omega)\| \leq v(\omega)$ a.s. for all $x \in X$.*

Then the family of functions $\omega \mapsto \Phi(b(x,\omega)), x \in X$, is a P-uniformity class.

Proof. Assume on the contrary that $\omega \mapsto \Phi(b(x,\omega)), x \in X$, is not P-stable. By Remark 2.4 then there exist $\alpha < \beta$ and $A \in \Sigma$ with $P(A) > 0$ such that for each $n \in \mathbb{N}$ and almost each $(s_1, \ldots, s_n) \in A^n$, for each subset I of $\{1, \ldots, n\}$ there is $x \in X$ with

$$\Phi(b(x,s_i)) < \alpha \quad \text{for} \quad i \in I \quad \text{and} \quad \Phi(b(x,s_j)) > \beta \quad \text{for} \quad j \notin I. \tag{3.1}$$

Let $\mathbb{R}^{m_y} = \bigcup_{i=1}^{\infty} \Pi_i$ be a subdivision of the type discussed above such that each of its cells has a diameter less than $\frac{\beta-\alpha}{L}$ where $L > 0$ denotes the Lipschitz modulus from Proposition 2.5.

Consider the family $\mathcal{I}_o$ of indicator functions $\omega \mapsto \blacktriangleright_{x,i}(\omega),\ x \in X,\ i \in \mathbb{N}$ of the sets $\{\omega \in \Omega : b(x,\omega) \in \Pi_i\}$. Since the closures of the Π_i are polyhedra and since, up to location in parallel hyperplanes, only finitely many facets occur, the family $\mathcal{I}_o$ arises from $\mathcal{I}$ by finitely many complementations and min-operations. Proposition 24 in [19] then yields that $\mathcal{I}_o$ is P-stable.

Let $M > (P(A))^{-1} \int v(\omega)P(d\omega)$ and let $A_M = \{\omega \in \Omega : v(\omega) \leq M\}$. By Markov's inequality,

$$P(A_M) \geq 1 - M^{-1} \int v(\omega)P(d\omega) > 1 - P(A).$$

Consequently the event $A_0 = A \cap A_M$ has a positive probability. By (ii), $\|b(x,\omega)\| \leq M$ for all $x \in X$ and all $\omega \in A_0$. In view of the local finiteness of $\bigcup_{i=1}^{\infty} \Pi_i$ there exists $N \in \mathbb{N}$, not depending on x, such that $b(x, A_o)$ intersects at most N different cells from $\bigcup_{i=1}^{\infty} \Pi_i$.

Now consider the family $\mathcal{I}_1$ of indicator functions given by

$$\omega \mapsto \max_{j=1,\ldots,N} \blacktriangleright_j(\omega), \quad \blacktriangleright_j(\omega) \in \mathcal{I}_o.$$

It coincides with the indicator functions of unions of at most N sets of the type $\{\omega \in \Omega : b(x,\omega) \in \Pi_i\}$. Again Proposition 24 in [19] implies that $\mathcal{I}_1$ is P-stable.

From (3.1) we conclude that there are no two points $b(x,s_i), b(x,s_j), i \in I, j \notin I$ belonging to the same cell in $\bigcup_{i=1}^{\infty} \Pi_i$, since otherwise

$$|\Phi(b(x,s_i)) - \Phi(b(x,s_j))| \leq L \cdot \|b(x,s_i) - b(x,s_j)\| \leq L \cdot \frac{\beta - \alpha}{L} = \beta - \alpha.$$

Hence for each $n \in \mathbb{N}$ and almost all $(s_1, \ldots, s_n) \in A_o^n$, for each subset $I \subseteq \{1, \ldots, n\}$ there is $x \in X$ such that no two points $b(x,s_i), b(x,s_j), i \in I, j \notin I$ belong to the same cell in $\bigcup_{i=1}^{\infty} \Pi_i$. Since the $b(x,s_i), i \in I$, belong to at most N different cells from $\bigcup_{i=1}^{\infty} \Pi_i$, there exists an indicator function $\blacktriangleright(\omega) \in \mathcal{I}_1$ such that $\blacktriangleright(s_i) = 1$ for all $i \in I$ and $\blacktriangleright(s_j) = 0$ for all $j \notin I$.

By Remark 2.4 this contradicts the P-stability of $\mathcal{I}_1$. Therefore the family of functions $\omega \mapsto \Phi(b(x,\omega)), x \in X$, is P-stable.

To obtain the assertion via Theorem 2.3 it remains to show that there exists $\bar{v}$ with $\int \bar{v}(\omega)P(d\omega) < \infty$ and $|\Phi(b(x,\omega))| \leq \bar{v}(\omega)$ a.s. whenever $x \in X$. Indeed, since assumptions (2.1) and (2.2) together imply that $\Phi(0) = 0$ we obtain with $\gamma_1 > 0, \gamma_2 > 0$ as in Proposition 2.5 (ii)

$$|\Phi(b(x,\omega))| = |\Phi(b(x,\omega)) - \Phi(0)| \leq \gamma_1 \|b(x,\omega)\| + \gamma_2 \leq \gamma_1 v(\omega) + \gamma_2 \quad \text{a.s.}$$

and we can take $\bar{v}(\omega) = \gamma_1 v(\omega) + \gamma_2$. 2

The following statement shows that assumption (i) in the above proposition holds for a fairly wide class of right-hand sides $b(x,\omega)$. Its proof will employ Vapnik-Červonenkis theory which is combinatorial by nature but has a strong impact on uniform convergence of empirical means (see [13, 18, 20], for instance).

A family $\mathcal{C}$ of sets is called a Vapnik-Červonenkis class (VČ class) if there exists $m \in \mathbb{N}$ such that for any finite set E with m elements not every subset $E_o \subset E$ arises as an intersection $E_o = E \cap C$ for some $C \in \mathcal{C}$. Now it holds that a family $\mathcal{F}$ of indicator functions is P-stable if the corresponding family of sets $\mathcal{C}$ is a VČ class. Indeed, suppose that $\mathcal{F}$ were not P-stable. For $m \in \mathbb{N}$ as in the definition of the VČ property we then obtain by Remark 2.4 that there are points $s_1, \ldots, s_m$ such that for each subset I of $\{1, \ldots, m\}$ there exists a function $f \in \mathcal{F}$ such that $f(s_i) = 1$ if and only if $i \in I$. In other words, for each subset I of $\{1, \ldots, m\}$ there exists a set $C = C_f \in \mathcal{C}$ such that $s_i \in C$ if and only if $i \in I$ contradicting the assumption that $\mathcal{C}$ is a VČ class.

Compared with P-stability the above sufficient condition in terms of VČ classes is rather restrictive since it does not depend on the probability measure P. What makes the condition useful, however, is that, in the literature, there are several techniques for verifying the VČ property. These can be readily applied as we will see in the following lemma.

Lemma 3.2. *If the family of functions $\omega \mapsto b(x, \omega), x \in X$, belongs to a finite-dimensional vector space then the family $\mathcal{I}$ of indicator functions introduced in Proposition 3.1 is P-stable.*

Proof. We employ Lemma 18 in [13], p.20, which states that families of level sets of real-valued functions in a finite-dimensional vector space of functions have the VČ property. Let $\{\xi_1(.), \ldots, \xi_d(.)\}$ denote a basis of the vector space containing $\{b(x, .) : x \in X\}$. For fixed $a \in \mathbb{R}^{m_y}$, the set $\{a^T\xi_1(.), \ldots, a^T\xi_d(.), \mathbb{1}\}$, with $\mathbb{1}$ denoting the constant function with value 1, generates a finite-dimensional vector space of real functions. Any set $\{\omega \in \Omega : a^T b(x, \omega) \leq a_o\}, x \in X, a_o \in \mathbb{R}$ now can be written as

$$\{\omega \in \Omega : \sum_{r=1}^{d} \lambda_r a^T \xi_r(\omega) - a_o \cdot \mathbb{1} \leq 0\},$$

i.e., as a lower level set of a real-valued function in a finite-dimensional vector space. Therefore, the family of sets corresponding to $\mathcal{I}$ is a VČ class, and $\mathcal{I}$ is P-stable.2

Remark 3.3. *In particular, the important case where*

$$b(x, \omega) = h(\omega) - T(\omega)x$$

is covered by the above lemma. Indeed, the set $\{h(\omega), t_1(\omega), \ldots, t_{n_x}(\omega)\}$, with $t_1(\omega), \ldots, t_{n_x}(\omega)$ denoting the columns of $T(\omega)$ generates a finite-dimensional vector space of functions that contains all the right-hand sides $b(x, .), x \in X$.

4 Concluding remarks

Uniform convergence of the empirical mean (1.5) to the expected recourse function (1.3) was established under essentially the condition that families of indicator functions associated with the right-hand side $b(x, \omega)$ are P-stable.Verification of this condition may be non-trivial since suitable sets of positive P-measure have to be identified. A sufficient condition for P-stability that does not involve the measure P can be formulated via Vapnik-Červonenkis theory. This condition, although coarse from formal viewpoint, already covers many practically important cases. For instance, it enables us to establish the P-stability of the relevant indicator functions by showing that $\{b(x,.) : x \in X\}$ is contained in a finite-dimensional vector space of functions (Lemma 3.2). In particular, this includes the linear case where $b(x,\omega) = h(\omega) - T(\omega)x$.

Altogether, Proposition 3.1 fairly extends our abilities to check uniform convergence of empirical approximations to expected recourse functions in stochastic programming. Up to now, the only related result we are aware of applied to the case $b(x,\omega) = h(\omega) - Tx$. It may be obtained as a conclusion from Proposition 3.1 in [16].

As is well known in stability analysis, uniform convergence of approximate objective functions has immediate consequences for the continuity of the optimal-value function and the upper semicontinuity of the solution set mapping. Without elaborating this, we here only mention that Proposition 3.1 enables us to prove such stability results in the context of estimation via empirical measures.

References

[1] Z. Artstein and R.J.-B. Wets, "Stability results for stochastic programs and sensors, allowing for discontinuous objective functions", *SIAM Journal on Optimization* 4(1994) 537-550.

[2] Z. Artstein and R.J.-B. Wets, "Consistency of minimizers and the SLLN for stochastic programs", *Journal of Convex Analysis* 2(1995) 1-17.

[3] B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer, *Nonlinear Parametric Optimization*, Akademie-Verlag, Berlin 1982.

[4] J.R. Birge and M.A.H. Dempster, "Stochastic programming approaches to stochastic scheduling", Preprint, Department of Industrial and Operations Engineering, The University of Michigan, Ann Arbor 1995.

[5] C.E. Blair and R.G. Jeroslow, "The value function of a mixed integer program: I", *Discrete Mathematics* 19(1977) 121-138.

[6] C.E. Blair and R.G. Jeroslow, "Constructive characterization of the value function of a mixed-integer program I", *Discrete Applied Mathematics* 9(1984) 217-233.

[7] C.C. Carøe and J. Tind, "L-shaped decomposition of two-stage stochastic programs with integer recourse", Technical Report, Institute of Mathematics, University of Copenhagen, 1995.

[8] P. Kall and S.W. Wallace, *Stochastic Programming*, J. Wiley & Sons, Chichester 1994.

[9] G. Laporte and F.V. Louveaux, "The integer L-shaped method for stochastic integer programs with complete recourse", *Operations Research Letters* 13(1993) 133-142.

[10] F.V. Louveaux and M.H. van der Vlerk, "Stochastic programs with simple integer recourse", *Mathematical Programming* 61(1993) 301-326.

[11] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization*, Wiley, New York 1988.

[12] G.Ch. Pflug, A. Ruszczyński and R. Schultz, "On the Glivenko-Cantelli problem in stochastic programming: linear recourse and extensions", Working Paper WP-96-020, International Institute for Applied Systems Analysis, Laxenburg 1996, *Mathematics of Operations Research*, submitted.

[13] D. Pollard, *Convergence of Stochastic Processes*, Springer-Verlag, New York 1984.

[14] A. Prékopa, *Stochastic Programming*, Kluwer Academic Publishers, Dordrecht 1995.

[15] R. Schultz, "On structure and stability in stochastic programs with random technology matrix and complete integer recourse", *Mathematical Programming* 70(1995) 73-89.

[16] R. Schultz, "Rates of convergence in stochastic programs with complete integer recourse", *SIAM Journal on Optimization* 6(1996) 1138-1152.

[17] R. Schultz, L. Stougie and M.H. van der Vlerk, "Solving stochastic programs with complete integer recourse: a framework using Gröbner bases", Preprint SC 95-23, Konrad-Zuse-Zentrum für Informationstechnik Berlin, 1995.

[18] G.R. Shorack and J.A. Wellner, *Empirical Processes with Applications to Statistics*, Wiley, New York 1986.

[19] M. Talagrand, "The Glivenko-Cantelli problem", *The Annals of Probability* 15(1987) 837-870.

[20] V.N. Vapnik and A.Y. Červonenkis, "Necessary and sufficient conditions for the uniform convergence of means to their expectations", *Theory of Probability and Applications* 26(1981) 532-553.