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A comparison of four approaches to the calculation of conservation laws

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A comparison of four approaches to the calculation of conservation laws

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Abstract

The paper compares computational aspects of four approaches to compute conservation laws of single differential equations or systems of them, ODEs and PDEs. The only restriction, required by two of the four corresponding computer algebra programs, is that each DE has to be solvable for a leading derivative. Extra constraints may be given. Examples of new conservation laws include non-polynomial expressions, an explicit variable dependence and conservation laws involving arbitrary functions. Examples involve the following equations: Ito, Liouville, Burgers, Kadomtsev-Petviashvili, Karney-Sen-Chu-Verheest, Boussinesq, Tzetzeica, Benney.

1 Introduction

As is well known, conservation laws play an important role in Mathematical Physics. The knowledge of conservation laws is useful in the numerical integration of partial differential equations (PDEs) [21], for example, to control numerical errors. Also, the investigation of conservation laws of the Korteweg de Vries equation was the starting point of the discovery of a number of techniques to solve evolutionary equations [26] (Miura transformation, Lax pair, inverse scattering technique, bi-Hamiltonian structures). The existence of a large number of conservation laws of a PDE (system) is a strong indication of its integrability. Conservation laws play an important role in the theory of non-classical transformations [23],[24] and in the theory of normal forms and asymptotic integrability [25]. Programs described below are able to find conservation laws involving the independent variables explicitly. Finding such conservation laws is a good challenge for the inverse scattering technique.

The purpose of the methods described below is to pose as few restrictions as possible on the differential equations (DEs) to be investigated. For example, it is not assumed that any Lie-symmetries are known, nor that the equations are equivalent to the Euler-Lagrange equations of a variational problem. Instead we attempt to solve the conservation law condition directly. The strategy will be to make a local ansatz involving only the dependent variables and their derivatives. Further, the order of the derivatives is bounded in order to obtain an over determined PDE problem which subsequently is solved with the computer algebra package Crack [33], [34].

In an earlier paper [35] three of the methods were discussed with emphasis put on the computer algebra algorithms involved. In this paper we present an additional fourth method and compare these methods in terms of complexity and functionality.

The rest of the paper is organized as follows. In section 2 a reminder on issues of the equivalence of conservation laws will provide the motivation for the four approaches which are explained in section 3 followed by an overview. In section 4 related computer algebra programs are shortly described and examples are given. Extensions of the basic usage of these programs are discussed in section 5. We start with comments on the equivalence of conservation laws.

2 The equivalence of conservation laws

We adopt the notation of the book of Olver [27] where the question of equivalence of conservation laws is described in more detail in chapter 4.3. Independent variables will be denoted by $x = (x^1, x^2, \ldots, x^p)$. The differential equations are $\Delta(x, u^{(n)}) = 0$ (i.e. $\Delta_1 = 0, \ldots, \Delta_q = 0$), for q functions $u = (u^1, u^2, \ldots, u^q)$, $u^{(n)}$ denoting u-derivatives of order up to n. We will use $_J$ as a multiple index denoting partial derivatives, for example, u^α_J will stand for an partial derivative of arbitrary order and D_J will denote multiple total differentiations. The multiplicity of partial derivatives with respect to one variable can be indicated with a number, for example, $\partial^{(5)}u/((\partial t)^2(\partial x)^3) = u_{2t3x}$.

The conservation law that is to be fulfilled by solutions of $\Delta = 0$ is $\operatorname{Div} P = 0$ with conserved current $P = (P^1, \ldots, P^p)$. This amounts to finding P^i such that $\operatorname{Div} P = 0$ modulo $\Delta = 0$ and $D_J\Delta = 0$. To have a way of counting conservation laws and of comparing them, they have to be put into an invariant form. Two conservation laws $\operatorname{Div} P = 0$ and $\operatorname{Div} \tilde{P} = 0$ are equivalent if $0 = \operatorname{Div}(P - \tilde{P}) = \operatorname{Div} R$ is a trivial conservation law.

i) The first kind of triviality is the case that $R = 0$ for all solutions of $\Delta = 0$, i.e. P and $\tilde{P}$ differ only by multiples of Δ and $D_J\Delta$. A way to cure this equivalence is to solve the system $\Delta = 0$ and its prolongations $D_J\Delta = 0$ for certain derivatives u^α_J and to substitute them in P. If the conservation laws are not yet calculated but one wants to prevent the calculation of equivalent conservation laws then the equations $\Delta = 0$ need not be solvable for some derivatives u_J. In that case one just drops the dependency of P on a leading derivative of u_J and all derivatives of u_J from the beginning of the calculation.

ii) The second kind of triviality occurs if $\operatorname{Div} R = 0$ for all functions $u = f(x)$, i.e. if R is a curl. The way to cure this kind of equivalence is to calculate characteristic functions Q_ν of a conservation law which are called integrating factors in the case of ordinary differential equations (ODEs) in the following way.

$$\begin{aligned}
& \operatorname{Div} P = 0 && (\text{mod } \Delta_\nu = 0, \;\; D_J\Delta_\nu = 0) && (1)\\
\iff \;\; \exists Q^J_\nu : \;\; & \operatorname{Div} P = \sum_{\nu,J} Q^J_\nu D_J \Delta_\nu && (\text{identically in } \textit{all } \; x, u^\alpha_J) && (2)\\
& = \sum_{\nu,J} D_J(Q^J_\nu \Delta_\nu) - D_J(Q^J_\nu)\Delta_\nu && (\text{repeatedly}) && (3)\\
& = \operatorname{Div} R + \sum_\nu Q_\nu \Delta_\nu && && (4)
\end{aligned}$$

It is known ([27], p. 272) that for a totally non degenerate system $\Delta = 0$, the equivalence class of conservation laws $\operatorname{Div} P = 0$ is determined uniquely by the characteristic functions Q_ν up to equivalence of type i). To calculate the Q_ν one uses the fact that the Euler operators $E_\nu = \sum_J (-D)_J \partial/\partial u^\nu_J$ acting on an expression give identically zero iff this expression is a divergence, where the D are total derivatives. Conditions for the Q_ν are therefore

$$\forall \nu : \quad 0 = E_\nu \left(\sum_\mu Q_\mu \Delta_\mu \right) = \sum_J (-D)_J \left(\frac{\partial}{\partial u^\nu_J} \sum_\mu Q_\mu \Delta_\mu \right). \tag{5}$$

On the space of solutions $\Delta_\mu = 0$ this gives

$$0 = \sum_{\mu,J} (-D)_J \left(Q_\mu \frac{\partial \Delta_\mu}{\partial u^\nu_J} \right) \Bigg|_{\Delta_\mu = 0} \qquad \forall \nu. \tag{6}$$

Conditions (6) are known as adjoint symmetry conditions which are necessary but not sufficient for the Q_μ to be characteristic functions of first integrals.

iii) For any two conservation laws $0 = \operatorname{Div} P$ and $0 = \operatorname{Div} \tilde{P}$, $0 = \operatorname{Div}(P + \tilde{P})$ is also a conservation law. By determining conservation laws with characteristic functions of successively increasing order, constant multiples of characteristic functions of lower order can be dropped.

iv) In the case of (systems of) ODEs the characteristic functions are called *integrating factors*, and P is a scalar, called a *first integral*. Any arbitrary function of first integrals is a first integral as well.

The four approaches described in the following four sections are to solve conditions (1), (4) with $R = 0$, (5) and (6).

3 The four approaches

3.1 A first approach

The first approach is to solve

$$\mathrm{Div}\, P = 0 \qquad (\mathrm{mod}\ \Delta_\nu = 0,\ \ D_J \Delta_\nu = 0) \tag{7}$$

directly.

The condition (7) is made over determined by restricting the P^i to be differential expressions in the u of some order k. Characteristic features of this approach are

(+) A single, short first order PDE is to be solved.

(0) Characteristic functions have to be computed from P in a straight forward calculation (described in [35]). This is done within the computer algebra program ConLaw1 which implements the first approach as part of the whole computation.

(−) It would be computationally expensive for a corresponding computer program to drop free functions which correspond to trivial conservation laws during the solution of (7)[1]. Hence, the condition (7) has to be solved in full generality and trivial conservation laws have to be dropped afterwards. That means that the task for the computer program is unnecessarily hard through the presence of the trivial conservation laws in the general solution. A rule of thumb says that the difficulty in solving a linear over determined PDE (system) depends less on the order or size of the PDE but more on the complexity of the result[2]. That means the trivial conservation laws will complicate the solution of (7), the more so the more independent variables are present.

(−) In most cases the expressions for the P^i are more complicated than the expressions for the characteristic functions Q_μ which by the above rule of thumb indicates a more difficult computation than the solution of equations involving only Q_μ.

To illustrate and compare all four approaches we will apply each to finding conservation laws of the sin-Gordon equation

$$u_{tx} - \sin(u) = 0. \tag{8}$$

If the program ConLaw1 is called to find conservation laws with conserved current P^t, P^x of order 0, then it will reply that it is not applicable. This is because Div P would be of first order in u, so equation (8) could not be used to substitute u_{tx} and therefore only trivial conservation laws would result.

Details of higher order investigations are given in table 1 below. $u^{(n)}$ stands for all derivatives of u of order 0 to n. $u_{tx}^{(n)}$ stands for all derivatives of u_{tx} up to order n, for example, $u_{tx}^{(1)}$ would be the derivatives u_{tx}, u_{2tx}, u_{t2x}. Finally, $u^{(n)}/u_{tx}^{(k)}$ stands for all derivatives of u up to order n apart from u_{tx} and all its derivatives up to order k. The conservation laws are given in the appendix (in the table only the equation number is referenced). For each conservation law in the appendix (apart from the first) there exists another one resulting from the exchange $t \leftrightarrow x$.

[1] An algorithm for that is given in [35].

[2] For example, if an over determined PDE (system) has no solution then a differential Gröber Basis calculation will quickly produce PDEs of lower and lower order until a contradiction is reached. On the other hand, if a PDE system has arbitrary functions in its general solution (as is the case with the PDE (7)) then computing a differential Gröbner Basis will *not* produce a system that is solvable by only integrating ODEs.

The times given in the table are measured on a 266 MHz Pentium PC running a 80 MByte REDUCE 3.6 session under LINUX using the Sep. 1998 version of the program CRACK for solving the over determined conditions. The 80 MByte were not necessary. For example, it is possible (using CONLAW2 which implements the 4^{th} method described below) to investigate up to 4^{th} order laws with 4 MByte and up to 7^{th} order laws with 8 MByte. To get this high in order with relatively low memory consumption, one has to give in CRACK the study of integrability conditions a higher priority than the integration of equations. The price is a higher computing time. The times in the last column are to be understood only as *very* rough indicators[3]. They depend sensitively on the order of priorities with which modules are to be used within the program CRACK (see the manual [33] and about its availability the end of the section 6).

When condition (7) is solved, the P^i that are calculated initially do not contain u_{tx} nor its derivatives. Only when the characteristic function Q is computed using (2)-(4) then u_{tx} is introduced through R in (4) . Finally CONLAW1, returns the conservation law in the form (9) below, i.e. with P^i and Q.

order of P^i	no of terms	independent variables, [no of var.]	functions to compute, [no of arg.]	cons. laws found	time to solve (7)
1	8	$t, x, u^{(2)}/u_{tx}$, [7]	$P^t, P^x(t, x, u^{(1)})$, [5]	(26),(27)	9 sec
2	12	$t, x, u^{(3)}/u_{tx}^{(1)}$, [9]	$P^t, P^x(t, x, u^{(2)}/u_{tx})$, [7]	(28)	38 sec
3	18	$t, x, u^{(4)}/u_{tx}^{(2)}$, [11]	$P^t, P^x(t, x, u^{(3)}/u_{tx}^{(1)})$, [9]	none[4]	34 min
4	26	$t, x, u^{(5)}/u_{tx}^{(3)}$, [13]	$P^t, P^x(t, x, u^{(4)}/u_{tx}^{(2)})$, [11]	low memory	

Table 1: The program CONLAW1 applied to compute conservation laws of the sin-Gordon equation.

3.2 A second approach

The next approach consists in solving

$$\mathrm{Div}\, P = \sum_\nu Q_\nu \Delta_\nu \tag{9}$$

directly, i.e. finding P^i, Q_μ that satisfy (9) identically in x^i, u^α_J. Equations $\Delta = D_J\Delta = 0$ are *not* used for substitutions in (9) but they are used to reduce dependencies of the Q_μ.

The problem becomes over determined by restricting the order of the Q_μ, i.e. $Q_\mu = Q_\mu(x, u^{(k)})$ for some k and by taking Q_μ mod $\Delta, D_J\Delta$, i.e. having Q_μ independent of one u-derivative (and their derivatives) from each one of the equations Δ_ν. If Q_μ would be allowed to depend on all variables which occur in (9) then this equation could simply be solved algebraically, by eliminating one of the Q_μ. But that would mean division through one Δ_μ and therefore Q_μ being singular for solutions of $\Delta_\mu = 0$.

The second approach has the following characteristics:

[3]For example, the computing times reported in [35] are now (10 months later) reduced by a factor of more than ten for higher orders.

[4]CRACK was not able to solve all equations completely because the general solution of (7) involves free functions (related to trivial conservation laws) which complicates the problem considerably for the computer program.

(+) The conservation law condition (9) is a single first order PDE as in the first approach.

(+) By calculating characteristic functions and furthermore characteristic functions modulo $\Delta = 0$, conservation laws are uniquely characterized.

(+) The effort in formulating conditions is as low as in the first approach.

(+) The P^i and Q_μ are computed in one go.

(0) The number of functions to compute is higher than in the first approach and also the number of jet-variables (derivatives of u) because no substitutions are done in (9). The resulting complication is not too big as more variables means a higher over determination and simplification.

(−) If the order of Δ is n and the order of Q_μ is chosen to be k then the order of P^i at the start of the computation has to be $\max(k,n) - 1$. In this approach the investigations with $k < n$ are not much simpler than the case $k = n$. This matters when the order n of Δ and the number p of variables x are high. Therefore this approach is not very efficient for low order conservation laws of high order equations.

For example, for zeroth order conservation laws ($k = 0$) of the Kadomtsev-Petviashvili equation (16) the P^i are taken initially as functions of the 23 variables $t, x, y,\ u, u_t,\ u_x, u_y,\ u_{tt}, \ldots, u_{yy},\ u_{ttt}, \ldots, u_{yyy}$ and the conservation law condition (9) is a condition in 38 variables (including the 4^{th} u-derivatives). That is a much harder problem than the corresponding conditions (5),(6). For example, in this case condition (5) is a single 4^{th} order PDE in also 38 variables but for only *one* function Q of only *four* variables!

(−) When looking for conservation laws with the first method, gradually increasing the order of the conserved current P gives each conservation law in its lowest order form, i.e. a form where P is of minimal order. This is not necessarily the case using the 2^{nd} method. The transformation (4) adding R to P may increase the order of P. This implies an increase of complexity having to go up in order to get the equivalent conservation law. To give an example, the Tzetzeica equation $u_{xt} = e^u - e^{-2u}$ (analysed in [30],[22]) has the conservation law

$$\begin{aligned} 0 \quad = \quad & D_t\left[3u_{xxx}^2 - 5u_{xx}^3 + 15u_{xx}^3 u_x^2 + u_x^6\right] + \\ & D_x\left[-3e^u\left(u_{xx}^2 + 2u_{xx}u_x^2 + 2u_x^4\right) - 3e^{-2u}\left(2u_{xx}^2 - 8u_{xx}u_x^2 + u_x^4\right)\right] \end{aligned}$$

with a third order conserved current. (In [22] an infinite list of conservation laws is given.) Bringing the above conservation law to the form (9) as it would be found with the second method, it becomes

$$\begin{aligned} & 6\left(u_{xxxxx} + 5u_{xxx}u_{xx} - 5u_{xxx}u_x^2 - 5u_{xx}^2 u_x + u_x^5\right)\left(u_{tx} - e^u + e^{-2u}\right) \\ = \quad & D_t\left[3u_{xxx}^2 - 5u_{xx}^3 + 15u_{xx}^2 u_x^2 + u_x^6\right] + \\ & D_x\, 3\left[2u_{tx}u_{xxxx} - 2u_{txx}u_{xxx} + 5u_{tx}u_{xx}^2 - 10u_{tx}u_{xx}u_x^2\right. \\ & \qquad +e^u\left(2u_{xxx}u_x - 2u_{xxxx} - 6u_{xx}^2 + 8u_{xx}u_x^2 - 2u_x^4\right) \\ & \qquad \left.+e^{-2u}\left(2u_{xxxx} + 4u_{xxx}u_x + 3u_{xx}^2 - 2u_{xx}u_x^2 - u_x^4\right)\right] \end{aligned}$$

with a 4^{th} order conserved current.

Applying the program ConLaw3 that corresponds to the above method to the sin-Gordon equation (8) gives the following table.

order of Q	no of terms	independent variables, [no of var.]	functions to compute, [no of arg.]	cons. laws found	time to solve (9)
0	10	$t, x, u^{(2)}$, [8]	$P^t, P^x(t, x, u^{(1)})$, [5] $Q(t, x, u)$, [3]	none	3 sec
1	10	$t, x, u^{(2)}$, [8]	$P^t, P^x, Q(t, x, u^{(1)})$, [5]	(26),(27)	8.3 sec
2	10	$t, x, u^{(2)}$, [8]	$P^t, P^x(t, x, u^{(1)})$, [5] $Q(t, x, u^{(2)}/u_{tx})$, [7]	none	0.2 sec
3	16	$t, x, u^{(3)}$, [12]	$P^t, P^x(t, x, u^{(2)})$, [8] $Q(t, x, u^{(3)}/u_{tx}^{(1)})$, [9]	low memory	

Table 2: The program ConLaw3 applied to compute conservation laws of the sin-Gordon equation.

3.3 A third approach

Instead of calculating the conserved current P^i directly, the third approach is to calculate characteristic functions Q_μ first and from them P^i afterwards using formulas of Anco & Bluman [2],[3],[5] in a form described in [35] or using repeatedly the Crack routine for integrating exact DEs. The condition as derived in [27],[2] is:

$$0 = \sum_J (-D)_J \left(\frac{\partial}{\partial u^\nu_J} \sum_\mu Q_\mu \Delta_\mu \right) \quad \forall \nu. \tag{10}$$

Typical features are:

(+) Equations (10) are equivalent to (9) and therefore necessary and sufficient.

(+) The usually more complicated P^i are eliminated and as in the 2^{nd} method, no trivial conservation laws are calculated which otherwise unnecessarily complicate the calculation.

(+) The highest u-derivatives in conditions (10) are of the order $2n$ where n is the order of the u-derivatives in $\sum_\mu Q_\mu \Delta_\mu$. The harder the problem, i.e. the higher n and the higher the number of variables, the more u-derivatives occur only explicitly in (10) and can be used for a direct separation (splitting). Higher over determination simplifies the solution of (10).

(−) Equations (10) consist of as many equations as there are dependent variables u^μ and the unknown functions Q_μ appear with n^{th} order derivatives.

(−) For an increasing order of the Q_μ, number of u^ν and number of x^i, the size of (10) can soon become unmanageable.

Applying the program ConLaw4 that corresponds to the above method to the sin-Gordon equation (8) gives the following table 3. The striking feature of this approach is the quick increase of the size of conditions. Apart from the order 0 case they increase by a factor of about 7 which itself is increasing slightly with the order. The size of conditions prevents going higher in the order. On the other hand, the completeness of conditions generated simplifies the solution in difficult cases and speeds up the solution of the over determined system as long as it is not too large from the beginning.

order	no of terms	independent variables, [no of var.]	functions to compute, [no of arg.]	cons. laws found	time to solve (10) h:min:sec
0	7	$t, x, u^{(1)}, u_{tx}$, [6]	$Q(t, x, u)$, [3]	none	0.7 sec
1	22	$t, x, u^{(2)}$, [8]	$Q(t, x, u^{(1)})$, [5]	(26)	2.8 sec
2	154	$t, x, u^{(3)}$, [17]	$Q(t, x, u^{(2)}/u_{tx})$, [7]	none	4.7 sec
3	1116	$t, x, u^{(4)}$, [24]	$Q(t, x, u^{(3)}/u_{tx}^{(1)})$, [9]	(28)	5 min 17 sec
4	8402	$t, x, u^{(5)}$, [34]	$Q(t, x, u^{(4)}/u_{tx}^{(2)})$, [11]	none	10 h 49 min[5]
5	64064	$t, x, u^{(6)}$, [41]	$Q(t, x, u^{(5)}/u_{tx}^{(3)})$, [13]	-	> 2 days

Table 3: The program ConLaw4 applied to compute conservation laws of the sin-Gordon equation.

3.4 A fourth approach

Projecting conditions (10) into the solution space we obtain

$$0 = \sum_{\mu,J} (-D)_J \left(Q_\mu \frac{\partial \Delta_\mu}{\partial u^\nu_J} \right) \Bigg|_{\Delta_\mu = 0} \qquad \forall \nu. \tag{11}$$

Characteristic features of this method are similar to those of the third method with the following modifications:

(+) The conditions usually involve fewer terms than in the third approach which can be decisive but as the conditions (11) are not sufficient, they are less over determined and may be harder to solve than those in the third approach.

(−) After computing the Q_μ, it has to be checked whether P^i exist that satisfy $\mathrm{Div}\, P = \sum_\nu Q_\nu \Delta_\nu$ ([2],[3],[4],[35]). If they do not exist then the Q_μ correspond to an adjoined symmetry but not to a conservation law.

(−) If the 4$^{\text{th}}$ method finds adjoined symmetries which are not conservation laws, then the question remains unanswered whether there are linear combinations of the adjoined symmetries which are conservation

[5] This time was nearly completely spent to formulate the condition and to separate it into 823 individual equations for Q. Then already the 3$^{\text{rd}}$ step gave that Q can not depend on 4$^{\text{th}}$ order derivatives.

laws. For example, the program ConLaw2 applying the third method finds 5 first integrals for the ODE $y'' + y = 0$ with an integrating factor at most linear in y':

$$\cos(x)^2 y' + \cos(x)\sin(x)y, \quad 2\cos(x)^2 y - 2\cos(x)y'\sin(x) - y, \quad \cos(x), \quad \sin(x), \quad y',$$

whereas the program using the 4$^{\mathrm{th}}$ method finds 4 such conservation laws with integrating factors

$$-2\cos(x)^2 y' - 2\cos(x)\sin(x)y + y', \quad \cos(x), \quad \sin(x), \quad y'$$

and 4 adjoint symmetries

$$\cos(x)y'y + \sin(x)y^2, \quad \cos(x)y^2 - y'\sin(x)y, \quad -\cos(x)^2 y + \cos(x)y'\sin(x), \quad y.$$

What becomes obvious is that the conditions of the 4$^{\mathrm{th}}$ approach are not sufficient as they have 8 instead of 5 solutions, and also that the 4$^{\mathrm{th}}$ approach can miss conservation laws. It did not see that the fourth + 2 times the third adjoint symmetry do give a conservation law.

Applying the program ConLaw2 that corresponds to the above method to the sin-Gordon equation (8) gives the following table. The typical feature of this approach is the slower increase of the size of conditions. Apart from the order 0 case they increase by a factor of about 2 which itself is increasing slightly with the order. Compared with the previous method the size of conditions grows slower which allows going higher in the order. Because the conditions that are generated are only necessary, not sufficient, they are slightly more difficult and expensive to solve. This causes longer running times for low order investigations. Time limitations could be overcome to some extend by faster computers.

order of Q	no of terms	independent variables, [no of var.]	functions to compute, [no of arg.]	cons. laws found	time to solve (11) h:min:sec
0	6	$t, x, u^{(1)}$, [5]	$Q(t, x, u)$, [3]	none	1 sec
1	21	$t, x, u^{(2)}/u_{tx}$, [7]	$Q(t, x, u^{(1)})$, [5]	(26)	4.3 sec
2	45	$t, x, u^{(3)}/u_{tx}^{(1)}$, [9]	$Q(t, x, u^{(2)}/u_{tx})$, [7]	none	12 sec
3	99	$t, x, u^{(4)}/u_{tx}^{(2)}$, [11]	$Q(t, x, u^{(3)}/u_{tx}^{(1)})$, [9]	(28)	50 sec
4	202	$t, x, u^{(5)}/u_{tx}^{(3)}$, [13]	$Q(t, x, u^{(4)}/u_{tx}^{(2)})$, [11]	none	2 min 43 sec
5	435	$t, x, u^{(6)}/u_{tx}^{(4)}$, [15]	$Q(t, x, u^{(5)}/u_{tx}^{(3)})$, [13]	(29)	16 min 10 sec
6	870	$t, x, u^{(7)}/u_{tx}^{(5)}$, [17]	$Q(t, x, u^{(6)}/u_{tx}^{(4)})$, [15]	none	49 min 20 sec
7	1836	$t, x, u^{(8)}/u_{tx}^{(6)}$, [19]	$Q(t, x, u^{(7)}/u_{tx}^{(5)})$, [17]	(30)	8 h
8	3643	$t, x, u^{(9)}/u_{tx}^{(7)}$, [21]	$Q(t, x, u^{(8)}/u_{tx}^{(6)})$, [19]	none	5 h 22 min
9	7434	$t, x, u^{(10)}/u_{tx}^{(8)}$, [23]	$Q(t, x, u^{(9)}/u_{tx}^{(7)})$, [21]	(31)	25 h

Table 4: The program ConLaw2 applied to compute conservation laws of the sin-Gordon equation.

3.5 Overview

Arranging the methods as in the table below one can compare rows I,II and columns A,B.

	A	B
I	$\mathrm{Div}\ P\|_{\Delta_\mu=0} = 0$	$0 = \sum_{\mu,J}(-D)_J \left(Q_\mu \frac{\partial \Delta_\mu}{\partial u^\nu_J}\right)\Big\|_{\Delta_\mu=0} \quad \forall \nu$
II	$\mathrm{Div}\ P = \sum_\nu Q_\nu \Delta_\nu$	$0 = \sum_J (-D)_J \left(\frac{\partial}{\partial u^\nu_J} \sum_\mu Q_\mu \Delta_\mu\right) \quad \forall \nu$

Table 5: The four approaches arranged in a table.

I-II: The conditions in row I are to be solved in the space of solutions ($|_{\Delta_\mu=0}$), in row II they are not. This means that methods of row I can not be applied if equations or constraints $\Delta_\mu = 0$ can not be solved for a leading derivative but methods of row II can. Due to these substitutions the conditions in row I have fewer terms and involve fewer jet-variables (derivatives of u) than conditions in row II. The complexity of conditions and the number of conservation laws up to some order obtained in row I depend on whether $\Delta_\mu = 0$ is used to substitute lower order u-derivatives by higher ones or higher ones by lower ones. There are two reasons for this.

1) Substitutions based on $0 = \Delta$ in Q may give extra restrictions for Q. For example, determining Q of conservation laws for the Korteweg de Vries equation $0 = \Delta = u_t - u_{xxx} - uu_x$ and restricting Q to be of 2^{nd} order, then a substitution $u_t = u_{xxx} + uu_x$ would imply $Q = Q(t, x, u, u_x, u_{xx})$, whereas a substitution $u_{xxx} = u_t - uu_x$ would not restrict Q.

2) If a lower u-derivative is substituted by higher ones using $0 = \Delta$ in the conservation law conditions in row I then such substitutions may increase the order of u-derivatives in which the conservation law conditions have to be satisfied identically. By that the desired effect of lowering the number of u-derivatives in which the conditions have to be fulfilled identically is lost. For example, condition IB for $\Delta = u_{tt} - u_{xxt}^2$ is $0 = Q_{tt} + 2(Qu_{xxt})_{xxt}$ which includes up to 6^{th} order u-derivatives (if Q is not of higher than 3^{rd} order). By substituting $u_{tt} = u_{xxt}^2$ the order would increase to seven.

Hence, substituting lower order u-derivatives by higher order u-derivatives gives more over determined conditions for a less general ansatz. Such conditions are easier to solve, which may allow higher orders of Q to be investigated. However, one then may miss conservation laws of some order in P or Q.

These aspects are not an issue in row II as no substitutions are made there.

A-B: In column A the single first order conservation law condition itself is to be solved, and in column B the integrability conditions of column A, which result when the conserved current P is eliminated are to be solved. Conditions in column B involve as many equations as there are functions u^μ and they are of the same order as the highest derivatives of u^μ in $\Delta_\mu = 0$. Conditions in column B are more straight forward to solve, they can be separated with respect to many high order jet-variables and yield highly over determined systems. The disadvantage of methods in column B is that already their formulation may exceed available computational resources. Another potential problem with using methods in column B is the following. If linear PDEs remain unsolved like the heat equation when investigating the Burgers equation (14) then the program will usually not be able to compute P from the Q_μ. A way out is to use methods IB or IIB to get Q_μ and to use that as input to get P from method IIA.

Differences between the approaches are amplified with problems that involve an increasing number of PDEs and an increasing number of independent variables.

4 The computer algebra programs

The names of computer algebra programs for the four approaches are: IA: ConLaw1,IB: ConLaw2, IIA: ConLaw3, IIB: ConLaw4. They and the program Crack for solving the over determined conditions are written in the computer algebra system REDUCE. Algorithms for extracting conservation laws from the general solution of the over determined system and for computing Q from P and P from Q are described in [35].

Compared to other computer algebra programs, the package Crack has a wide variety of techniques for solving over determined PDE-systems. This allows the following new features as compared with other computer programs, a list of which and a short description is given in [13]:

- In all four computer programs P as well as Q are computed.

- By solving systems of over determined differential equations it is possible to find conservation laws with non-polynomial, even non-rational P, Q.
- If memory requirements are not too high then the program will make a definite statement about the existence of conservation laws of a given order. In the majority of these cases the program will find the explicit form of the conservation law, otherwise it will return unsolved equations.
- It is possible to find conservation laws with an explicit dependence of P, Q on the independent variables.
- There is no limit on the number of DEs nor the number of independent variables to be investigated for conservation laws other than a limit through the complexity of computations. Although not demonstrated in this paper, the program is able to handle ordinary differential equations (ODEs) as well.
- It is possible to determine values of parameters in the DE such that conservation laws exist.
- For each of the four programs ConLaw1..4 an ansatz for P^i and/or Q^μ can be input to specify conservation laws to be calculated.

The recently published program of Göktaş and Hereman [9] makes a polynomial ansatz for conservation laws and finds the coefficients in this ansatz by solving a linear algebraic system of equations. Compared with that, the programs ConLaw1..4 are able to find more general conservation laws and to make a definitive statement in case the order is not too high to complete the computations. On the other hand, the program of Göktaş and Hereman was recently extended to handle differential-difference systems [10],[11],[12].

Before showing examples which highlight the special abilities of ConLaw1..4 a comment to the treatment of ODEs shall be made. Although all methods and programs are applicable equally well to ODEs, the form of the ansatz for the integrating factor or for the first integral to be made will usually be different. An nth order ODE has always first integrals of order $n-1$ and any arbitrary function of first integrals is a first integral as well. In order to obtain an over determined system of conditions, the ansatz for a conservation law must not contain functions of all variables $x, y, y', \ldots, y^{(n-1)}$ but, for example, a polynomial in $y^{(n-1)}$ with arbitrary functions of $x, y, y', \ldots, y^{(n-2)}$ as coefficients or any other combination of functions of less than $n+1$ variables, see also [5] for more details.

Further examples:

Example:

The Ito equations for two functions $u = u(t,x)$, $v = v(t,x)$ read [16]

$$\begin{aligned} u_t &= u_{xxx} + 6uu_x + 2vv_x \\ v_t &= 2(uv)_x. \end{aligned}$$

Conserved densities P^t for the first 7 conservation laws calculated by the corresponding program ConLaw1 which in turn calls Crack to solve condition (7), are

$$u, \quad v, \quad u^2+v^2, \quad u_x^2 - 2u^3 - 2uv^2, \quad (4uv^2 - v_x^2)/v^3,$$
$$u_{xx}^2 - 10uu_x^2 - 4vv_xu_x + 5u^4 + 6u^2v^2 + v^4, \quad ((2vv_{xx} - 4uv^2 - 3v_x^2)^2 + 16v^6)/v^7$$

(P^x is not shown due to its length). Somewhat surprisingly 2 of the 7 conservation laws have a non-polynomial expression for P^t and as far as the author knows therefore have not been known so far.

Example:

The following equations [17] describe low - frequency Alfvén waves propagating parallel to an external magnetic field in a relativistic electron-positron plasma [31]. Typical for them is the symmetry with respect to interchanging the two functions $u = u(t,x), v = v(t,x)$ due to the same charge-to-mass ratio for both kinds of particles. The equations are

$$\begin{aligned} \Delta_1 = u_t + r_x = 0, &\quad \text{with} \quad r = u(u^2+v^2) + u_{xx}, \\ \Delta_2 = v_t + s_x = 0, &\quad \text{with} \quad s = v(u^2+v^2) + v_{xx}. \end{aligned} \tag{12}$$

The equations themselves have the form of conservation laws. We find the following additional ones:

$$
\begin{aligned}
4u\Delta_1 + 4v\Delta_2 &= D_t[2(u^2+v^2)] + \\
& \quad D_x[4uu_{xx} - 2u_x^2 + 4vv_{xx} - 2v_x^2 + 3(u^2+v^2)^2] \\
4r\Delta_1 + 4s\Delta_2 &= D_t[(u^2+v^2)^2 - 2u_x^2 - 2v_x^2] + \\
& \quad D_x\left[4u_tu_x + 4v_tv_x + 2u_{xx}^2 + 2v_{xx}^2 + 4(u^2+v^2)\times\right. \\
& \qquad \left.\big((3(u^2+v^2)t - x)(uu_t+vv_t) + uu_{xx} + vv_{xx}\big)\right] \\
4(xu-3tr)\Delta_1 &+ 4(xv-3ts)\Delta_2 = \\
& \quad D_t\left[3t\big(\big(2u_x^2 + 2v_x^2 - (u^2+v^2)^2\big) + 2x(u^2+v^2)\right] + \\
& \quad D_x 2\left[(uu_t+vv_t)\left(-x^2 + (u^2+v^2)\left(6tx - 9t^2(u^2+v^2)\right)\right)\right. \\
& \qquad -3t(u^2+v^2)^3 + 3x(u^2+v^2)^2 + 2x(uu_{xx}+vv_{xx}) - 3tr^2 \\
& \qquad \left. -3ts^2 - 2uu_x - 2vv_x - xu_x^2 - xv_x^2 - 6tu_tu_x - 6tv_tv_x\right]
\end{aligned}
$$

Whereas the first two are known [31], the last one shows an explicit x, t-dependence and is new. Further investigation provides that no conservation laws exist with the characteristic functions Q_μ of 3^{rd} or 4^{th} order (if u_t, v_t are substituted due to (12)).

Example:
The following equation of Gibbons and Tsarev [8]

$$0 = z_{xx} + z_y z_{xy} - z_x z_{yy} + 1 \tag{13}$$

is unusual in that it has already 5 conservation laws of first order. Characteristic functions contain x, y explicitly. Up to first order they are:

$$1, \quad z_y, \quad 3z_y^2 + 2z_x + 3x, \quad 2z_y^3 + 3z_xz_y + 4z_yx + y,$$

$$10z_y^4 + 6z_x^2 + 24z_xz_y^2 + 20z_xx + 30z_y^2x + 12z_yy + 2z + 15x^2,$$

$$3z_y^5 + 6z_x^2z_y + 10z_xz_y^3 + 18z_xz_yx + 4z_xy + 12z_y^3x + 6z_y^2y + 12z_yx^2 + 2z_yz + 6xy.$$

Example:
The Liouville equation for a function $u = u(x, y)$ reads

$$\Delta = u_{xy} - e^u.$$

Conservation laws of order zero found by ConLaw2 are

$$(f_x + fu_x)\Delta = D_x(-e^uf) + D_y(f_xu_x + fu_x^2/2), \quad f = f(x) \quad \text{arbitrary}$$

$$(g_y + gu_y)\Delta = D_y(-e^ug) + D_x(g_yu_y + gu_y^2/2), \quad g = g(y) \quad \text{arbitrary.}$$

Because the ansatz made is investigated in full generality, any free functions in the conservation law will be found if the conditions can be solved completely by Crack. Otherwise the remaining conditions are returned as in the following example.

Example:
The Burgers equation in the form

$$\Delta = u_t - u_{xx} - \frac{1}{2}u_x^2 = 0, \quad u = u(t, x) \tag{14}$$

has zeroth order conservation laws

$$fe^{u/2}\Delta = D_t(2fe^{u/2}) + D_x(e^{u/2}(2f_x - fu_x)) \tag{15}$$

with $f = f(t, x)$ satisfying the linear reverse heat equation $0 = f_t + f_{xx}$.[6]

The occurrence of free functions in the conservation law indicates linearizability of $\Delta = 0$, which is the case for both previous examples. The following example involves more than 2 variables.

Example:
The Kadomtsev-Petviashvili equation for $u = u(t, x, y)$ with the abbreviation

$$w = u_t + 2uu_x + u_{xxx}$$

is

$$0 = \Delta = w_x - u_{yy}. \tag{16}$$

Its zeroth order conservation laws include an arbitrary function $c = c(t)$:

$$\begin{aligned}
c\Delta &= D_x(cw) + D_y(-cu_y) && (17)\\
cy\Delta &= D_x(cyw) + D_y(cu - cyu_y) && (18)\\
(2cx + c_ty^2)\Delta &= D_t(-2cu) + \\
&\quad D_x\left((2cx + c_ty^2)w - 2cu_{xx} - 2cu^2\right) + && (19)\\
&\quad D_y\left(-(2cx + c_ty^2)u_y + 2c_tuy\right) \\
(6cxy + c_ty^3)\Delta &= D_t(-6cyu) + \\
&\quad D_x\left((6cxy + c_ty^3)w - 6cyu_{xx} - 6cyu^2\right) + && (20)\\
&\quad D_y\left(-(6cxy + c_ty^3)u_y + 3c_tuy^2 + 6cxu\right).
\end{aligned}$$

It is somewhat remarkable that although equation (16) does not involve u_t but only u_{xt} nevertheless the conserved density P^t in the last two conservation laws involves u and not u_x.

In the following section we give examples for an extension of our method to compute non-local conservation laws and report on the possibility to determine parameters in the equation such that conservation laws exist.

5 Extending applicability

5.1 Non-local conservation laws

The implementations of the four methods have a common limitation: the characteristic functions Q and the conserved current P must depend functionally only on a finite number of derivatives of the u. No dependencies on integrals are possible. The same restriction is usually made when generators of Lie-symmetries are determined for differential equations. Whereas this restriction is less severe when calculating symmetries of PDEs, it is a serious restriction for the determination of conservation laws. To give an example, the Burgers equation in the form

$$\Delta = u_t - u_{xx} - uu_x = 0, \quad u = u(t, x) \tag{21}$$

has as low order conservation law only the trivial one $D_tu - D_x(u_x + u^2/2) = 0$. In order to include dependencies on $\int u\,dx$ one could set $u = v_x$ for some function $v(x, t)$ and investigate conservation laws depending on derivatives of v and also v itself. For the Burgers equation such a substitution alone is not enough. In addition one has to realize that (21) can be integrated with respect to x to $f(t)_t = v_t - v_{xx} - v_x^2$ for some function $f = f(t)$. Renaming $v - f \to u$ gives (14) and its conservation laws (15).

To give a further example, we consider the Boussinesq equation describing surface water waves whose horizontal scale is much larger than the depth of the water [1],[14]

$$u_{tt} - u_{xx} + 3uu_{xx} + 3u_x^2 + \alpha u_{xxxx} = 0. \tag{22}$$

[6] Although already used in [2], [35] this example is shown again as it also serves to demonstrate an extension to non-local conservation laws in section 5.

Calculating conservation laws, using (22) to substitute u_{xxxx}, the only characteristic functions Q up to 4^{th} order are $1, x, t, xt$. On the other hand, substituting $u = v_x$, integrating (22) with respect to x and renaming $v - f \to v$ gives

$$v_{tt} - v_{xx} + 3v_x v_{xx} + \alpha v_{xxxx} = 0 \tag{23}$$

having 2 conservation laws with characteristic functions $1, t$ which x-differentiated give the conservation laws above with characteristic functions $1, t$. In addition two new conservation laws with characteristic functions v_x, v_t result. Repeating this step again: $v = w_x$, x-integration of (23), $w - f \to w$ gives

$$w_{tt} - w_{xx} + 3/2 w_{xx}^2 + \alpha w_{xxxx} = 0 \tag{24}$$

with three third order conservation laws. Two of them have characteristic functions w_{xxx}, w_{xxt} which correspond to the above conservation laws with characteristic functions v_x, v_t. In addition one extra conservation law with $Q = w_{txx} - w_{txx}w_{xx} + w_{tx}w_{xxx} - \frac{2}{3}w_{ttt}$ exists.

A third example is the Kadomtsev-Petviashvili equation already discussed above[7]. After a substitution $u = v_x$, x-integration of (16) and $v - f \to v$ the equation is

$$0 = [v_t + v_{3x} + v_x^2]_x - v_{2y}.$$

Apart from conservation laws equivalent to (17),(18) three new conservation laws result with characteristic functions

$$-c_{2t}y^2 - 2c_t x + 4cv_x$$
$$-c_{3t}y^3 - 6c_{2t}xy + 12c_t y v_x + 24cv_y,$$
$$-c_{4t}y^4 - 12c_{3t}xy^2 + 24c_{2t}y^2v_x - 12c_{2t}x^2 + 48c_t x v_x + 96c_t y v_y + 48c_t v + 144cv_t.$$

Conserved currents are omitted due to their length. Repeating this transformation again does not yield conservation laws with characteristic functions of order less than three.

The purpose of this paragraph was to show that even if computer algebra programs ConLaw, Crack do only allow the investigation of local conservation laws depending on a finite number of derivatives of the unknown functions, we still may be able to enlarge the range of search by a contact transformation and integration of the PDE.

In the next section we extend the computation of conservation laws to the computation of parameters such that conservation laws exist.

5.2 Differential equations with parameters

In applications it is common that the DEs contain parameters and usually it would be desirable to know conservation laws which are valid for all possible values of these parameters. But as the example below shows, often conservation laws exist only for special values of parameters. Even if these parameter values are not of interest from the application side of view, the conservation laws valid for these values can at least be used, for example, to test numerical code. Another purpose for determining parameters together with conservation laws could be to find integrable equations from a more general class of equations.

The problem to determine parameters such that conservation laws exist is potentially much harder than determining conservation laws which are valid for any values of these parameters. This is because the problem becomes non-linear. Expressions may become unmanageably large and many sub cases may have to be considered. To use ConLaw1..4 for such calculations one only has to specify in its call the names of parameters to be computed (more details in the ConLaw manual).

Example:
The 5'th order Korteweg - de Vries equation

$$u_t + \alpha u^2 u_x + \beta u_x u_{2x} + \gamma u u_{3x} + u_{5x} = 0 \tag{25}$$

[7]The hint to try KP for this extension was given by Alan Fordy.

with constant parameters α, β, γ includes well known special cases [7], [9], [15], [19], [28]: for $\alpha = 30, \beta = 20, \gamma = 10$ the Lax equation [20], for $\alpha = 5, \beta = 5, \gamma = 5$ an equation due to Sawata, Kotera [29] and Dodd and Gibbon [6], for $\alpha = 20, \beta = 25, \gamma = 10$ an equation due to Kaup [18] and Kupershmidt, for $\alpha = 2, \beta = 6, \gamma = 3$ an equation due to Ito [16].

The following zeroth and first order conservation laws are calculated with ConLaw1 (omitting P^x due to its length in the last two of these conservation laws):

- $Q = 1, \;\; P^t = u$
- $\alpha = \beta = \gamma = 0$: As (25) becomes linear, a conservation law is obtained with a characteristic function $Q = Q(x,t)$ satisfying the adjoint PDE $Q_t + Q_{5x} = 0$ with $P^t = Qu$.
- $\alpha = 0, \gamma = \beta/3$: $Q = x^2, \;\; P^t = x^2 u$
- $\alpha = 0, \gamma = \beta/3$: $Q = x, \;\; P^t = xu$
- $\gamma = \beta/2$: $Q = 2u, \;\; P^t = u^2$
- $\alpha = \frac{1}{10}(-2\beta^2 + 7\beta\gamma - 3\gamma^2)$:
  $Q = 60u_{xx}t(\beta - 3\gamma) + 6u^2t(2\beta^2 - 7\beta\gamma + 3\gamma^2) + 60x$
  $P^t = 30u_x^2t(-\beta + 3\gamma) + u^3t(4\beta^2 - 14\beta\gamma + 6\gamma^2) + 60ux$
- $\alpha = \frac{1}{10}(-2\beta^2 + 7\beta\gamma - 3\gamma^2)$:
  $Q = 30u_{xx} + 3u^2(2\beta - \gamma),$
  $P^t = -15u_x^2 + u^3(2\beta - \gamma)$

We find the same conservation laws as found by the program of Göktaş and Hereman and in addition a few conservation laws with explicit x, t-dependence.

6 Summary

Four approaches to find conservation laws have been compared with respect to their complexity and other characteristic features.

In a number of examples, conservation laws have been given, some of them new, which show that the programs ConLaw1..4 and Crack can be used to find local, not necessarily polynomial, conservation laws with explicit variable dependence and free functions. The programs are, in principle, applicable to problems with arbitrarily many equations, functions and variables.

The programs including a manual and a test file are available via ftp from `ftp.maths.qmw.ac.uk`, directory `pub/tw`. A demo web page which allows the use of ConLaw for problems of restricted size, is accessible from `http://cathode.maths.qmw.ac.uk/demos.html`. The package will be submitted to the REDUCE network library.

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8 Appendix: Conservation Laws of the sin-Gordon equation

In this appendix conservation laws for the sin-Gordon equation

$$u_{tx} - \sin(u) = 0$$

are shown as they have been computed by ConLaw1-4 and as they are refered to in tables above. They are not new, we provide them only to illustrate computer results. Except for the first conservation law, for all the following there is an additional conservation law due to the $x \leftrightarrow t$ symmetry. These results are further examples of the ability of the programs to compute non-polynomial conservation laws.

$$
\begin{aligned}
2(tu_t - xu_x)(u_{tx} - \sin(u)) &= D_t\left[2\cos(u)t - u_x^2 x\right] + D_x\left[-2\cos(u)x + u_t^2 t\right] && (26)\\
2u_t(u_{tx} - \sin(u)) &= D_t\left[2\cos(u)\right] + D_x\left[u_t^2\right] && (27)\\
(8u_{3t} + 4u_t^3)(u_{tx} - \sin(u)) &= D_t\left[4\cos(u)u_t^2 + 8u_{tx}u_{2t} - 8u_{2t}\sin(u)\right] + D_x\left[-4u_{2t}^2 + u_t^4\right] && (28)
\end{aligned}
$$

$$
\begin{aligned}
& 2(-8u_{5t} - 20u_{3t}u_t^2 - 20u_{2t}^2u_t - 3u_t^5)(u_{tx} - \sin(u)) \qquad (29)\\
= \; & D_t 2\left[-8\cos(u)u_{3t}u_t + 4\cos(u)u_{2t}^2 - 3\cos(u)u_t^4 - 8u_{tx}u_{4t} - 20u_{tx}u_{2t}u_t^2\right.\\
& \quad \left. +8u_{4t}\sin(u) + 8u_{3t}u_{2tx} + 12u_{2t}u_t^2\sin(u)\right]\\
+ \; & D_x\left[-8u_{3t}^2 + 20u_{2t}^2u_t^2 - u_t^6\right]
\end{aligned}
$$

$$
\begin{aligned}
& 8(-16u_{7t} - 56u_{5t}u_t^2 - 224u_{4t}u_{2t}u_t - 168u_{3t}^2u_t - 280u_{3t}u_{2t}^2 - 70u_{3t}u_t^4 - 140u_{2t}^2u_t^3 - 5u_t^7)\\
& \times(u_{tx} - \sin(u)) \qquad (30)\\
= \; & D_t 8\left[16u_{6t}\sin(u) - 16u_{tx}u_{6t} - 16\cos(u)u_{5t}u_t + 16\cos(u)u_{4t}u_{2t} - 8\cos(u)u_{3t}^2\right.\\
& \quad -40\cos(u)u_{3t}u_t^3 - 20\cos(u)u_{2t}^2u_t^2 - 5\cos(u)u_t^6 - 56u_{tx}u_{4t}u_t^2 - 112u_{tx}u_{3t}u_{2t}u_t\\
& \quad -56u_{tx}u_{2t}^3 - 70u_{tx}u_{2t}u_t^4 + 16u_{5t}u_{2tx} - 16u_{4t}u_{3tx} + 40u_{4t}u_t^2\sin(u) + 56u_{3t}u_{2tx}u_t^2\\
& \quad \left. +160u_{3t}u_{2t}u_t\sin(u) + 40u_{2t}^3\sin(u) + 30u_{2t}u_t^4\sin(u)\right]\\
+ \; & D_x\left[64u_{4t}^2 - 224u_{3t}^2u_t^2 + 112u_{2t}^4 + 280u_{2t}^2u_t^4 - 5u_t^8\right]
\end{aligned}
$$

$$
\begin{aligned}
& 2(-128u_{9t} - 576u_{7t}u_t^2 - 3456u_{6t}u_{2t}u_t - 7296u_{5t}u_{3t}u_t - 6720u_{5t}u_{2t}^2 - 1008u_{5t}u_t^4\\
& \quad -4416u_{4t}^2u_t - 24192u_{4t}u_{3t}u_{2t} - 8064u_{4t}u_{2t}u_t^3 - 5824u_{3t}^3 - 6048u_{3t}^2u_t^3 - 24864u_{3t}u_{2t}^2u_t^2\\
& \quad -840u_{3t}u_t^6 - 6384u_{2t}^4u_t - 2520u_{2t}^2u_t^5 - 35u_t^9)\\
& \times(u_{tx} - \sin(u)) \qquad (31)\\
= \; & D_t 2\left[128u_{8t}\sin(u) - 128\cos(u)u_{7t}u_t + 128\cos(u)u_{6t}u_{2t} - 128\cos(u)u_{5t}u_{3t} + 64\cos(u)u_{4t}^2\right.\\
& \quad -448\cos(u)u_{5t}u_t^3 - 1344\cos(u)u_{4t}u_{2t}u_t^2 - 1568\cos(u)u_{3t}^2u_t^2 - 1344\cos(u)u_{3t}u_{2t}^2u_t\\
& \quad -560\cos(u)u_{3t}u_t^5 + 336\cos(u)u_{2t}^4 - 840\cos(u)u_{2t}^2u_t^4 - 35\cos(u)u_t^8 - 1008u_{t,x}u_{4t}u_t^4\\
& \quad -576u_{t,x}u_{6t}u_t^2 - 2304u_{t,x}u_{5t}u_{2t}u_t - 4992u_{t,x}u_{4t}u_{3t}u_t - 4416u_{t,x}u_{4t}u_{2t}^2 - 128u_{t,x}u_{8t}\\
& \quad -5184u_{t,x}u_{3t}^2u_{2t} - 4032u_{t,x}u_{3t}u_{2t}u_t^3 - 4256u_{t,x}u_{2t}^3u_t^2 - 840u_{t,x}u_{2t}u_t^6 + 128u_{7t}u_{2t,x}\\
& \quad -128u_{6t}u_{3t,x} + 448u_{6t}u_t^2\sin(u) + 128u_{5t}u_{4t,x} + 576u_{5t}u_{2t,x}u_t^2 + 2688u_{5t}u_{2t}u_t\sin(u)\\
& \quad -576u_{4t}u_{3t,x}u_t^2 + 4480u_{4t}u_{3t}u_t\sin(u) + 1152u_{4t}u_{2t,x}u_{2t}u_t + 4032u_{4t}u_{2t}^2\sin(u)\\
& \quad +560u_{4t}u_t^4\sin(u) + 1920u_{3t}^2u_{2t,x}u_t + 5824u_{3t}^2u_{2t}\sin(u) + 3264u_{3t}u_{2t,x}u_{2t}^2\\
& \quad \left. +1008u_{3t}u_{2t,x}u_t^4 + 4480u_{3t}u_{2t}u_t^3\sin(u) + 3360u_{2t}^3u_t^2\sin(u) + 280u_{2t}u_t^6\sin(u)\right]\\
+ \; & D_x\left[-128u_{5t}^2 + 576u_{4t}^2u_t^2 - 1280u_{3t}^3u_t - 3264u_{3t}^2u_{2t}^2 - 1008u_{3t}^2u_t^4 + 2128u_{2t}^4u_t^2\right.\\
& \quad \left. +840u_{2t}^2u_t^6 - 7u_t^{10}\right]
\end{aligned}
$$