Special cases of the hypergraph assignment problem

Masterarbeit

von

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Zusammenfassung

Das Hyperassignment-Problem, das in der hier behandelten Version von Borndörfer und Heismann in [BH12] eingeführt wurde, kann als eine Verallgemeinerung des perfekten Matching-Problems in Graphen gesehen werden.

Gegeben ist ein Hypergraph $\mathcal{H} = (V, W, \mathcal{A})$, wobei V und W zwei gleich große, disjunkte Knotenmengen sind und eine Menge an Hyperkanten $\mathcal{A} \subseteq 2^{V \cup W}$, sodass $|a \cap V| = |a \cap W|$ für jede Hyperkante $a \in \mathcal{A}$ gilt. Zusätzlich sind Kosten $c_a \in \mathbb{R}$ auf den Hyperkanten $a \in \mathcal{A}$ gegeben. Gesucht wird nun ein perfektes Matching, das Borndörfer und Heismann auch Hyperassignment nennen, mit minimalem Gewicht bezüglich der Gewichtsfunktion c.

Nach einer kurzen Einleitung und Zusammenfassung einiger wichtiger Resultate der Linearen Optimierung werden in dieser Arbeit zunächst einige Grundlagen der Theorie der Hypergraphen analog zur Graphentheorie entwickelt. Danach werden in Kapitel 4 balancierte Hypergraphen behandelt, die eine Verallgemeinerung bipartiter Graphen darstellen, sowie normale Hypergraphen, welche genau die Klasse bilden, deren Matching-Polytop ganzzahlig ist.

Anschließend wird das Hyperassignment-Problem in partitionierten Hypergraphen untersucht. Ein partitionierter Hypergraph ist ein Hypergraph $\mathcal{H} = (V, W, \mathcal{A})$, in dem V und W partitioniert werden können, also $V = V_1 \cup V_2 \cup \ldots \cup V_k$ und $W = W_1 \cup W_2 \cup \ldots \cup W_r$, sodass zusätzlich für jede Hyperkante $a \in \mathcal{A}$ Indizes $i \in \{1, \ldots, k\}$ und $j \in \{1, \ldots, r\}$ existieren mit $a \cap V \subseteq V_i$ und $a \cap W \subseteq W_j$. Besonders interessant sind partitionierte Hypergraphen, bei denen die Parts höchstens Größe sieben haben, da diese in der Praxis vorkommen. Jedoch ist das Hyperassignment Problem selbst für partitionierte Hypergraphen, in denen jeder Part höchstens aus zwei Knoten besteht, $\mathcal{NP}$-schwer.

Der Schwerpunkt des fünften Kapitels liegt auf der Untersuchung des Matching- sowie des perfekten Matching-Polytops partitionierter Hypergraphen, insbesondere solcher mit Parts der Größe zwei. Für einige dieser Hypergraphen kann die Dimension des perfekten Matching-Polytops angegeben werden, was für allgemeine Hypergraphen sehr schwer ist. Außerdem werden gültige Ungleichungen und Facetten der beiden Polytope beschrieben.

Zusätzlich wird gezeigt, dass die triviale LP-Relaxierung der folgenden IP-Formulierung des Hyperassignment-Problems eine unbeschränkte Ganzzahligkeitslücke besitzt:

$$\begin{aligned} & \min\nolimits_{a \in \mathcal{A}} \; c_a x_a \\ \text{unter} \quad & \sum_{a \in \mathcal{A} : v \in A} x_a = 1 \;\; \forall v \in V \cup W \\ & x_a \in \{0, 1\} \;\; \forall a \in \mathcal{A}. \end{aligned}$$

Sucht man nur ein kostenminimales Matching, so besitzt die triviale LP-Relaxierung

nur eine Ganzzahligkeitslücke von drei für partitionierte Hypergraphen mit Parts der Größe zwei.

Neben der theoretischen Seite werden zum Schluss die Ergebnisse der Berechnung minimaler gebrochener Beispiele für partitionierte Hypergraphen mit Parts der Größe zwei, die insgesamt höchstens zwölf Knoten besitzen, vorgestellt.

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Contents

1. Motivation

Regularity plays an important role in rail transport. Passenger usually prefer regular schedules, i.e., the schedule on Monday should be nearly the same as that on Tuesday, Wednesday, etc. If the schedule is regular, then also the vehicles serving the trips should be similar on each day.

The problem to assign railway vehicles to trips in a given timetable is known as Railway Vehicle Rotation Planning. The timetable repeats itself every week and most trips appear nearly every day. This regularity should be transferred to the schedule of the railway vehicles. For example, if on Monday the trip 1091 from Berlin to Munich is followed by the trip 1090 from Munich to Berlin, then this should be the case on all other days on which both trips exists, in this case from Monday to Friday.

It is clear that regular schedules are easier to handle in practice than non-regular ones. Therefore, Borndörfer, Reuther, Schlechte and Weider introduced a model using hypergraphs, see [BRSW11]. Furthermore, Borndörfer and Heismann investigated a simpler version of this problem in [BH11] on which we will focus now.

As above, a weekly repeating timetable is given, which consists of trips from a departure station to another station together with the departure day, the departure time, and its duration. We have to assign one vehicle to every trip. During a trip the vehicle does not change, so we are only interested in the departure and the arrival location, the departure time, and the duration of the trip, and not in the intermediate stops of the trip. The arrival location of a trip need not be the departure location of the following trip because a vehicle can do a "deadhead trip" to another train station without any passengers.

The aim is to construct an assignment of each trip to a trip served after it with the same vehicle. Starting with one fixed trip, we get a sequence of trips served by the same vehicle, which eventually has to become periodic as the timetable is periodic. Thus, the trips of the timetable are partitioned into disjoint rotations each operated by a different vehicle. The cost of a rotation is the sum of the operation costs of each pair of consecutive trips, depending on the duration and distance of the "deadhead trip", as well as the duration of the break between the trips.

The assignment of the trips can be modelled using perfect matchings in a bipartite graph $G = (V, E)$. The vertex set V consists of two disjoint copies V_1 and V_2 of the trips. There is an edge between a trip $v \in V_1$ and a trip $w \in V_2$ if and only if a vehicle can operate trip w after trip v. Furthermore, the cost of an edge are the operation costs of the corresponding pair of consecutive trips. A perfect matching of minimum costs corresponds to a rotation plan of minimum operation cost. Finding a perfect matching in a bipartite graph can be accomplished in polynomial time, so we can compute a rotation plan in polynomial time.

However, this approach does not consider regularity. For that, we group all trips that differ only by the departure day and call them a "train". This is a natural construction, for example, all trips from Berlin Südkreuz via Frankfurt to Munich Main Station starting at 6:00 and arriving at 13:27 are called ICE 1091, regardless of the weekday they start. A passenger that uses this connection to Frankfurt on different weekdays always knows that 1091 is the right train and does not have to remember a different number for every day.

Now, adding regularity can be done in the following way:

Suppose $v_1, \ldots, v_n$ and $w_1, \ldots w_n$ are two trains such that each pair $\{v_i, w_i\}$ can be operated consecutive. Regularity means that if one edge $\{v_i, w_i\}$ is contained in a perfect matching, we would like that all edges $\{v_1, w_1\}, \ldots, \{v_n, w_n\}$ are elements of this perfect matching. For that, we introduce a *hyperedge* $\tilde{e} = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ and give this hyperedge costs $c_{\tilde{e}}$ smaller than the sum of the costs of all edges $\{v_i, w_i\}$, $i = 1, \ldots, n$. Now, instead of using all edges $\{v_1, w_1\}, \ldots, \{v_n, w_n\}$ we can use the hyperedge $\tilde{e}$, thereby getting a reward of $\sum_{i=1}^{n} c_{\{v_i, w_i\}} - c_{\tilde{e}}$ for regularity.

The advantage of this method is that we still have a linear cost function. However, we have to work in hypergraphs, which are much more complicated than ordinary graphs. It turns out that finding a minimum cost perfect matching in a hypergraph is $\mathcal{NP}$-hard in general.

For more information on the application and a more elaborated model for Railway Vehicle Rotation Planning see [BRSW11]. In this thesis we will focus on the problem of finding a perfect matching in a hypergraph.

After a short summary of some basic results of linear programming used later, we give an introduction to the theory of hypergraphs.

In Chapter 4 we focus on balanced and normal hypergraphs, which can be seen as a generalization of bipartite graphs. For these classes of hypergraphs the perfect matching polytope has an easy description and a perfect matching can be found in polynomial time. Furthermore, a Hall type condition for the existence of a perfect matching in a balanced hypergraph exists for which we find a generalization to normal hypergraphs.

In Chapter 5 we look at a special class of hypergraphs, arising from the modelling of the Vehicle Rotation Planning Problem described above. The main emphasis lies on the investigation of the matching and perfect matching polytope of this class of hypergraphs. Besides theoretical results on the dimension, faces and facets, and the integrality gap, we present computational results for hypergraphs with few vertices.

2. Linear Programming and Polyhedral Theory

This chapter gives an overview of a small part of the theory of linear programming concentrating on concepts that are used in this thesis. The notion, definitions, and theorems follow the lecture notes of Grötschel, see [Grö10]. In particular, by $\mathbb{K}$ we denote the real numbers $\mathbb{R}$ or the rational numbers $\mathbb{Q}$.

Proofs are omitted, they can be found in [Grö10] or [Sch99].

2.1. Polyhedra and Polytopes

A linear program (LP) consists of a linear function $f : \mathbb{K}^n \to \mathbb{K}$ and a system of linear inequalities. The goal is to find a solution of this system that minimizes (or maximizes) f. As f is linear, it can be represented by a vector $c \in \mathbb{K}^n$. So every (LP) can be written as

$$\min\ c^T x \text{ subject to } Ax \leq b,$$

where $A \in \mathbb{K}^{m\times n}$ and $b \in \mathbb{K}^m$.

The set of solutions of a system of linear inequalities is a geometric object, called polyhedron.

Definition. A subset $P \subseteq \mathbb{K}^n$ is a *polyhedron* if there exist a matrix $A \in \mathbb{K}^{m\times n}$ and a vector $b \in \mathbb{K}^m$ such that

$$P = \{x \in \mathbb{K}^n | Ax \leq b\} = \bigcap_{k=1}^{m} \{x \in \mathbb{K}^n | A_{k.}x \leq b_k\}.$$

If $b = 0$, P is called a *convex cone*. A bounded polyhedron is called a *polytope*.

An important property of a polyhedron $P \subseteq \mathbb{K}^n$ is that its projection onto some linear subspace U of dimension less than n is still a polyhedron. Of course, this also holds for polytopes because projecting a bounded set gives again a bounded set. All investigated polyhedra in this thesis are bounded and thus polytopes.

The next definitions are needed to obtain another way of representing a polyhedron.

Definition. A vector $x \in \mathbb{K}^n$ is a *linear combination* of vectors $x_1, \ldots, x_k \in \mathbb{K}^n$ if there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$ with

$$x = \sum_{i=1}^{k} \lambda_i x_i.$$

Moreover, x is called a *conical combination* of $x_1, \dots, x_n$ if $\lambda_i \geq 0$ for all $1 \leq i \leq k$ and an *affine combination* if $\sum_{i=1}^{k} \lambda_i = 1$. A linear combination that is both conical and affine is called a *convex combination.*

For a set $\emptyset \neq S \subseteq K^n$ denote by $\text{lin}(S), \text{cone}(S), \text{aff}(S), \text{conv}(S)$ the *linear, conical, affine, convex hull* of S, respectively, i.e., the set of all vectors in $\mathbb{K}^n$ that can be written as a finite linear, conical, affine or convex combination of elements in S. The affine and the convex hull of the empty set are defined to be empty, whereas the linear hull and the conical hull of the empty set are $\{0\}$.

Theorem 2.1.1. *Let $P := \{x \in \mathbb{K}^n | Ax \leq b\}$ be a polyhedron defined by a matrix $A \in \mathbb{K}^{m \times n}$ and a vector $b \in \mathbb{K}^m$. There exist finite sets $V, E \subseteq \mathbb{K}^n$ such that*

$$P = \text{conv}(V) + \text{cone}(E).$$

On the other hand, every subset that can be written as $\text{conv}(V) + \text{cone}(E)$ *for finite subsets V, E of $\mathbb{K}^n$ is a polyhedron.*

The above theorem implies that a polytope is just the convex hull of finitely many points. This fact can be used to model discrete optimization problems. In such a problem, a finite set S and a family of subsets $\mathcal{T}$ is given. Furthermore, every $s \in S$ has costs c_s and the objective is to find an element $T \in \mathcal{T}$ of minimum cost $c(T) := \sum_{s \in T} c_s$. By Theorem 2.1.1, the convex hull of all vectors $\chi^T \in \mathbb{K}^S$ with $T \in \mathcal{T}$ defined by

$$\chi_s^T := \begin{cases} 1, & \text{if } s \in T \\ 0, & \text{else} \end{cases}$$

is a polytope in $\mathbb{K}^S$. Thus, we can formulate every discrete optimization problem as a linear program.

An important tool to decide whether a system of equations and inequalities has a solution is Farkas' Lemma. There are many equivalent formulations, the most useful one for this thesis is the following:

Theorem 2.1.2. *Let $A \in \mathbb{K}^{m \times n}$ and $b \in \mathbb{K}^m$. Then exactly one of the two following statements holds:*

1. *There exists a vector $x \in \mathbb{K}^n$ with $Ax = b$ and $x \geq 0$.*

2. *There exists a vector $y \in \mathbb{K}^m$ with $Ay \geq 0$ and $b^T y < 0$.*

2.2. Faces, Facets and Vertices

As shown in the last section, a polytope P can be written as the solution set of a system of linear inequalities or as the convex hull of finitely many points. However, these representations are not unique. There are many systems of linear inequalities representing P and many sets S whose convex hull is P. In this section it is shown how to represent P by a minimal system $Ax \leq b$ and a minimal set S.

Definition. Let $P \subseteq \mathbb{K}^n$ be a polyhedron. A subset F of P is called a *face* of P if there exists a vector $a \in \mathbb{K}^n$ and a scalar $\alpha \in \mathbb{K}$ such that $a^T x \leq \alpha$ holds for all $x \in P$ and

$$F = P \cap \{x \in \mathbb{K}^n | a^T x = \alpha\}.$$

Then $a^T x \leq \alpha$ is a *valid inequality* for P and F is said to be *induced* by this inequality.
A maximal face $F \neq P$ with respect to inclusion of sets is called a *facet*. Moreover, x is called a *vertex* of P if $\{x\}$ is a face.

If $P = \{x \in \mathbb{K}^n | Ax \leq b\}$ is a polyhedron, then every inequality $a^T x \leq \alpha$ that is valid for P and induces a non-empty face satisfies $u^T A = a^T$ and $u^T b = \alpha$ for some $u \in \mathbb{K}^m_+$, i.e., it is an affine combination of the inequalities in $Ax \leq b$. Furthermore, $a^T x = \alpha$ is equivalent to $A_{i.} x = b_i$ for all i with $u_i > 0$. So the following definition makes sense:

Definition. Let $P = \{x \in \mathbb{K}^n | Ax \leq b\}$ with $A \in \mathbb{K}^{m \times n}$ and $b \in \mathbb{K}^m$ be a polyhedron. For an arbitrary subset F of P the *equality set* is defined by

$$\mathrm{eq}(F) := \{i \in \{1, \ldots, m\} | A_{i.} x = b_i \ \forall \ x \in F\}.$$

On the other hand, every set $I \subseteq \{1, \ldots, m\}$ induces a face

$$\mathrm{fa}(I) := \{x \in P | A_{i.} x = b_i \ \forall i \in I\}.$$

Remark. It can be shown that every face F is of the form $\mathrm{fa}(I)$ and $F = \mathrm{fa}(\mathrm{eq}(F))$.

We have seen that every face of a polytope of the form $\{x \in \mathbb{K}^n | Ax \leq b\}$ comes from an inequality which is an affine combination of the inequalities in $Ax \leq b$. Of course, it is possible that an inequality of the system $Ax \leq b$ can be expressed as an affine combination of other inequalities in this system. Deleting such an inequality does not change the polytope. This motivates the following definition:

Definition. Let $P = \{x \in \mathbb{K}^n | Ax \leq b\}$ with $A \in \mathbb{K}^{m \times n}, b \in \mathbb{K}^m$ be a polyhedron. An inequality $A_{i.} x \leq b_i$ is called *redundant* with respect to $Ax \leq b$ if

$$P = \{x \in \mathbb{K}^n | A_{k.} x \leq b_k \ \forall \ k \in \{1, \ldots, m\} \setminus \{i\}\}$$

and *implicit* if $i \in \mathrm{eq}(P)$.

For a given polyhedron, it is natural to look for an irredundant system of linear inequalities and equations, i.e., $Ax \leq b$ and $A'x = b'$ such that A' has full row rank and $Ax \leq b$ contains no redundant inequalities with respect to $Ax \leq b, A'x = b'$.

It turns out that an irredundant system only contains inequalities defining facets. To this end, let $\mathcal{F}$ be the set of facets of a given polyhedron $P = \{x \in \mathbb{K}^n | Ax \leq b\}$. Then there exists an index set $I \subseteq \{1, \ldots, m\} \setminus \mathrm{eq}(P)$ of size $\mathcal{F}$ with

$$F \in \mathcal{F} \;\Leftrightarrow\; F = \mathrm{fa}(\{i\}) \text{ for exactly one } i \in I.$$

Such a set I is called a facet index set.

Theorem 2.2.1. *Let $P = \{x \in \mathbb{K}^n | Ax \leq b\}$ with $A \in \mathbb{K}^{m \times n}, b \in \mathbb{K}^m$ be a non-empty polyhedron, $I \subseteq \{1, \ldots, m\} \setminus \mathrm{eq}(P)$ and $J \subseteq \mathrm{eq}(P)$. The system $A_{I.}x \leq b_I, A_{J.}x = b_J$ is irredundant if and only if I is a facet index set and $A_{J.}$ is a rank$(A_{\mathrm{eq}(P).}) \times n$ matrix of full row rank.*

To characterize facets of a polyhedron the notion of a dimension is needed.

Definition. Let $P \subseteq \mathbb{K}^n$ be a polyhedron and $\mathcal{A}_P$ be the smallest affine subspace of $\mathbb{K}^n$ containing P. Then the *dimension* of P, $\dim(P)$, is defined as the affine dimension of $\mathcal{A}_P$.

If P is given as the solution set of a system of linear inequalities, the dimension can be calculated explicitly:

Theorem 2.2.2. *Let $P = \{x \in \mathbb{K}^n | Ax \leq b\}$ with $A \in \mathbb{K}^{m \times n}, b \in \mathbb{K}^m$ be a polyhedron. Then*

$$\dim(P) = n - \mathrm{rank}(A_{\mathrm{eq}(P).}).$$

Now, we can state a theorem that characterizes facets of a polyhedron using the dimension of a facet:

Theorem 2.2.3. *Let $P \subseteq \mathbb{K}^n$ be a polyhedron and $\emptyset \subsetneq F \subsetneq P$ be a face of P. The following are equivalent:*

1. *F is a facet of P.*
2. $\dim(F) = \dim(P) - 1$.
3. *There are $\dim(P)$ affine independent vectors in F.*

As facets lead to a minimal representation of the form $Ax \leq b$, vertices lead to a minimal representation of a polytope of the form $\mathrm{conv}(S)$. For a general polyhedron P also a minimal set E such that $P = \mathrm{conv}(S) + \mathrm{cone}(E)$ can be characterized. We omit details for this as only polytopes appear in this thesis.

The next theorem gives some characterizations for vertices of a polyhedron.

Theorem 2.2.4. *Let $P = \{x \in \mathbb{K}^n | Ax \leq b\} \subseteq \mathbb{K}^n$ be a polyhedron. A vector $x \in P$ is a vertex if and only if one of the equivalent statements holds:*

1. *The face $\{x\}$ has dimension zero.*
2. *$rank(A_{\mathrm{eq}(\{x\}).}) = n$.*
3. *x cannot be written as a proper convex combination of elements in P, i.e., if $x = \lambda y + (1-\lambda)z$ for $y, z \in P$ with $0 < \lambda < 1$, then $y = z$.*

The last statement of Theorem 2.2.4 implies that if $P = \mathrm{conv}(S)$, then all vertices are elements of S. It can be shown that one can even choose S to be just the set of all vertices of P to get a minimal representation of P.

Theorem 2.2.4, 2. gives the following two conclusions for a polyhedron of the form $P = \{x \in \mathbb{K}^n | Ax = b, x \geq 0\}$:

Corollary 2.2.5. *Let $x \in P$ with P as above. Then x is a vertex of P if and only if the column vectors of A corresponding to entries $x_i > 0$ are linearly independent.*

Corollary 2.2.6. *Let $x \in P$ be a vertex of the polyhedron P, $k \in \{1, \ldots, n\}$ be an index with $x_k = 0$ and $\pi : \mathbb{K}^n \to \mathbb{K}^{n-1}$ be the projection to the $(n-1)$-dimensional subspace spanned by all standard basis vectors e_i for $i \neq k$. Then $\pi(x)$ is a vertex of the polyhedron $\pi(P)$.*

These two facts will be used several times in this thesis as we will investigate polytopes of the above form.

3. Introduction to Hypergraphs

The following chapter is a short introduction to the theory of hypergraphs with a special focus on commonalities and differences to graph theory.

After some basic definitions, matchings in hypergraphs are defined and the Hyperassignment Problem is introduced. It consists of finding a minimum cost perfect matching, which is also called hyperassignment, in hypergraphs having a special structure. Furthermore, the relation of the Hyperassignment Problem to the Stable Set Problem is described. For some special graph classes the Stable Set Problem can be solved in polynomial time, giving hypergraph classes for which the Hyperassignment Problem can be solved efficiently.

3.1. Basic Definitions

A hypergraph can be seen as a generalization of an ordinary graph, where in contrast to the edges of a graph hyperedges can connect more than two vertices.

Definition. A *hypergraph* $\mathcal{H} = (V, \mathcal{E})$ consists of a non-empty set of vertices (or nodes) V and a multiset $\mathcal{E}$ of subsets of the vertices. An element of $\mathcal{E}$ is called a *hyperedge* and the *rank* of a hypergraph $r(\mathcal{H})$ is $\max_{e \in \mathcal{E}} |e|$.

As in ordinary graphs, a hyperedge e is said to be *incident* to a vertex v (or a vertex v is incident to a hyperedge e) if $v \in e$. Two hyperedges $e_1, e_2 \in \mathcal{E}$ that intersect or two vertices $v_1, v_2 \in V$ that have a common incident hyperedge are called *adjacent*.

Furthermore, $\delta(v)$ denotes the multiset of all hyperedges incident to v. Its size $\deg_{\mathcal{H}}(v) := |\delta(V)|$ is called the *degree* of a vertex v and $\Delta(\mathcal{H}) := \max_{v \in V} \deg_{\mathcal{H}}(v)$ is the *maximum degree* of $\mathcal{H}$.

Remark. Usually $\mathcal{E}$ does not contain any multiple elements but it can be useful to multiply hyperedges. So in this definition $\mathcal{E}$ is a multiset and not just a subset of the power set 2^V. An element $e \in \mathcal{E}$ of size two will be called an *edge* and $\mathcal{H}$ is called a graph if all hyperedges are edges. *Proper hyperedges* are hyperedges of size greater than two.

Generalizing directed graphs is more difficult and there are several approaches (e.g., [GLP93]). Here the definition of Borndörfer and Heismann ([BH12]) is used (they decided to call this generalization bipartite hypergraph).

Definition. A bipartite hypergraph $\mathcal{D} = (V, W, \mathcal{A})$ is a hypergraph whose set of vertices consists of two disjoint sets V and W of the same cardinality and a set of hyperedges $\mathcal{A}$. Furthermore, every hyperedge $a \in \mathcal{A}$ satisfies $|a \cap V| = |a \cap W| > 0$.

The "direction" is not directly visible in this definition. The sets V and W can be seen as two disjoint copies of the same vertex set and a hyperedge a goes from its tail $a \cap V$ to its head $a \cap W$.

Many concepts of graph theory can be transferred to hypergraphs. In the following only a few are mentioned which will be useful later. A detailed introduction is given in [Ber89].

Definition. A *partial* hypergraph of $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph restricted to a subset of hyperedges, i.e., $\mathcal{H}[\mathcal{E}'] = (V, \mathcal{E}')$, in which every element of $\mathcal{E}'$ is an element of $\mathcal{E}$.

A *subhypergraph* induced by a subset S of the vertices is the hypergraph $\mathcal{H}|S$ with vertex set S and hyperedges $e \cap S$ for every $e \in \mathcal{E}$ with $e \cap S \neq \emptyset$.

A *partial subhypergraph* is a partial hypergraph of a subhypergraph (or the other way around).

Remark. The definition of a subhypergraph does not agree with that of subgraph in an ordinary graph. In a subhypergraph we do not only restrict the set of vertices to some subset, but we also change the hyperedges from e to $e \cap S$ if the later set is nonempty. For example, if $G = (V, E)$ is a graph and S a subset of V, then the induced subgraph $G[S]$ contains all edges $e \in E$ with both vertices in S, but the subhypergraph $G|S$ contains additionally the loops $e \cap S$ for all edges e having exactly one vertex in S.

Definition. The *dual* $\mathcal{H}^* = (V^*.\mathcal{E}^*)$ of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ without multiple hyperedges has the set $\mathcal{E}$ as its set of vertices and for every $v \in V$ a hyperedge $\{e \in \mathcal{E} | v \in e\}$.

When defining paths and cycles in hypergraphs, a difficulty arises as a hyperedge can be incident to not only its predecessor and its successor, but also to other vertices of the sequence (this cannot happen in graphs). So one has to distinguish two kinds of paths and cycles.

Definition. A sequence $P = v_0, e_1, v_1, \ldots, e_l, v_l$ is called a *path* of *length* l in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ if $v_i \in V$, $0 \leq i \leq l$, and $e_i \in \mathcal{E}$, $1 \leq i \leq l$, are pairwise distinct, respectively, and $v_{i-1}, v_i \in e_i$, $1 \leq i \leq l$. If furthermore $v_0 = v_l$, a path is called a *cycle*.

P is called a strong path (or cycle) if $|e_i \cap \{v_0, \ldots, v_l\}| = 2$ for every hyperedge $e_i \in \{e_1, \ldots, e_l\}$.

Later, the notion of a strong cycle will be useful in defining a special class of hypergraphs which can be seen as a counterpart of bipartite graphs.

Definition. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ has the *Helly property* if every set of pairwise intersecting hyperedges is intersecting; i.e, they have a common vertex.

Ordinary graphs containing a triangle do not have the Helly property but all other graphs do. This is the case because if the Helly property holds, then for every three vertices $v_1, v_2, v_3 \in V$ the set of edges being incident to at least two of the three vertices is intersecting, so w.l.o.g. all these edges contain v_1. As edges in graphs are incident to exactly two vertices, this means that there is no edge between v_2 and v_3, so the graph is triangle-free.

Colorings in hypergraphs are defined slightly different than in graphs. Adjacent edges or vertices can have the same color, only “monochromatic” vertices or edges are forbidden.

Definition. A function $c : \mathcal{E} \to \{1, \ldots, k\}$ is an *edge coloring* of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ in k colors if any vertex with degree at least two is incident to hyperedges $e, f \in \mathcal{E}$ with $c(e) \neq c(f)$.

A *vertex coloring* in k colors is a function $c : V \to \{1, \ldots, k\}$ such that for every hyperedge e with $|e| > 1$ there are two vertices $v_1, v_2 \in V$ with $c(v_1) \neq c(v_2)$.

For $1 \leq i \leq k$ the set $C_i := c^{-1}(\{i\})$ is called *edge* or *vertex color class*, respectively.

Remark. An edge coloring in $\mathcal{H}$ corresponds to a vertex coloring in the dual $\mathcal{H}^*$ and vice versa.

Definition. A hypergraph $\mathcal{H}$ has the *colored edge property* if there exists an edge coloring in $\Delta(H)$ colors such that $c(e) \neq c(f)$ for all adjacent hyperedges $e, f \in \mathcal{E}$. Furthermore, $\mathcal{H}$ is called *k-colorable* if $\mathcal{H}$ admits a vertex coloring in k colors.

By the last remark, a hypergraph has the colored edge property if and only if its dual admits a vertex coloring in $\Delta(\mathcal{H})$ colors such that two distinct adjacent vertices are colored differently.

3.2. Representation of Hypergraphs

There are two standard possibilities of representing hypergraphs by graphs. The first one is called bipartite representation, which can be used to decide algorithmically, whether a hypergraph has a strong odd cycle.

Definition. For a hypergraph $\mathcal{H} = (V, \mathcal{E})$ without multiple edges we define a bipartite graph $G = (S, E)$, called the *bipartite representation* of $\mathcal{H}$, as follows:

The set S is the disjoint union of V and $\mathcal{E}$. There is an edge between a vertex $v \in V$ and a hyperedge $e \in \mathcal{E}$ if and only if $v \in e$.

Another possibility is to forget about the vertices and just look at hyperedges and how they are related.

Definition. The *line graph* $L(\mathcal{H}) = (V_\mathcal{E}, E_\mathcal{E})$ of a hypergraph is a graph with node set $V_\mathcal{E}$ which contains one vertex for every hyperedge and an edge between v_e and v_f if e and f have a nonempty intersection.

Remark. An odd cycle in a hypergraph corresponds to a cycle of length two modulo four in the bipartite representation. If additionally the odd cycle is strong, the corresponding cycle in the bipartite representation is an subgraph induced by the set of vertices of that cycle.

The set of hyperedges of an odd cycle induces an odd cycle of the same length in the line graph. However, this cycle can have a chord even if the original cycle is strong because two edges of a strong cycle can intersect in a vertex which does not belong to the strong cycle.

Another way of representing hypergraphs uses 0-1-matrices as in graphs.

Definition. The *incidence matrix* $A = (a_{v,e})_{v \in V, e \in \mathcal{E}} \in \mathbb{R}^{V \times \mathcal{E}}$ of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is defined by

$$a_{v,e} := \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{else} \end{cases}.$$

A hypergraph can be drawn in the following way:

- Each vertex is represented by a little disc.
- Each hyperedge is drawn as a closed curve containing exactly the vertices defining this hyperedge.

Figure 3.1 gives an example of a drawing of a hypergraph. This hypergraph has the incidence matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Its line graph and bipartite representation are shown in Figure 3.2.

3.3. Matchings

The definition of a matching in a hypergraph coincides with that of a matching in a graph.

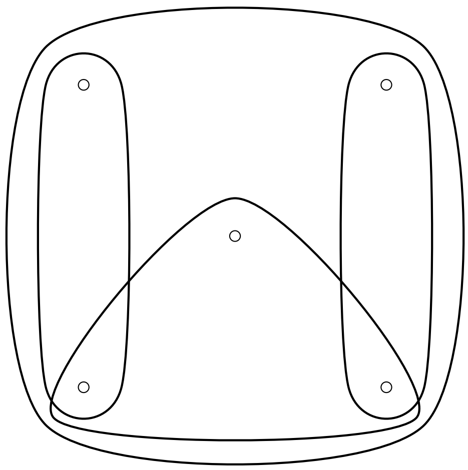


Figure 3.1.: Example of a hypergraph

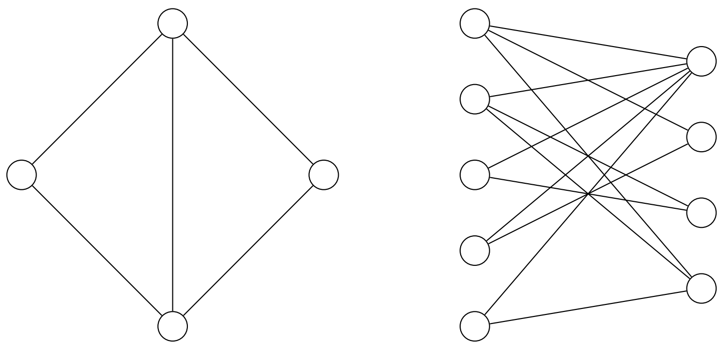


Figure 3.2.: Line graph (left) and bipartite representation (right)

Definition. A subset M of $\mathcal{E}$ is called a *matching* of $\mathcal{H}$ if every vertex $v \in V$ has at most degree one in the partial hypergraph induced by M. If every vertex has exactly degree one, the matching M is called *perfect*. A *maximal matching* is a matching M such that M is not a proper subset of another matching.

In contrast to graphs it is not clear what a maximum matching should be, as a matching with maximal number of hyperedges must not be a matching in which the number of covered vertices is maximal. So there are two kinds of maxima criteria:

Definition. A matching M with maximal number of hyperedges is called *$\mathcal{E}$-maximum matching* and $\gamma_{\mathcal{E}}(\mathcal{H}) := |M|$ is called the *$\mathcal{E}$-matching number*.

A *V-maximum matching* is a matching M such that the number of vertices $V(M)$ with degree one in $\mathcal{H}[M]$ is maximal and $\gamma_V(\mathcal{H}) := |V(M)|$ is the *V-matching number*.

For example, the hypergraph displayed in Figure 3.1 has one V-maximum matching, namely the hyperedge containing all five vertices. But this is not $\mathcal{E}$-maximum because the two hyperedges of size two form a matching that has more hyperedges.

The dual concept of a matching is a vertex cover. As there are two different notions of maximum matchings, there are also two kinds of vertex covers:

Definition. An *$\mathcal{E}$-vertex cover S* is a set of vertices $S \subseteq V$ such that every hyperedge is incident to at least one vertex in S. A *minimum $\mathcal{E}$-vertex cover S* is of minimal size among all $\mathcal{E}$-vertex covers and $\tau_{\mathcal{E}}(\mathcal{H}) := |S|$ is called *$\mathcal{E}$-vertex cover* number.

A *V-vertex* cover S is a multiset of vertices $S \subseteq V$ such that every hyperedge $e \in \mathcal{E}$ is incident to at least $|e|$ vertices in S, where multiple vertices are counted with multiplicity. A *minimum V-vertex cover S* is of minimal size among all V-vertex covers and $\tau_V(\mathcal{H}) := |S|$ is called *V-vertex cover number*.

The inequality $\gamma_{\mathcal{E}}(\mathcal{H}) \leq \tau_{\mathcal{E}}(\mathcal{H})$ holds because every $\mathcal{E}$-vertex cover must contain at least one vertex for every hyperedge of a $\mathcal{E}$-maximum matching.

Similarly, $\gamma_V(\mathcal{H}) \leq \tau_V(\mathcal{H})$ holds. As every vertex V-vertex cover S must contain at least $|e|$ (multiple) vertices for every hyperedge e contained in a V-maximum matching M, the size of S is at least $\sum_{m \in M} |m| = \gamma_V(\mathcal{H})$.

Classes of hypergraphs for which equality holds will be investigated in the next chapter.

3.4. The Hyperassignment Problem

Perfect matchings in bipartite hypergraphs are also called hyperassignments. This comes from the fact that one can see a perfect matching M as an assignment of subsets of V to subsets of W by assigning the set $m \cap V$ to $m \cap W$ for every $m \in M$.

The hypergraph assignment problem (abbr. HAP) is the following:

Definition. Given a bipartite hypergraph $\mathcal{D} = (V, W, \mathcal{A})$ and some cost function $c : \mathcal{A} \to \mathbb{R}$, find a minimum cost hyperassignment, i.e., a hyperassignment $H \subseteq \mathcal{A}$ such that

$$c(H) := \sum_{a \in H} c(a) \leq \sum_{a \in H'} c_a = c(H')$$

holds for all hyperassignment H' of $\mathcal{D}$ whenever $\mathcal{D}$ admits a hyperassignment, else decide that no hyperassignment exists.

HAP can be formulated as an integer linear optimization problem, where the cost function c is interpreted as a vector $c \in \mathbb{R}^{\mathcal{A}}$:

$$\begin{aligned} &\min c^T x \\ \text{subject to } &\sum_{a \in \delta(v)} x_a = 1 \ \forall v \in V \cup W \qquad (3.1) \\ &x_a \in \{0, 1\} \ \forall \ a \in \mathcal{A} \qquad (3.2) \end{aligned}$$

Lemma 3.4.1. *$\mathcal{D}$ admits a hyperassignment if and only if the above integer linear program is feasible. Furthermore, every optimal solution of the ILP corresponds to a minimum cost hyperassignment and vice versa.*

Proof. Let H be a hyperassignment of $\mathcal{D}$, then $x \in \mathbb{Z}^{\mathcal{A}}$ with

$$x_a = \begin{cases} 1, & a \in H \\ 0, & a \notin H \end{cases}$$

satisfies (3.1) and (3.2), so it is a feasible solution to the ILP.

Now let $x \in \{0, 1\}^{\mathcal{A}}$ be a feasible solution of the integer program, then the set $H := \{a | x_a = 1\}$ is a hyperassignment of $\mathcal{D}$ because (3.1) implies that every vertex $v \in V \cup W$ is incident to exactly one hyperedge of H.

The second claim holds, since for every solution x of the ILP $c^T x$ is equal to the cost of the corresponding hyperassignment H in $\mathcal{D}$. □

3.5. Special Classes of Hypergraphs

A special class of hypergraphs are r-uniform hypergraphs where every hyperedge has size r. If additionally V can be partitioned into r sets $V_1, \ldots, V_r$ such that $|e \cap V_i| = 1$ for every hyperedge e and every $1 \leq i \leq r$, the hypergraph is called r-partite. In particular, for $r = 2$ this is just a bipartite graph.

In a r-uniform hypergraph every $\mathcal{E}$-maximum matching is V-maximum and the other way around. Furthermore $\gamma_V(\mathcal{H}) = r\gamma_{\mathcal{E}}(\mathcal{H})$. Matchings in uniform hypergraphs are a deep field of study (see e.g., [Für81], [FKS93]).

Another special class are regular hypergraphs, in which every vertex has the same degree.

A graph based hypergraph is a hypergraph, such that every proper hyperedge e is the disjoint union of edges. For example, the hypergraphs constructed in Chapter 1 to solve the Railway Vehicle Rotation Planning problem are graph based.

There is another important class of bipartite hypergraphs, arising from this application:

Definition. A *partitioned hypergraph* is a bipartite hypergraph $\mathcal{D} = (V, W, \mathcal{A})$ with vertex sets V and W that can be partitioned into non-empty sets $\{V_1, \ldots, V_k\}$ and $\{W_1, \ldots, W_l\}$ such that for every $a \in \mathcal{A}$ there exist indices $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, l\}$ with $a \cap V \subseteq V_i$ and $a \cap W \subseteq W_j$. The sets $\{V_1, \ldots, V_k\}, \{W_1, \ldots, W_l\}$ are called a *partitioning* of $\mathcal{D}$ in this case.

A partitioning $\{V_1, \ldots, V_k\}, \{W_1, \ldots, W_l\}$ is *finer* than partitioning $\{V'_1, \ldots, V'_m\}$, $\{W'_1, \ldots, W'_n\}$ if for all $1 \leq i \leq m$ there is a $1 \leq j \leq k$ with $V'_i \subseteq V_j$ and for all $1 \leq i' \leq n$ there is a $1 \leq j' \leq l$ with $W'_{i'} \subseteq W_{j'}$ and at least one subset is proper. A *finest partitioning* is one for which no finer partitioning exists.

Lemma 3.5.1. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a bipartite hypergraph. Then there is a unique finest partitioning.*

Proof. The existence is clear since $V_1 = V$, $W_1 = W$ is a partitioning for every bipartite hypergraph.

Suppose there are two different finest partitionings $\{V_1, \ldots, V_k\}, \{W_1, \ldots, W_l\}$ and $\{V'_1, \ldots, V'_m\}, \{W'_1, \ldots, W'_n\}$. Look at the sets $V_{ij} = V_i \cap V'_j$ for all pairs of indices with $1 \leq i \leq k$, $1 \leq j \leq m$ and $W_{rs} = W_r \cap W'_s$ for all $1 \leq r \leq l$, $1 \leq s \leq n$. Then those of the sets that are nonempty form a partitioning of V and W, respectively, that is finer than the two originally partitionings. This shows that there must be a unique finest partitioning. $\square$

By the last lemma the next definition is well defined:

Definition. Let $\mathcal{D} = (V, \mathcal{A})$ be a partitioned hypergraph. The *maximum part size* of $\mathcal{D}$ is the maximum cardinality of a set in the finest partitioning.

Of course, the maximum size $|a \cap V|$ where a ranges over all hyperedges is a lower bound for the maximum part size. Actually, every bipartite hypergraph can be transformed into a partitioned hypergraph with maximum part size equal to that number such that there is a one-to-one correspondence between hyperassignments in the original bipartite hypergraph and the new partitioned hypergraph (see [BH12]):

Theorem 3.5.2. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a bipartite hypergraph and $c : \mathcal{A} \to \mathbb{R}$ be a cost function on the hyperedges. There exists a partitioned hypergraph $\mathcal{D}' = (V', W', \mathcal{A}')$ with maximum part size $\min_{a \in \mathcal{A}} |a \cap V| = \min_{a \in W} |a \cap W|$ and a cost function $c' : \mathcal{A}' \to \mathbb{R}$ such that there is a cost preserving bijection between the hyperassignments in $\mathcal{D}$ and $\mathcal{D}'$ with respect to c and c'.*

Unfortunately, even for bipartite hypergraphs with maximum part size two the hyperassignment problem turns out to be $\mathcal{NP}$-hard (see [BH12]).

3.6. HAP and the Stable Set Problem

Every matching in a hypergraph corresponds to a stable set in the line graph. To get a correspondence between perfect matchings and certain stable set, a weight function is used.

Theorem 3.6.1. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph, $L(\mathcal{H}) = (V_\mathcal{E}, E_\mathcal{E})$ its line graph and $d : V_\mathcal{E} \to \mathbb{R}$ be the weight function with $d(v_e) := |e|$ for all $v_e \in V_\mathcal{E}$.*

Every perfect matching $M \subseteq \mathcal{E}$ of $\mathcal{H}$ gives a stable set $S := \{v_m | m \in M\}$ of maximum d-weight in $L(\mathcal{H})$

On the other side, if there exists a stable set S of weight $d(S) = |V|$, then every maximum d-weight stable set in $L(D)$ corresponds to a perfect matching in $L(D)$.

Proof. If M is a perfect matching, then $S_M := \{v_m | m \in M\}$ is a stable set of weight $\sum_{m \in M} |m| = |V|$.

On the other hand, if S is a stable set of weight $|V|$, then $M_S := \{e | v_e \in S\}$ is a matching covering $\sum_{m \in M} |m| = d(S) = |V|$ vertices, so it is perfect.

It remains to show that $L(D)$ cannot have a stable set of larger weight than $|V|$. Suppose a stable set $T \subseteq V_\mathcal{E}$ with $d(T) > |V|$ exists. No two hyperedges of the corresponding set of hyperedges M_T intersect, otherwise T would not be stable, so M_T is a matching.

This gives $|V| < d(T) = \sum_{e \in M_T} |e| = |\{v \in V|\ v \text{ is covered by some } e \in M_T\}| \leq |V|$, a contradiction. $\square$

The theorem above shows that deciding whether a bipartite hypergraph has a hyperassignment or not can be reduced to the stable set problem. However, the real problem is to find a hyperassignment of minimum weight. This can be done by adjusting the weight function of the line graph.

Theorem 3.6.2. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a bipartite hypergraph, $c : \mathcal{A} \to \mathbb{R}_+$ be a cost function on the hyperedges, and $L(\mathcal{D}) = (V_\mathcal{A}, E)$ be the line graph of $\mathcal{D}$. We define a weight function $w : V_\mathcal{A} \to \mathbb{R}$ on the vertices of $L(\mathcal{D})$ by $w(v_a) := K \cdot |a| - c(a)$, where $K := \sum_{a \in \mathcal{A}} c(a) + 1 \in \mathbb{R}_+$ is a constant.*

If $\mathcal{D}$ has a hyperassignment, then every hyperassignment M of minimum cost corresponds to a maximum weight stable set in $L(D)$ with respect to c and w, respectively.

Proof. It suffices to show that $\sum_{v_a \in S} |a| \geq \sum_{v_a \in T} |a|$ for every maximum weight stable set S and every stable set T of $L(\mathcal{D})$. Then a stable set S of maximum w-weight has also maximum weight with respect to the weight function $d(v_a) := |a|$. This implies by the proof of the last theorem that S corresponds to a V-maximum matching M_S.

In particular, if $\mathcal{D}$ admits a hyperassignment, M_S covers all vertices, i.e., M_S is a hyperassignment.

Suppose there is a maximum weight stable set S and a stable set T such that $\sum_{v_a \in S} |a| < \sum_{v_a \in T} |a|$. As S has maximum weight, we have that

$$\sum_{v_a \in S} (K \cdot |a| - c(a)) = w(S) \geq w(T) = \sum_{v_a \in T} (K \cdot |a| - c(a)).$$

This gives the following contradiction

$$\begin{aligned} 0 &\leq w(S) - w(T) \\ &= K \cdot \left(\sum_{v_a \in S} |a| - \sum_{v_a \in T} |a| \right) + \sum_{v_a \in T} c(a) - \sum_{v_a \in S} c(a) \\ &\leq -K + \sum_{v_a \in T} c(a) \leq -1. \end{aligned}$$

□

On the other side, a maximum size stable set can be calculated using HAP:

Theorem 3.6.3. *For a given graph $G = (V, E)$ let $\mathcal{D} = (E \times \{0,1\}, \mathcal{A})$ be the bipartite hypergraph with hyperedges $a_v := \{(e,0) | e \ni v\} \cup \{(e,1) | e \ni v\}$ for all $v \in V$ and edges $a_e := \{(e,0),(e,1)\}$ for every edge $e \in E$. Define $c : \mathcal{A} \to \mathbb{R}$ by $c(a_v) = -1$ and $c(a_e) = 0$.*

A minimum cost hyperassignment in $\mathcal{D}$ gives a maximum size stable set in G, and conversely.

Proof. Let M be a minimum cost hyperassignment with respect to c defined above and set $S := \{v \in V | a_v \in M\}$. Suppose S is not stable, then there is an edge e connecting two vertices $v_1, v_2 \in S$, but e corresponds to the two vertices $(e,0)$ and $(e,1)$ in $\mathcal{D}$ which are therefore incident to both hyperedges $a_{v_1}, a_{v_2} \in M$. This is a contradiction to M being a matching. So S is a stable set of size $-c(M)$.

On the other side, if S is a stable set in G then all hyperedges a_v for $v \in S$ together with hyperedges a_e for edges e not covered by S form a hyperassignment in $\mathcal{D}$ of cost $-|S|$.

Together we get that a maximum size stable set in G corresponds to a minimum cost hyperassignment in $\mathcal{D}$. □

There are some classes for which the stable set problem can be solved in polynomial time:

Perfect graphs: Hypergraphs corresponding to this class of graphs will be treated in the next chapter.

Claw-free graphs: A claw-free graph is a graph $G = (V, E)$ containing no $K_{1,3}$ as an induced subgraph. The stable set polytope of this class is not fully understood but there is an algorithm running in polynomial time which solves even the weighted stable set problem.

t-**perfect graphs:** A graph $G = (V, E)$ is called t-perfect if

$$\mathrm{STAB}(G) : = \mathrm{conv}(\{\chi^S \in \mathbb{R}^V | S \subseteq V \text{ is a stable set}\})$$

is equal to

$$\mathrm{CSTAB}(G) : = \{x \in \mathbb{R}^V | 0 \leq x_v \leq 1 \ \forall\ v \in V, x_u + x_v \leq 1 \ \forall\ \{uv\} \in E,$$
$$x(V(C)) \leq \frac{|C| - 1}{2} \ \forall \text{ odd cycles } C\}.$$

t-perfect graphs are important because odd cycle constraints can be separated in polynomial time:

Let $x \in \mathbb{R}^V$ with $0 \leq x \leq 1$ and $x_u + x_v \leq 1$ for all edges $\{u, v\} \in E$. To check whether x satisfies all odd cycle constraints we construct a length function $l : E \to \mathbb{R}_+$ by setting $l(e) := 1 - x_u - x_v \geq 0$ for all $e = uv \in E$. Now we search a shortest odd cycle C in G with respect to l (this can be done in polynomial time). If $l(C) \geq 1$, then x satisfies all odd cycle inequalities. Otherwise, we have found an odd cycle constraint that cuts off x.

The polynomial-time solvability of the separation problem implies that also the optimization problem can be solved in polynomial time (cf. [GLS81]).

No characterization of t-perfect graphs is known yet. However, there are some examples:

- Almost bipartite graphs: A graph $G = (V, E)$ such that deleting one vertex $v \in V$ makes G bipartite, so v is contained in every odd cycle.
- Graphs without an odd K_4 as a subgraph (see [GS98]): A odd K_4 is a subdivision of K_4 such that all triangles have become odd circuits.

4. Balanced and Normal Hypergraphs

In this chapter we will investigate balanced and normal hypergraphs. They share some nice properties with bipartite graphs, for example, their matching and perfect matching polytope can be described by the trivial inequalities $x \geq 0$ and the degree inequalities $x(\delta(v)) \leq 1$ and $x(\delta(v)) = 1$, respectively. In particular, for balanced or normal bipartite hypergraphs the Hyperassignment Problem can be solved in polynomial time.

Furthermore, Kőnig's Theorem holds for balanced hypergraphs and a Hall-type condition for the existence of a perfect matching can be formulated.

Definition. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ without strong odd cycles is called *balanced*. Equivalently, the incidence matrix A of $\mathcal{H}$ does not contain an odd square submatrix of the form

$$\begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

after permuting some rows and columns.

Every partial hypergraph and every subhypergraph of a balanced hypergraph is balanced again because they obviously cannot have any strong odd cycles. Also the dual of a balanced hypergraph is balanced:

If $C = v_0, e_1, v_1, \dots, e_l, v_l = v_0$ is a strong odd cycle in $\mathcal{H}$, then $e_1, v_1, \dots, e_l, v_l, e_1$ is an odd cycle in $\mathcal{H}^*$. C is strong in $\mathcal{H}^*$ because every hyperedge e_{v_i} corresponding to v_i is only incident to vertices e in $\mathcal{H}^*$ with $v_i \in e$, so v_i is only incident to e_{i-1} and e_i in C. This shows that $\mathcal{H}$ is balanced if $\mathcal{H}^*$ is balanced, since $(\mathcal{H}^*)^* = \mathcal{H}$ also the other direction holds.

Remark. In a bipartite hypergraph $\mathcal{D} = (V, W, \mathcal{A})$ two consecutive vertices of a cycle can be both in V (or W) or one lies in V and the other in W. In the first case, the two vertices of V (or W) are connected by a proper hyperedge. In the remaining, such a hyperedge will be called a tail-to-tail hyperedge. This denotation comes from [CN95], where similar circles for Leontief flow problems are investigated.

Because of the structure of a bipartite hypergraph, every odd cycle uses an odd number of tail-to-tail hyperedges.

Theorem 4.0.4. *A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is balanced if and only if every subhypergraph of $\mathcal{H}$ is 2-colorable.*

Proof. If $\mathcal{H}$ is not balanced, then there exists a strong odd cycle C in $\mathcal{H}$. The subhypergraph induced by the set of vertices contained in C is not 2-colorable.

As every subhypergraph of a balanced hypergraph is balanced again, it suffices to show that every balanced hypergraph is 2-colorable. Let $\mathcal{H}$ be balanced, we will give an algorithm due to Cameron and Edmonds, see [CE90].

- For $k = 1, \ldots, |V|$ we do the following:
- As long as not all vertices are colored and there is no monochromatic hyperedge of size at least two, we color the k-th vertex and label it as follows:
  1. If there is an edge e, whose incident vertices are colored in one color except one vertex, then color this vertex with the other color, label it with k and "forced by hyperedge e".
  2. Otherwise, color arbitrarily one of the uncolored vertices, label it with k and "freely colored".

Clearly, this algorithm terminates after at most $|V|$ steps. Suppose the above algorithm stopped, because there was a monochromatic hyperedge $e \in \mathcal{E}$ with $|e| > 1$. We construct a strong odd cycle using the labelling obtained from the algorithm as in [CE90].

First we define for every colored vertex v a sequence alternating between vertices and hyperedges. The sequence $F(v) = v_1, e_1, v_2, \ldots, e_k, v_{k+1}$ starts with with $v_1 = v$ and ends with a freely colored vertex v_{k+1}. Every vertex v_i is followed by the hyperedge e_i, which forced v_i's coloring, and every hyperedge e_i is followed by the vertex of e_i with the highest number smaller than that of v_i w.r.t. the labelling.

Let w be the highest and w' be the second highest numbered vertex contained in the monochromatic hyperedge e. As soon as w' was colored, w is forced by e, but the algorithm did not color w in a different color than w', so it colored other "forced" vertices. This shows that the freely colored end vertex of $F(w')$ is the same as that of $F(w)$. Let v' be the first common vertex of $F(w)$ and $F(w')$. Now, we can use $F(w)$ and $F(w')$ to construct a path from w via v' to w'. As w and w' have the same color and the color of the vertices of the path alternate between two colors, closing the path with e gives a strong cycle of odd length. $\square$

Remark. The algorithm of the above proof cannot decide whether a hypergraph is balanced or not. Even though it will give a correct 2-coloring if the given hypergraph is balanced, it can also give a correct 2-coloring of a non-balanced hypergraph. For example, for a bipartite hypergraph $\mathcal{D} = (V, W, \mathcal{A})$ the algorithm can color in every "free" step a vertex of V with blue, then it might do some "forced" red colorings of vertices of W that are connected by an edge to a new blue vertex, after that there might be some "forced" blue colorings of vertices of V, etc. If there are some uncolered vertices of W left after the coloring of V, all remaining vertices can be colored red. This will give a correct 2-coloring of a bipartite hypergraph, whether it is balanced or not.

Theorem 4.0.5. *Every balanced hypergraph $\mathcal{H}$ has the colored edge property.*

Proof. The claim is trivial for $\Delta(\mathcal{H}) = 1$.

So we can assume $k := \Delta(\mathcal{H}) \geq 2$. Let $S_1, S_2, \ldots, S_k$ be the color classes of a vertex coloring c of $\mathcal{H}^* = (V^*, \mathcal{E}^*)$ where some classes could be empty. This is possible since $\mathcal{H}^*$ is balanced and thus 2-colorable.

If for every hyperedge in $\mathcal{H}^*$ all adjacent vertices are in different color classes, c is an edge coloring of $\mathcal{H}$ such that $c(e) \neq c(f)$ for adjacent hyperedges $e \neq f$.

Otherwise, we use an idea in [CE90] to construct a "better" coloring. Suppose there is a hyperedge e^* of $\mathcal{H}^*$ which has two vertices of the same color. This implies that there are classes S_q, S_p with $|e^* \cap S_q| \geq 2$ and $e^* \cap S_p = \emptyset$ because $|e^*| \leq k$. The subhypergraph of $\mathcal{H}^*$ induced by $S_p \cup S_q$ is balanced and thus it admits a 2-coloring. Let S'_p and S'_q be the color classes of this coloring. Replacing S_p and S_q by S'_p and S'_q yields a new coloring c'. c' uses one more color for the vertices of e^* than c, but as least as many colors as c for the vertices of all other hyperedges of $\mathcal{H}^*$.

Repeating the above procedure finally gives a vertex coloring of $\mathcal{H}^*$ as desired. □

In the remaining of this chapter, Kőnig and Hall type theorems for balanced hypergraphs will be shown; moreover, the polyhedral structure of their matching polytopes will be investigated.

4.1. Integral Polyhedra

Balanced hypergraphs are important in a polyhedral point of view since the LP-relaxation of their matching and covering polytopes is integral.

Just as in Lemma 3.4.1, the convex hull of the characteristic vectors of all matchings, perfect matchings and covering of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ is given by

$$IP_M(\mathcal{H}) := \operatorname{conv}(\{x \in \{0,1\}^{\mathcal{E}} | Ax \leq 1\}), \tag{4.1}$$

$$IP_{PM}(\mathcal{H}) := \operatorname{conv}(\{x \in \{0,1\}^{\mathcal{E}} | Ax = 1\}), \tag{4.2}$$

$$IP_C(\mathcal{H}) := \operatorname{conv}(\{x \in \{0,1\}^{\mathcal{E}} | Ax \geq 1\}), \tag{4.3}$$

where A is the incidence matrix of $\mathcal{H}$. The integer polytopes $IP_M(\mathcal{H}), IP_{PM}(\mathcal{H})$ and $IP_C(\mathcal{H})$ will be called matching polytope, perfect matching polytope and covering polytope, respectively.

Let LP_M, LP_{PM}, LP_C denote the polytopes obtained from the above systems of linear inequalities by replacing $x \in \{0,1\}^{\mathcal{E}}$ by $x \in \mathbb{R}^{\mathcal{E}}$ and $0 \leq x \leq 1$. They will be called fractional matching polytope, fractional perfect matching polytope and fractional covering polytope, respectively.

Theorem 4.1.1. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. Then the following four claims are equivalent:*

1. *$\mathcal{H}$ is balanced.*
2. *For every partial subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ the perfect matching polytope $LP_{PM}(\mathcal{H}')$ is integral.*
3. *For every partial subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ the covering polytope $LP_C(\mathcal{H}')$ is integral.*
4. *For every partial subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ the matching polytope $LP_M(\mathcal{H}')$ is integral.*

Proof. (1) $\Rightarrow$ (2) : Every partial subhypergraph is balanced, so it suffices to show that $LP_{PM}(\mathcal{H})$ is integral for every balanced hypergraph $\mathcal{H} = (V, \mathcal{E})$. For this we use induction on $|V| + |\mathcal{E}|$.

If the polytope is empty, there is nothing to show. Now suppose $LP_{PM} \neq \emptyset$ and let x^* be a vertex of it. If $x^*_e = 0$ for some $e \in \mathcal{E}$, then projecting x^* to $\mathbb{R}^{\mathcal{E}\setminus e}$ yields a vertex of $LP_{PM}(\mathcal{H}[\mathcal{E} \setminus e])$ by Corollary 2.2.6. Using induction shows that x^* is integral.

Otherwise, we will show that $\mathcal{H}$ contains no cycles, so there is at least one vertex $v \in V$ of degree one giving the equation $x^*_e = 1$ for the unique hyperedge e incident to v. It follows that there are no $e' \neq e$ with $e \cap e' \neq \emptyset$, otherwise $x^*_{e'} = 0$. The vector obtained by projecting x^* to $\mathbb{R}^{\mathcal{E}\setminus\{e\}}$ is a vertex of the perfect matching polytope of $\mathcal{H}' := (V \setminus e, \mathcal{E} \setminus \{e\})$. By induction, this vector and thus x^* is integral. So it is enough to show that there are no cycles in $\mathcal{H}$.

First, we show that $\mathcal{H}$ does not contain any strong even cycle. Otherwise, projecting x^* to the perfect matching polytope of the partial subhypergraph induced by the strong even cycle yields a vertex for which the vectors χ^e corresponding to positive entries in x^* are not linearly independent, because the matrix with these vectors as columns is (up to permutation):

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

and it has determinant zero, as the number of rows is even.

Because $\mathcal{H}$ is balanced, it clearly does not contain any strong odd cycle. So suppose there is a cycle in $\mathcal{H}$ which is not strong. Pick a shortest one, say $C = v_0, e_1, \dots, e_l, v_l$. Then there is an hyperedge e_i containing at least three vertices v_{i-1}, v_i and some v_j. W.l.o.g. $j > i$, then $C' := v_0, e_1, \dots, v_{i-1}, e_i, v_j, e_{j+1}, \dots, e_l, v_l$ is a shorter cycle. Hence, by the choice of C the cycle C' must be strong, this is a contradiction.

(2) $\Rightarrow$ (3) : Let x^* be a vertex of $LP_C(\mathcal{H})$. $A_{\mathrm{eq}(x^*).}$ is the incidence matrix of partial subhypergraph $\mathcal{H}'$ of $\mathcal{H}$. $\mathcal{H}'$ is induced by the set of vertices and hyperedges corresponding to the rows and columns of $A_{\mathrm{eq}(x^*).}$, respectively. So, $\mathcal{H}'$ is a balanced hypergraph, too, implying that x^* is a vertex of $LP_{PM}(\mathcal{H}')$, and therefore x^* is integral.

(3) $\Rightarrow$ (4) : Similar to the last one.

(4) $\Rightarrow$ (1) : If $\mathcal{H}$ is not balanced, it contains a partial subhypergraph $\mathcal{C}$ corresponding to a strong odd cycle of length $l \in \mathbb{Z}$. The vector x^* with all entries equal to $\frac{1}{2}$ is a solution of

$$\max\ 1^T x \text{ subject to } x \in LP_M(\mathcal{C})$$

with value $\frac{l}{2}$. But every matching of $\mathcal{C}$ has size at most $\frac{l-1}{2}$, so $LP_M(\mathcal{C})$ is not integral. □

Balanced hypergraphs have the very strong property, that their (perfect) matching polytope is integral for every partial subhypergraph. There is a larger class of hypergraphs for which the three polytopes above are integer, so called normal hypergraphs.

Definition. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is called *normal* if it has the Helly property and $L(\mathcal{H})$ is perfect.

Remark. The usual definition of a normal hypergraph requires that all partial hypergraphs have the colored edge property, which is equivalent to the above definition. This version was chosen, because nowadays the strong perfect graph theorem can be used (see [CRST06]), making the proof of the next theorem easier.

It turns out that normal hypergraphs are exactly the class of hypergraphs, for which the matching polytope is integer.

Theorem 4.1.2. *The following two statements are equivalent for a hypergraph $\mathcal{H}$:*

1. *$\mathcal{H}$ is normal.*
2. *$P_M(\mathcal{H})$ is integral.*

Proof. (2) $\Rightarrow$ (1) : First suppose $\mathcal{H}$ has not the Helly-property. Then there are three hyperedges e_1, e_2, e_3 such that every two of them intersect but not all three. The vector $x \in \mathbb{R}^{\mathcal{E}}$ with entries $x_{e_i} = \frac{1}{2}$ for $i = 1, 2, 3$ and $x_e = 0$ else lies in $LP_M(\mathcal{H})$, but is not a convex combination of characteristic vectors of matchings.

Now, suppose $L(\mathcal{H})$ is not perfect. If $L(\mathcal{H})$ has an odd hole, $\mathcal{H}$ has a strong odd cycle, therefore $LP_M(\mathcal{H})$ would not be integral.

If $L(\mathcal{H})$ has an odd anti-hole of size k, then setting

$$x_e := \begin{cases} \frac{1}{k-3} & \text{if e lies in the odd anti-hole} \\ 0 & \text{else} \end{cases}$$

gives a vector $x \in LP_M(\mathcal{H})$, which is not in the convex hull of the characteristic vectors of matchings.

(1) $\Rightarrow$ (2) : First observe that the matching polytope is equal to the stable set polytope of the line graph $L(\mathcal{H}) = (\mathcal{E}, E)$, which is:

$$\text{STAB}(L(\mathcal{H})) = \text{conv}(\{\chi^S \in \mathbb{R}^{\mathcal{E}} | S \text{ is a stable set}\}).$$

This follows from the fact that every stable set in $L(\mathcal{H})$ is a matching in $\mathcal{H}$ and vice versa.

The line graph $L(\mathcal{H})$ is perfect. Thus, its stable set polytope is determined by the non-negativity constraints $x_e \geq 0$ for all vertices $e \in \mathcal{E}$ of $L(\mathcal{H})$ and by the clique inequalities $x(Q) \leq 1$ for all cliques Q of $L(\mathcal{H})$. Every clique of the line graph is contained in one of the form $\delta(v)$ for some $v \in V$, because $\mathcal{H}$ has the Helly property. This, together with the first observation, gives

$$\begin{aligned} IP_M(\mathcal{H}) &= \text{STAB}(L(\mathcal{H})) = \text{QSTAB}(L(\mathcal{H})) \\ &= \{x \in \mathbb{R}^{\mathcal{E}} | x \geq 0,\ x(\delta(v)) \leq 1\ \forall v \in V\} \\ &= LP_M(\mathcal{H}), \end{aligned}$$

which is the claim. □

Remark. The last two theorems imply that a balanced hypergraph is normal.

Hypergraphs failing to be normal because they do not have the Helly-property are not much more difficult.

Theorem 4.1.3. *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with incidence matrix A such that $L(\mathcal{H})$ is perfect. Then*

$$IP_M(\mathcal{H}) = \{x \in \mathbb{R}^{\mathcal{E}} | Ax \leq 1, x(Q) \leq 1 \textit{ for all cliques } Q \textit{ in } L(\mathcal{H}), 0 \leq x \leq 1\}.$$

Proof. For every clique Q we add one dummy vertex v_Q contained in every hyperedge of Q. The resulting hypergraph $\mathcal{H}_Q$ has the Helly-property and $L(\mathcal{H}_Q) = L(\mathcal{H})$ is perfect, thus $\mathcal{H}_Q$ is normal. Furthermore, there is a natural bijection between the hyperedges of $\mathcal{H}$ and that of $\mathcal{H}_Q$ preserving matchings which shows that

$$IP_M(\mathcal{H}) = IP_M(\mathcal{H}_Q).$$

The last polytope is integral and equal to

$$\{x \in \mathbb{R}^{\mathcal{E}} | Ax \leq 1, x(Q) \leq 1 \text{ for all cliques } Q \text{ in } L(\mathcal{H}), 0 \leq x \leq 1\}$$

proving the claim. □

Remark. Clique inequalities cannot be separated in polynomial time. But for perfect graphs the orthonormal representation constraints imply the clique inequalities and it is possible to separate them, see ([GLS93]). Moreover, for partitioned hypergraphs of fixed maximum part size d the clique inequalities can be separated directly, see [Hei10].

4.2. Kőnig's Theorem and a Hall Condition

A bipartite graph $G = (V, E)$ with color classes U and W has a matching covering U if and only if $|N(S)| \geq |S|$ for all $S \subseteq U$, where $N(S)$ is the set of neighbours of S. This is a well known application of Hall's marriage theorem for the existence of a transversal in a SDR to matchings in bipartite graphs. To derive such a "Hall"-type condition for perfect matching in hypergraphs one has to reformulate the condition:

Theorem 4.2.1. *A bipartite graph $G = (V, E)$ has a perfect matching if and only if for all pairs of disjoint node sets R, B with $|R| > |B|$ there exists an edge $e \in E$ such that $|e \cap R| > |e \cap B|$.*

Proof. Let U and W be the color classes of G.

First, suppose G has no perfect matching. If U and W are not of the same size, w.l.o.g. $|U| > |W|$, then $R = U$ and $B = W$ violates the condition of the theorem. So, we might assume $|U| = |W|$. By Hall's Theorem there exists a set $S \subseteq U$ with $|S| > |N(S)|$. Now, choosing $R = S$ and $B = N(S)$ gives a pair violating the right hand side of the theorem.

On the other hand, if there exist disjoint node sets R, B with $|R| > |B|$ and $|e \cap R| \leq |e \cap B|$ for every edge $e \in E$, then G has no perfect matching. To see this, take $S_U = R \cap U$ and $S_W = R \cap W$. As every edge which has one node in R must have the other one in B, it follows that $N(S_U) \subseteq B \cap W$ and $N(S_W) \subseteq B \cap U$, giving

$$|N(S_U)| + |N(S_W)| \leq |B| < |R| = |S_U| + |S_W|.$$

So at least one of the sets S_U, S_W violates the classical Hall condition; thus G has no perfect matching. □

Now we can state a Hall condition for balanced hypergraphs:

Theorem 4.2.2. *Let $\mathcal{H} = (V, \mathcal{E})$ be a balanced hypergraph. $\mathcal{H}$ has a perfect matching if and only if for all disjoint node sets R, B with $|R| > |B|$ there exists a hyperedge $e \in \mathcal{E}$ such that $|e \cap R| > |e \cap B|$.*

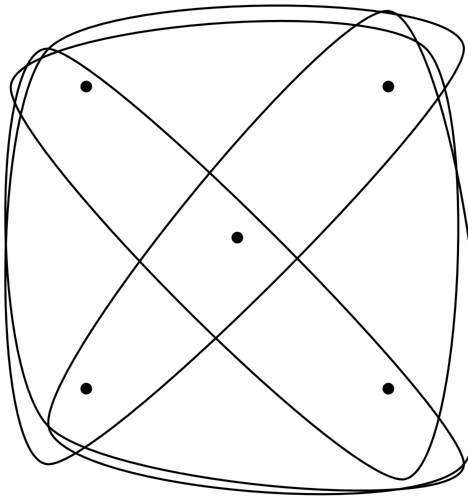


Figure 4.1.: The hypergraph $\mathcal{H}_4$

If a hypergraph (not necessarily balanced) has a perfect matching M, then for every pair of disjoint sets $R, B \subseteq V$ with $|R| > |B|$ we have:

$$\sum_{e \in M} |e \cap R| = |R| > |B| = \sum_{e \in M} |e \cap B|.$$

So there is at least one edge $e \in M$ with $|e \cap R| > |e \cap B|$. This shows the necessity of the above condition for all hypergraphs, not just balanced ones.

Conforti et al. give a polyhedral proof for sufficiency in [CCKV96] and Huck and Triesch a combinatorial one in [HT02].

Looking at the polyhedral proof, which uses integrality of the perfect matching polytope and Farkas' Lemma, one might think that Theorem 4.2.2 could still hold for normal hypergraphs. This is not the case, as the following example shows:

Example. For every natural number $n \geq 3$, let $\mathcal{H}_n = (V_n, \mathcal{E}_n)$ be the hypergraph with vertex set $\{1, \dots, n, n+1\}$ and a hyperedge $S \cup \{n+1\}$ for every subset S of $\{1, \dots, n\}$ that has size $n-1$. For example, Figure 4.1 shows $\mathcal{H}_4$.

Every two hyperedges of $\mathcal{H}_n$ intersect in the $(n+1)$-th vertex, so $\mathcal{H}_n$ is Helly and its line graph is K_n which is perfect. Thus $\mathcal{H}_n$ is a normal hypergraph and $\mathcal{H}_n$ has no perfect matching. However, for every pair of disjoint set $R, B \subseteq V$ with $|R| > |B|$ there is an $e \in \mathcal{E}_n$ with $|e \cap R| > |e \cap B|$.

For $B = \emptyset$ this is obvious, so from now on let $|B| \geq 1$.

If $n+1 \notin R$, there is either an $n-1$ subset S of $\{1, \dots, n\}$ with $R \subseteq S$ or $R = \{1, \dots, n\}$. In the first case, choosing $e = S \cup \{n+1\} \in \mathcal{E}_n$ gives a hyperedge with

$$|e \cap R| = |R| > |B| \geq |B \cap e|.$$

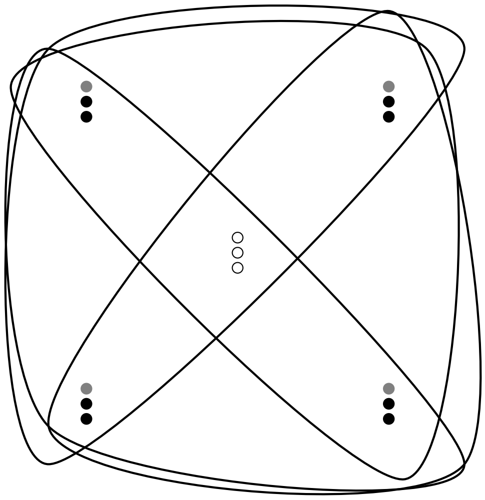


Figure 4.2.: $\mathcal{H}_4^3$

In the other case, $B = \{n+1\}$ and therefore $|R \cap e| = n - 1 > 1 = |B \cap e|$ holds for all $e \in \mathcal{E}_n$.

Now, let $n + 1 \in R$. The set $R \setminus \{n + 1\}$ has size at most $n - 1$ as $B \neq \emptyset$. So there exists an $n-1$ element subset S of $\{1, \ldots, n\}$ with $R \setminus \{n+1\} \subseteq S$. Choosing $e = S \cup \{n+1\} \in \mathcal{E}_n$ gives

$$|e \cap R| = |R| > |B| \geq |e \cap B|.$$

This shows that Theorem 4.2.2 fails for normal hypergraphs.

However, if we copy each vertex $n - 1$ times and add each copy of a vertex $v \in V_n$ to all hyperedges $e \in \mathcal{E}_n$ containing v we get a pair R, B violating the Hall condition. To see this, we put exactly one copy of every vertex $1, \ldots, n$ into R and all $n - 1$ copies of vertex $n + 1$ into B. With this choice $|e \cap R| = n - 1 = |e \cap B|$ holds for every hyperedge.

For example, in the case $n = 4$ we get the hypergraph shown in Figure 4.2. Taking R to be the set of grey vertices and B to be the set of non-filled vertices gives a pair violating the Hall condition.

The following theorem shows how the idea of the example can be transferred to all normal hypergraphs:

Theorem 4.2.3. *Let $\mathcal{H} = (V, \mathcal{E})$ be a normal hypergraph and $\mathcal{H}^N = (V_N, \mathcal{E}_N)$ be the hypergraph with $V^N := \{(v,i) | v \in V,\ 1 \leq i \leq N\}$ and $\mathcal{E}^N := \{e^N | e \in \mathcal{E}\}$ where the hyperedges are defined by $e^N := \{(v,i) | v \in e,\ 1 \leq i \leq N\}$.*

$\mathcal{H}$ has no perfect matching if and only if there exists a natural number N such that $\mathcal{H}^N = (V^N, \mathcal{E}^N)$ does not satisfy the Hall condition for hypergraphs.

Proof. Suppose $\mathcal{H}$ has no perfect matching. Let A be the incidence matrix of $\mathcal{H}$. The system $Ax = 1$ cannot have a solution $x \geq 0$, otherwise it has an integer solution because of Theorem 4.1.2. By Farkas' Lemma there exists a vector $z \in \mathbb{Q}^V$ with $z^T A \geq 0$ and $z^T 1 < 0$. Scaling z, we can assume $z \in \mathbb{Z}^V$.

Now, we take $N := \max_{v \in V} |z_v|$ and define a pair (R, B) of disjoint node sets by

$$R := \{(v,i) | z_v < 0,\ 1 \leq i \leq |z_v|\},$$
$$B := \{(v,i) | z_v > 0,\ 1 \leq i \leq z_v\}.$$

R is larger than B because $z^T 1 = \sum_{v \in V} z_v = |B| - |R| < 0$. Moreover, $z^T A \geq 0$ implies $|e \cap R| \leq |e \cap B|$ for all $e \in \mathcal{E}^N$. This shows that R, B is a pair violating Hall's condition in $\mathcal{H}^N$.

On the other hand, if there is an N such that $\mathcal{H}^N$ does not satisfy Hall's condition, then $\mathcal{H}^N$ has no perfect matching and thus $\mathcal{H}$ has none. □

Hall's condition for balanced hypergraphs implies that a balanced hypergraph with maximum degree Δ can be partitioned into Δ disjoint matchings. Using the proof of this from [CCV06] and Theorem 4.2.3, it can be proven for normal hypergraphs, too.

Theorem 4.2.4. *Let $\mathcal{H} = (V, \mathcal{E})$ be a normal hypergraph, then $\mathcal{H}$ can be partitioned into $\Delta(\mathcal{H})$ disjoint matchings.*

Proof. For $\Delta(\mathcal{H}) = 1$ the edges of $\mathcal{H}$ form a matching.

Now let $\Delta := \Delta(\mathcal{H}) > 1$. For every vertex v with $\deg(v) < \Delta(\mathcal{H})$ add $\deg(v) - \Delta(\mathcal{H})$ copies of the one element hyperedge $\{v\}$. After that, $\mathcal{H}$ still has the Helly-property and $L(\mathcal{H})$ remains perfect.

Assume $\mathcal{H}$ has no perfect matching. Theorem 4.2.3 shows that there is a number $N \in \mathbb{N}$ such that $\mathcal{H}^N$ does not fulfil Hall's condition. So there exists a pair R, B with $|R| > |B|$ and $|e \cap R| \leq |e \cap B|$ for all $e \in \mathcal{E}^N$. As $\deg_{\mathcal{H}^N}((v,i)) = \deg_{\mathcal{H}}(v) = \Delta$ for all $v \in V$ and all $1 \leq i \leq N$, $\mathcal{H}^N$ is also Δ-regular. Adding the inequalities $|e \cap R| \leq |e \cap B|$ for all $e \in \mathcal{E}^N$ gives $\Delta|R| \leq \Delta|B|$, contradicting $|R| > |B|$.

Therefore, $\mathcal{H}$ must have a perfect matching M. Deleting all hyperedges in M from $\mathcal{E}$ gives a hypergraph with smaller maximum degree. So the assertion follows by induction on Δ. □

Remark. The above theorem shows that a normal hypergraph has the colored edge property (take the matchings as color classes). Clearly, every partial hypergraph of a normal hypergraph is normal again since its line graph is a subgraph of the original line graph, and the Helly-property still holds after removing some hyperedges. Thus, every partial hypergraph of a normal hypergraph has the colored edge property.

Kőnig's Theorem follows already from the fact that the matching polytope of balanced hypergraphs is integral but there is also a nice combinatorial proof by Scheidweiler and Triesch ([ST11]).

Theorem 4.2.5. *Let $\mathcal{H} = (V, \mathcal{E})$ be a balanced hypergraph. Then*

$$\gamma_{\mathcal{E}}(\mathcal{H}) = \tau_{\mathcal{E}}(\mathcal{H}).$$

Proof. $\gamma_{\mathcal{E}}(\mathcal{H}) \leq \tau_{\mathcal{E}}(\mathcal{H})$ was shown in Chapter 2. Suppose there is a balanced hypergraph for which the inequality is strict. Among these choose a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $|V| + |\mathcal{E}|$ minimal.

If there is a vertex $v \in V$ with $v \in V(M)$ for every $\mathcal{E}$-maximum matching M, then $\mathcal{H} - v := (V \setminus v, \mathcal{E}')$ with $\mathcal{E}' := \{e | e \in \mathcal{E} : v \notin e\}$ has $\mathcal{E}$-matching number $\gamma_{\mathcal{E}}(\mathcal{H}) - 1$ and $\mathcal{E}$-vertex cover number $\tau_{\mathcal{E}}(\mathcal{H}) - 1$. $\mathcal{H}$ is a minimal counterexample, so the last two numbers are equal. This implies

$$\gamma_{\mathcal{E}}(\mathcal{H}) = \tau_{\mathcal{E}}(\mathcal{H}),$$

contradicting the assumption.

Now, suppose that for every $v \in V$ there is a $\mathcal{E}$-maximum matching M_v such that v is not covered by M_v. We choose one such matching for each vertex $v \in e'$ where $e' \in \mathcal{E}$ is a fixed hyperedge. Using these matchings and the fixed edge e', we define the hypergraph $\mathcal{H}' := (V', \mathcal{E}')$ by it's set of vertices $V' := \bigcup_{v \in e'} V(M_v) \cup e'$ and the set of hyperedges $\mathcal{E}'$ which consists of the fixed hyperedge e' and of k copies of every $e \in \mathcal{E}$, where k is the number of matchings M_v $(v \in e')$ containing e. So $\mathcal{H}'$ is a hypergraph with $\sum_{v \in e'} |M_v| + 1 = |e'|\gamma_{\mathcal{E}}(\mathcal{H}) + 1$ hyperedges and maximum degree $\Delta(\mathcal{H}') \leq |e'|$. By construction, $\mathcal{H}'$ is obtained from a partial hypergraph of $\mathcal{H}$ by multiplying some hyperedges, thus it is also balanced. In particular $\mathcal{H}'$ has the colored edge property which shows that the hyperedges can be colored with $|e'|$ colors such that no two adjacent hyperedges get the same color. The $|e'|$ color classes are matchings of $\mathcal{H}'$ and at least one class has size $\gamma_{\mathcal{E}} + 1$. This class is also a matching of $\mathcal{H}$ as it cannot have multiple hyperedges. But then we have found a matching with more than $\gamma_{\mathcal{E}}$ hyperedges, a contradiction. □

Remark. The proof actually holds for normal hypergraphs because only the colored edge property is used, and multiplying hyperdeges does not destroy normality (c.f. [Ber89]).

Theorem 4.2.5 implies Kőnig's Theorem for the case of V-maximum matchings and V-minimum vertex covers. This can be done by a reduction of Berge, see [Ber89] in Chapter 5, p. 180.

Corollary 4.2.6. *Let $\mathcal{H} = (V, \mathcal{E})$ be a balanced hypergraph. Then*

$$\gamma_V(\mathcal{H}) = \tau_V(\mathcal{H}).$$

Proof. We construct a new hypergraph $\mathcal{H}'$ from $\mathcal{H}$ by replacing each vertex $v \in V$ by $m = \max_{e \in \delta(v)} |e|$ copies $v^1, \ldots, v^m$ and each hyperedge $e = \{v_1, \ldots v_k\} \in \mathcal{E}$ by $|e| = k$ new hyperedges $e^i := \{v_1^i, \ldots, v_k^i\}$, $1 \leq i \leq k$.

By the construction of $\mathcal{H}'$, V-maximum matchings of $\mathcal{H}$ are transformed into $\mathcal{E}$-maximum matchings of $\mathcal{H}'$ and V-minimum vertex covers of $\mathcal{H}$ into E-minimum vertex covers of $\mathcal{H}'$. The hypergraph $\mathcal{H}'$ is also balanced, so Theorem 4.2.5 implies the claim of the corollary. $\square$

Scheidweiler and Triesch give an interesting estimation of the V-matching number of a balanced hypergraph which also holds for normal hypergraphs (using the same proof).

Theorem 4.2.7. *Given a normal hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a natural number $q \in \mathbb{N}$ with*

$$\sum_{v \in V} (\Delta(\mathcal{H}) - \deg_{\mathcal{H}}(v)) \leq q\Delta(\mathcal{H}) - 1,$$

then $\gamma_V(\mathcal{H}) \geq |V| - q + 1$ holds.

Proof. By Theorem 4.2.4, $\mathcal{E}$ can be partitioned into $\Delta(\mathcal{H})$ matchings $M_1, \ldots, M_{\Delta(\mathcal{H})}$. Suppose each matching covers at most $|V| - q$ vertices. This implies that

$$\sum_{v \in V} \deg_{\mathcal{H}}(v) = \sum_{i=1}^{\Delta(\mathcal{H})} \sum_{v \in V} \deg_{\mathcal{H}[M_i]}(v) \leq \Delta(\mathcal{H})(|V| - q),$$

so

$$\sum_{v \in V} (\Delta(\mathcal{H}) - \deg_{\mathcal{H}}(v)) \geq |V|\Delta(\mathcal{H}) - (\Delta(\mathcal{H})(|V| - q)) = q\Delta(\mathcal{H}),$$

which is a contradiction to the assumption. $\square$

The last theorem shows that a normal hypergraph has a matching covering many vertices if the degrees of the vertices differ not too much from each other, so it is a sharpened version of Theorem 4.2.4. Another interesting fact is that it is possible to multiply certain hyperedges to obtain the right value of $\gamma_V(\mathcal{H})$, for details see [Sch11].

4.3. Complexity

4.3.1. Testing Balancedness

Checking whether a hypergraph is balanced or not can be accomplished in polynomial time. For this, the bipartite representation is used. This bipartite graph does not contain an induced cycle of length $4k+2$ for a $k \in \mathbb{N}$ if and only if the given hypergraph is balanced.

It is clear that a cycle of length $4k + 2$ uses $2k + 1$ vertices of each side of the bipartition. So an easy algorithm to check whether a bipartite graph contains such an induced cycle is to take all pairs of $2k+1$ subsets of the two partitions and test if the graph induced by these sets of vertices is a cycle. Calculating an induced subgraph of a graph and deciding whether it is a cycle can be done in polynomial time, for example, by looking at the degrees of the vertices, which must be equal to two, and then check if the subgraph is connected, which takes only linear time. However, there are too many subsets to test. If $\mathcal{H} = (V, \mathcal{E})$ and $n = \lfloor (\min(|V|, |\mathcal{E}|) - 1)/2 \rfloor$, then there are

$$\sum_{k=1}^{n} \binom{|V|}{2k+1} \cdot \binom{|\mathcal{E}|}{2k+1}$$

possibilities.

To obtain a polynomial running time more sophisticated tools are needed. Conforti, Cornuéjols and Rao gave the first polynomial-time algorithm using decomposition results. Later Zambelli ([Zam05]) found an easier algorithm with running time $\mathcal{O}((|V| + |\mathcal{E}|)^9)$.

Although deciding whether a graph is balanced or not lies in $\mathcal{P}$, there is no algorithm known that can be practically used for large instances.

4.3.2. Hall Condition

Conforti, Di Summa and Zambelli gave in [CSZ07] an algorithm that can be used to find a perfect matching or a pair R, B violating the Hall condition in a balanced hypergraph. For this we need the following notions:

Let $A \in \mathbb{R}^{m \times n}$ be a matrix. We denote by A^i the $(m-1) \times n$ matrix obtained from A by removing the i-th row of A. Further, we call the system $Ax = 1, x \geq 0$ minimally infeasible if it has no solution but $A^i x = 1, x \geq 0$ is feasible for every $1 \leq i \leq m$.

Theorem 4.3.1. *There is a polynomial time algorithm which gives for every hypergraph $\mathcal{H} = (V, \mathcal{E})$ one of the following outputs:*

- *$\mathcal{H}$ is not balanced.*
- *A pair $R, B \subseteq V$ violating the Hall condition.*
- *A perfect matching.*

Proof. Let $A \in \mathbb{R}^{V \times \mathcal{E}}$ be the incidence matrix of $\mathcal{H}$.

First, calculate a vertex x^* of $LP_{PM}(\mathcal{H})$. If x^* exists, then it is either fractional, implying that $\mathcal{H}$ is not balanced, or it is integral, giving the perfect matching $M := \{e | x^*_e = 1\}$ of $\mathcal{H}$.

Otherwise, we do the following:

If $\mathcal{A}^v = 1, x \geq 0$ is infeasible, remove row v (so we delete vertex v from $\mathcal{H}$ but we keep hyperedges which become empty). Iterate until we get a hypergraph $\tilde{\mathcal{H}}$ with incidence matrix $\tilde{A}$ for which the system $\tilde{A}x = 1, x \geq 0$ is minimally infeasible.

For every row i of $\tilde{A}$ we calculate a vertex x^i of $\tilde{A}^i x = 1, x \geq 0$. If some x^i is fractional then $\mathcal{H}$ is not balanced. If all x^i are integral, we set $R := \{i | A_{i.} x^i = 0\}$ and

$B := \{i | A_{i.}x^i \geq 2\}$. Conforti et al. show in [CSZ07] that in a balanced hypergraph, such that the system defining LP_{PM} is minimally infeasible, taking R and B as above gives a pair violating the Hall condition. So, we test if R and B indeed violate the Hall Condition. If not, $\tilde{\mathcal{H}}$ and thus $\mathcal{H}$ are not balanced.

The algorithm runs in polynomial time, since calculating a vertex of a polytope can be done in polynomial time and the number of removed rows is bounded by $|V|$. $\square$

Remark. The algorithm above cannot be used to decide whether a hypergraph is balanced. If $\mathcal{H}$ is not balanced and has a perfect matching the algorithm can give the first or the last output. Furthermore, for a hypergraph without perfect matching there can be a pair violating the Hall condition, even if the hypergraph is not balanced.

5. Partitioned Hypergraphs

In this chapter partitioned hypergraphs are investigated by looking at the structure of their matching and perfect matching polytope. The first section gives theoretical results on the dimension, valid inequalities, facets and the integrality gap of both polytopes, whereas the second section summarizes computational results for partitioned hypergraphs with maximum part size two having few vertices.

First, we define an infinite sequence $\mathcal{D}_n$ of bipartite hypergraphs where n is an even natural number:

Definition. $\mathcal{D}_n$ has disjoint vertex sets $V_n = \{v_1, \dots, v_n\}$ and $W_n = \{w_1, \dots w_n\}$. Each of the two vertex sets is partitioned into $\frac{n}{2}$ parts of size two, say $V_{2n}^i = \{v_{2i-1}, v_{2i}\}$ and $W_{2n}^i = \{w_{2i-1}, w_{2i}\}$ $(1 \leq i \leq \frac{n}{2})$. The set of hyperedges $\mathcal{A}_n$ of $\mathcal{D}_n$ consists of n^2 edges $\{v_i, w_j\}$ for $1 \leq i, j \leq n$ and $(\frac{n}{2})^2$ proper hyperedges of the form $V_n^i \cup W_n^j$ for all $1 \leq i, j \leq \frac{n}{2}$.

Remark. The graph induced by the set of edges is the complete bipartite graph $K_{n,n}$. If we are just looking at the hyperedges and shrink each part of V and W to one vertex, then the hyperedges become edges between the parts and the resulting graph is $K_{\frac{n}{2},\frac{n}{2}}$. In a complete bipartite graph $K_{k,k}$ the edge set can be partitioned into k perfect matchings. Thus the hyperedges of $\mathcal{D}_n$ can be partitioned into $n + \frac{n}{2}$ perfect matchings. This shows that for every hyperedge there is a hyperassignment containing it and one missing it.

We draw a partitioned hypergraph of maximum part size two in the following way:

- The vertices and edges are drawn as usual, i.e., a vertex is represented as a disc and an edge as a line joining its endvertices.

- A part is represented by an ellipse around the corresponding two vertices.

- A proper hyperedge is represented by a line joining the two parts that the hyperedge connects.

Figure 5.1 shows $\mathcal{D}_6$ as an example; the complete bipartite hypergraph with 12 vertices partitioned into 6 parts of size two.

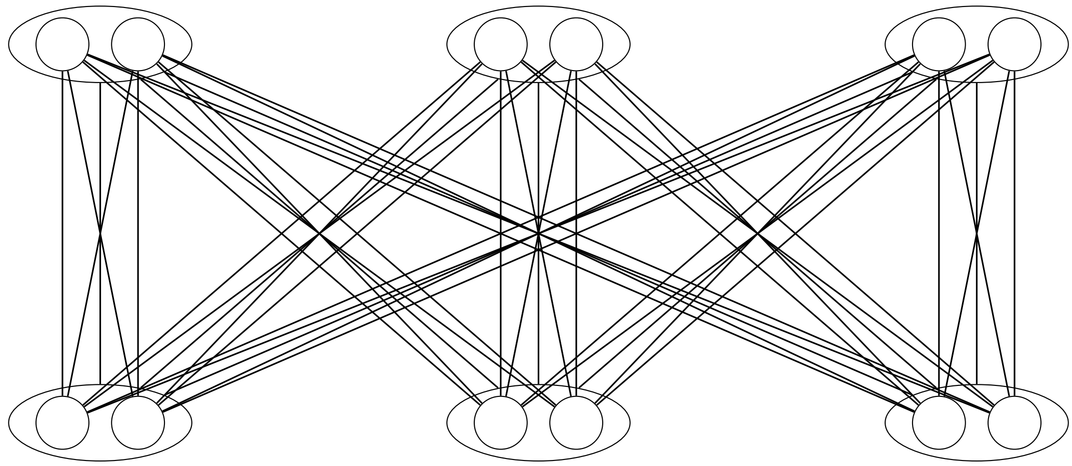


Figure 5.1.: The hypergraph $\mathcal{D}_6$

5.1. Polyhedral Results

In this section we will investigate the matching polytope, the perfect matching polytope, and their fractional variants (as defined in chapter 3). We focus on partitioned hypergraphs of part size two. These are partial subhypergraphs of some $\mathcal{D}_n$.

Although it is difficult to calculate the dimension of the perfect matching polytope for general hypergraphs, we give some results for partial hypergraphs of $\mathcal{D}_n$, exploiting known results on the dimension of the perfect matching polytope of bipartite graphs.

After that, we look at the extended formulation of (HAP) in partitioned hypergraphs found by Borndörfer and Heismann in [BH12]. We show that the extended formulation implies all clique inequalities.

Then we summarize different classes of inequalities that are known for the matching and the perfect matching polytope. In the first case, some facets can be characterized. In the second case, we give some necessary conditions for some classes of inequalities to define facets.

At the end of this section, we show that the integrality gap of the natural LP-relaxation of the Hyperassignment Problem is unbounded, whereas the integrality gap of the LP-relaxation of the maximum weight matching problem is equal to three for partitioned hypergraphs of part size two.

5.1.1. Dimension

First of all, we need to know the dimension of a polytope, for example, to prove that certain inequalities define facets. This is simple for the matching polytope but not for the perfect matching polytope.

Theorem 5.1.1. *$IP_M(\mathcal{D}_n)$ has full dimension.*

Proof. $\{\chi^{\emptyset}\} \cup \{\chi^a | a \in \mathcal{A}_n\}$ is a set of $|\mathcal{A}_n| + 1$ affinely independent vectors, so $IP_M(\mathcal{D}_n) \subseteq \mathbb{R}^{\mathcal{A}_n}$ has full dimension. □

Remark. The same argument shows that $IP_M(\mathcal{H})$ is full dimensional for every hypergraph $\mathcal{H}$.

The dimension of the perfect matching polytope is more difficult to calculate, as it is $\mathcal{NP}$-hard to decide whether a hypergraph has a perfect matching. However, the dimension of the fractional perfect matching polytope can be calculated much easier as we have a full description in terms of inequalities and equations.

Theorem 5.1.2. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a bipartite, graph based hypergraph such that every hyperedge $a \in \mathcal{A}$ is contained in a perfect matching. Let G be the underlying graph, i.e. $G = (V \cup W, E)$ with edge set $E = \{a \in \mathcal{A} | \, |a| = 2\}$. Denote by k the number of connected components of G. Then*

$$\dim(LP_{PM}(\mathcal{D})) = |\mathcal{A}| - 2|V| + k.$$

Proof. As no inequality of the form $x_a \geq 0$ is implicit, the dimension of $LP_{PM}(\mathcal{D})$ is equal to $|\mathcal{A}| - \text{rank}(A)$ where A is the incidence matrix of $\mathcal{D}$. Furthermore, the rank of A is equal to the maximum number of linear independent columns of A. Every column of A that corresponds to a hyperedge is the sum of the columns corresponding to a set of underlying disjoint edges because $\mathcal{D}$ is graph based. Thus, A has the same rank as the incidence matrix of G.

By Theorem 18.6 in [Sch03], the dimension of the perfect matching polytope of G is equal to $|E| - (|V| + |W|) + k$. So the incidence matrix of G has rank $|V| + |W| - k = 2|V| - k$. This shows that

$$\dim(LP_{PM}(\mathcal{D})) = |\mathcal{A}| - \text{rank}(A) = |\mathcal{A}| - 2|V| + k.$$

□

Corollary 5.1.3. *The dimension of $LP_{PM}(\mathcal{D}_n)$ is $\frac{5}{4}n^2 - 2n + 1$ for any even $n \in \mathbb{N}$.*

We know that $\mathcal{D}_n$ has a perfect matching and that every hyperedge is contained in one. This can be used to extend the proof of Theorem 18.6 in [Sch03] about the dimension of the perfect matching polytope of a bipartite graph to the dimension of $IP_{PM}(\mathcal{D}_n)$.

Theorem 5.1.4. *The dimension of $IP_{PM}(\mathcal{D}_n)$ is $\frac{5}{4}n^2 - 2n + 1$ for any even $n \in \mathbb{N}$.*

Proof. The perfect matching polytope of $\mathcal{D}_n$ lies in the fractional perfect matching polytope of $\mathcal{D}_n$, therefore its dimension is at most $\dim LP_{PM}(\mathcal{D}_n) = \frac{5}{4}n^2 - 2n + 1$.

For the reverse inequality, choose a vector x in the relative interior of $IP_{PM}(\mathcal{D}_n)$. Then $0 < x_a < 1$ for all $a \in \mathcal{A}_n$ because each hyperedge is contained in one but not in every perfect matching. Let T be a set of edges of $\mathcal{D}_n$ corresponding to a spanning tree of $K_{n,n}$. We show that by adjusting the x-values of T we can obtain a vector x' in the perfect matching polytope with $x'_{a^*} = x_{a^*} \pm \epsilon$ for a fixed $a^* \in \mathcal{A}_n \setminus T$ and $x'_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup a^*)$:

First, let a^* be an edge, i.e., $|a^*| = 2$, then the partial hypergraph induced by T together with a^* contains exactly one cycle C. Choose some $\epsilon > 0$ such that ϵ is smaller than $\min_{a \in C} x_a$ and $\min_{a \in C}(1 - x_a)$. C has even length because $\mathcal{D}[T \cup \{a^*\}]$ is bipartite as a graph. Adding and subtracting alternately ϵ to x_a for all edges $a \in C$ gives a new vector $x' \in IP_{PM}(\mathcal{D}_n)$ as desired. In this way the value of x_{a^*} can by changed by a small $\epsilon > 0$.

Next, let $a^* \in \mathcal{A}_n \setminus T$ be a proper hyperedge. Then a^* can be written as the disjoint union of two edges a^1 and a^2. There are three cases:

1. $a^1, a^2 \in T$:

   As above, there is some $\tilde{x} \in IP_{PM}(\mathcal{D}_n)$ such that $\tilde{x}_{a^1} = x_{a_1} \pm \epsilon$ for an $\epsilon > 0$ with 2ϵ smaller than $\min_{a \in T \cup \{a^1, a^2\}} x_a$ and $\min_{a \in T \cup \{a^1, a^2\}}(1 - x_a)$ and $\tilde{x}_a = x_a$ for all $a \in \mathcal{A} \setminus (T \cup \{a^1\})$.

   Repeating the argument for $\tilde{x}$ and a^2 gives a vector $x'' \in IP_{PM}(\mathcal{D}_n)$ with $x''_{a^2} = \tilde{x}_{a^2} \pm \epsilon = x_{a^2} \pm \epsilon$ and $x''_a = \tilde{x}_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^2\})$. Observe that $x''_a = \tilde{x}_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^1, a^2\})$ and $x''_a = x_a \pm \epsilon$ for $a = a^1$ or $a = a^2$. Thus, x' defined by

$$x'_a = \begin{cases} x''_a, & a \in T \\ x_a \pm \epsilon, & a = a^* \\ x_a, & \text{else} \end{cases}$$

   is an element of the perfect matching polytope with $x'_{a^*} = x_a \pm \epsilon$ and $x'_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^*\})$.

2. $a^1, a^2 \in T$:

   Let $\epsilon > 0$ be smaller than $\min\{x_{a^*}, x_{a^1}, x_{a^2}, 1 - x_{a^*}, 1 - x_{a^1}, 1 - x_{a^2}\}$, then $x' \in \mathbb{R}^{\mathcal{A}_n}$ defined by:

$$x'_a := \begin{cases} x_a \pm \epsilon, & \text{for } a = a^* \\ x_a \mp \epsilon, & \text{for } a = a^1, a^2 \\ x_a, & \text{else} \end{cases}$$

   lies in the perfect matching polytope of $\mathcal{D}_n$ and fulfils $x'_{a^*} = x_a \pm \epsilon$ and $x'_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^*\})$.

3. $a^1 \in T$ and $a^2 \notin T$:

If a^1 lies not in the circle C defined $T \cup \{a^2\}$, then a vector x' as above can be constructed by adjusting the values of C and x_{a^1} separately by some $\pm\epsilon$.

Otherwise, alternately assign $\pm\epsilon$ to the hyperedges of C with $\epsilon > 0$ small enough.

If both a^1 and a^2 received the same value, say both got $+\epsilon$, then x' defined by adjusting the x-values of $C \setminus \{a^1, a^2\}$ by the assigned $\pm\epsilon$ and x_{a^*} by $+\epsilon$ gives a vector $x' \in IP_{PM}(\mathcal{D}_n)$ with $x'_{a^*} = x_{a*} + \epsilon$ and $x'_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^*\})$. In the symmetric case that a^1 and a^2 received value $-\epsilon$, we get an $x' \in IP_{PM}(\mathcal{D}_n)$ with $x'_{a^*} = x_{a*} - \epsilon$ and $x'_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^*\})$.

Now, suppose we assigned $+\epsilon$ to a^1 and $-\epsilon$ to a^2. We alternately adjust the values x_a for all $a \in C$ by the given value $\pm\epsilon$, except for $a = a^1$ and $a = a^2$. For $a = a^1$ we add 2ϵ and for $a = a^2$ we do not change x_a. Furthermore, we subtract ϵ from x_{a^*}. This gives a vector $x' \in \mathbb{R}^{\mathcal{A}_n}$ with $x'_{a^*} = x_{a*} - \epsilon$ and $x'_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^*\})$. It is clear that $x'(\delta(v)) = x(\delta(v)) = 1$ for all $v \in V \cup W$ with $v \notin a^1 \cup a^2$. For $v \in a^1$ we get

$$x'(\delta(v)) = x(\delta(v)) + 2\epsilon - \epsilon - \epsilon = 1$$

and for $v \in a^2$ we get

$$x'(\delta(v) = x(\delta(v)) + \epsilon - \epsilon = 1.$$

So x' lies in the perfect matching polytope.

In the same way we can construct a vector $x' \in IP_{PM}(\mathcal{D}_n)$ if we have assigned $-\epsilon$ to a^1 and $+\epsilon$ to a^2. Then $x'_{a^*} = x_{a*} + \epsilon$ and $x'_a = x_a$ for all $a \in \mathcal{A}_n \setminus (T \cup \{a^*\})$ holds.

Geometrically, this means that it is possible to make a little step from an interior point in the direction given by some $e \in \mathcal{A}_n \setminus T$ without leaving the polytope, so $\dim(IP_{PM}(\mathcal{D}_n))$ must be at least $|\mathcal{A}_n \setminus T| = \frac{5}{4}n^2 - 2n + 1$. $\square$

For a subhypergraph of $\mathcal{D}_n$ Theorem 18.6 in [Sch03] yields the following:

Theorem 5.1.5. *Let $\mathcal{D}$ be a graph based partial hypergraph of $\mathcal{D}_n$ such that every $a \in \mathcal{A}$ is contained in a perfect matching. Furthermore, assume that the subgraph of $\mathcal{D}$ induced by the set of edges in $\mathcal{D}$ is connected and that the graph obtained by shrinking all parts and connecting to parts by an edge if there is a hyperedge between the two parts in $\mathcal{D}$. Then*

$$\dim(IP_{PM}(\mathcal{D})) = |\mathcal{A}| - 2n + 1$$

holds.

Proof. The dimension of $LP_{PM}(\mathcal{D})$ is equal to $|\mathcal{A}|-2n+1$. As $IP_{PM}(\mathcal{D})$ is contained in $LP_{PM}(\mathcal{D})$, we get $\dim(IP_{PM}(\mathcal{D})) \leq |\mathcal{A}| - 2n + 1$.

For the other direction, observe that in the proof of the "$\geq$" part in Theorem 5.1.4 we only used that each $a \in \mathcal{A}$ is contained in a perfect matching but also missed by a perfect matching and that each proper hyperedge can be written as the disjoint union of two edges. By assumption each hyperedge is contained in a perfect matching and $\mathcal{D}$ is graph based which implies the last property.

Suppose there is a hyperedge $a^* \in \mathcal{A}$ contained in every matching, then a hyperedge adjacent to a^* cannot lie in a perfect matching. This means that a^* is isolated. But then one of the graphs assumed to be connected in the assertion would not be connected. Thus every hyperedge is contained in a perfect matching and missed by one. □

5.1.2. An Extended Formulation

Borndörfer and Heismann give an extended formulation for (HAP) in partitioned hypergraphs using so called "configurations"([Hei10], [BH12]):

Definition. Let $\mathcal{D} = (V, W, \mathcal{A})$ be a partitioned hypergraph with parts $\{V_1, \dots, V_k\}$ and $\{W_1, \dots, W_l\}$. A subset $C \subseteq \mathcal{A}$ is a *configuration* of a part $P \in \{V_1, \dots, V_k\}$ (or $P \in \{W_1, \dots, W_l\}$) if $a_1 \cap a_2 = \emptyset$ for two different hyperedges $a_1, a_2 \in C$ and $\bigcup_{a \in C} a \cap V = P$ (or $\bigcup_{a \in C} a \cap W = P$). Let $\mathcal{C}_P$ denote the set of configurations of part P and $\mathcal{C}_V := \bigcup_{i=1}^{k} \mathcal{C}_{V_i}$ and $\mathcal{C}_W := \bigcup_{i=1}^{l} \mathcal{C}_{W_i}$ the set of configurations of V and W, respectively. Furthermore, $\mathcal{C} := \mathcal{C}^V \cup \mathcal{C}^W$ is defined to be the set of all configurations.

Now we can state the Configuration ILP for a partitioned hypergraph $\mathcal{D} = (V, W, \mathcal{A})$ with costs $c : \mathcal{A} \to \mathbb{R}$:

$$\min_{x \in \mathbb{R}^{\mathcal{A}}} c^T x$$

$$\text{subject to} \sum_{a \in \delta(v)} x_a = 1 \ \forall v \in V \cup W \tag{5.1}$$

$$\sum_{C \in \mathcal{C}_V : a \in C} y_C = x_a \ \forall a \in \mathcal{A} \tag{5.2}$$

$$\sum_{C \in \mathcal{C}_W : a \in C} y_C = x_a \ \forall a \in \mathcal{A} \tag{5.3}$$

$$x, y \geq 0 \tag{5.4}$$

$$x \in \mathbb{Z}^{\mathcal{A}} \tag{5.5}$$

$$y \in \mathbb{Z}^{\mathcal{C}_V} \times \mathbb{Z}^{\mathcal{C}_W} \tag{5.6}$$

Denote by $LP_{\mathcal{C}}(\mathcal{D}) := \{(x, y) \in \mathbb{R}^{\mathcal{A}} \times (\mathbb{R}^{\mathcal{C}_V} \times \mathbb{R}^{\mathcal{C}_W}) |\ (x, y) \text{ fulfils (5.1)-(5.4)}\}$ the polytope corresponding to the LP-relaxation of the Configuration ILP.

Projecting a solution (x, y) of the Configuration ILP to $x \in \mathbb{R}^{\mathcal{A}}$ yields a solution of (HAP) and each solution of (HAP) can be extended to a solution of the Configuration ILP (see [BH12] for a proof). So this ILP is a correct extended formulation of the hyperassignment problem. Moreover, the projection of $LP_{\mathcal{C}}$ to $\mathbb{R}^{\mathcal{A}}$ is stronger than LP_{PM} because it implies also the clique inequalities:

Theorem 5.1.6. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a partitioned hypergraph and $(x, y) \in LP_{\mathcal{C}}(\mathcal{D})$. Then $x(Q) \leq 1$ holds for all cliques $Q \subseteq \mathcal{A}$ of $L(\mathcal{D})$.*

Proof. The proof is due to [BH12].

First, we show that for every clique Q there exists a part P with $Q \subseteq \delta(P)$. If Q is empty, there is nothing to show. So let $a_1 \in Q$ with $a_1 \cap V \subseteq V_i$ and $a_1 \cap W \subseteq W_j$. Every hyperedge intersecting a_1 must intersect at least one of the two parts V_i and W_j. If there is an edge $a_2 \in Q$ with $a_2 \cap W \not\subseteq W_j$, then any other edge $a \in Q$ must intersect V_i, otherwise $a \cap a_2 = \emptyset$ or $a \cap a_1 = \emptyset$. The same argument holds for the case $a_2 \cap V \not\subseteq V_i$.

W.l.o.g. $Q \subseteq \delta(P)$ for a part $P \subseteq V$. For very vertex $v \in P$ the following holds:

$$
\begin{aligned}
1 &= \sum_{a \in \delta(v)} x_a \\
&= \sum_{a \in \delta(v)} \sum_{C \in \mathcal{C}_V : a \in C} y_C \\
&= \sum_{C \in \mathcal{C}_V : a \in C} |\delta(v) \cap C| y_C \\
&= \sum_{C \in \mathcal{C}_P} y_C \text{ , as } |\delta(v) \cap C| = 1 \text{ for all } C \in \mathcal{C}_P \text{, and 0 else,} \\
&= \sum_{a \in Q} \sum_{C \in \mathcal{C}_P : a \in C} y_C + \sum_{C \in \mathcal{C}_P : C \cap Q = \emptyset} y_C \ (\star) \\
&\geq \sum_{a \in Q} \sum_{C \in \mathcal{C}_P : a \in C} y_C \\
&= \sum_{a \in Q} x_a = x(Q).
\end{aligned}
$$

$(\star)$ holds because every configuration $C \in \mathcal{C}_P$ contains at most one hyperedge a with $a \in Q$. $\square$

The last theorem shows that the projection $LP_C(\mathcal{D})$ to $\mathbb{R}^{\mathcal{A}}$ yields a better relaxation of $IP_{PM}(\mathcal{D})$ than $LP_{PM}(\mathcal{D})$, but the number of variables has increased.

For a partitioned hypergraph $\mathcal{D} = (V, W, \mathcal{A})$ of maximum part size two the number of configurations is bounded by $\mathcal{O}(|V||\mathcal{A}|)$. To see this, we determine the maximum number of configurations containing a fixed hyperedge $a \in \mathcal{A}$. If a is a hyperedge, then $a \cap V = V_i$ for some part V_i of V, so $\{a\}$ is already a configuration and there are no other configurations $C \in \mathcal{C}_V$ containing a. Similarly, $\{a\}$ is the only configuration of $\mathcal{C}_W$ containing a. If a is an edge and $a \cap V \subseteq V_i$ is a part of size two, then all configurations $C \in \mathcal{C}_V$ containing a are of the form $\{a, a'\}$ where a' is an edge connecting the unique vertex of $V_i \setminus a$ with a vertex of $W \setminus a$. This implies that at most $|W| - 1 = |V| - 1$ configurations $C \in \mathcal{C}_V$ contain a. Using the same argument shows that a lies in at most $|V| - 1$ configurations $C \in C_W$. Together we get that there are less than $2|V||\mathcal{A}|$ configurations.

Although for fixed part size d the number of configurations is bounded by a polynomial in the number of hyperedges and vertices, the degree of this polynomial grows fast for increasing d. Therefore, Heismann gives in [Hei10] an efficient algorithm to optimize over the LP-relaxation of the configuration formulation without enumerating all configurations.

5.1.3. Valid Inequalities and Facets

We start looking at the matching polytope, which is easier to handle. The theorems are formulated in terms of $\mathcal{D}_n$ but the proofs hold for other hypergraphs, too.

Theorem 5.1.7. *Every trivial inequality $x_a \geq 0$ defines a facet of $IP_M(\mathcal{D}_n)$.*

Proof. Fix a hyperedge $a^* \in \mathcal{A}_n$. The $|\mathcal{A}_n| - 1$ linearly independent vectors χ^a with $a \in \mathcal{A}_n \setminus \{a^*\}$ are elements of the matching polytope and satisfy $x_{a^*} = 0$.

So $\{x \in IP_M(\mathcal{D}_n) | x_{a^*} = 0\}$ is a facet. □

Theorem 5.1.8. *A clique inequality $x(Q) \leq 1$ defines a facet of $IP_M(\mathcal{D}_n)$ if and only if Q is a maximal clique.*

Proof. If Q' is a clique containing Q as a proper subset then $x(Q') \leq 1$ implies $x(Q) \leq 1$.

Now let Q be a maximal clique in $\mathcal{D}_n$. So Q is a maximal clique in $L(\mathcal{D}_n)$ and thus $x(Q) \leq 1$ defines a facet of the stable set polytope of $L(D)$ (see [GLS93]). The dimension $\mathrm{STAB}(L(D))$ is $|\mathcal{A}_n|$, so there are $|\mathcal{A}_n| - 1$ linearly independent vectors in $\mathrm{STAB}(L(D))$ satisfying $x(Q) = 1$. These vectors lie also in the matching polytope of $\mathcal{D}_n$, so $x(Q) \leq 1$ defines a facet for $IP_M(\mathcal{D}_n)$. □

Another interesting class of inequalities are odd-cycle inequalities:

$$x(C) \leq \frac{|C| - 1}{2}$$

for an odd cycle C. It is not easy to say when an odd-cycle inequality is facet-defining. If C has length three, it is just a clique and therefore Theorem 5.1.8 can be used. For $|C| \geq 5$ it is clear that C should at least be a strong cycle:

Theorem 5.1.9. *If an odd cycle inequality is not strong, then it is redundant with respect to $Ax \leq 1, x \geq 0$ and all odd-cycle inequalities.*

Proof. Suppose the odd cycle $C = (v_0, e_1, v_1, \ldots, e_l, v_l = v_0)$ is not strong. Then there is a hyperedge e_i containing three vertices v_{i-1}, v_i, v_j. We can assume $j > i$, so $C' := (v_0, e_1, \ldots, v_{i-1}, e_i, v_j, e_{j+1}, \ldots, e_l, v_l)$ and $C'' := (v_i, e_{i+1} \ldots, v_j, e_i, v_i)$ are two cycles with $|C'| + |C''| = |C| + 1$ as hyperedge e_i is used twice. This implies that either both cycles are even or both are odd.

If both cycles are odd, then the inequality corresponding to C is redundant since it is implied by the cycle-inequalities of C' and C'':

$$x(C) \leq x(C') + x(C'') \leq \frac{|C'| + |C''| - 2}{2} = \frac{|C| - 1}{2}.$$

The case that both cycles are even remains to be shown. Without loss of generality we may assume $j = l-1$. Then it holds that $C' := (v_0, e_1, \ldots, v_{i-1}, e_i, v_{l-1}, e_l, v_l = v_0)$ and $C'' := (v_i, e_{i+1} \ldots, e_{l-1}, v_{l-1}, e_i, v_i)$. C' has even length, so i must be odd. Now,

$$x_{e_{2s-1}} + x_{e_{2s}} \leq 1$$

for all $s = 1, \ldots, \frac{i-1}{2}$ implies that

$$\sum_{k=1}^{i-1} x_{e_k} \leq \frac{|C'| - 2}{2}.$$

Similarly,

$$\sum_{k=i+1}^{l-2} x_{e_k} \leq \frac{|C''| - 2}{2}$$

holds. Together with $x_{e_i} + x_{e_{l-1}} + x_{e_l} \leq 1$ this yields

$$x(C) = \sum_{k=1}^{l} x_{e_k} \leq \frac{|C'| - 2}{2} + \frac{|C''| - 2}{2} + 1 = \frac{|C'| + |C''| - 2}{2} = \frac{|C| - 1}{2},$$

which shows that the cycle inequality corresponding to C is redundant. □

Now we turn to the perfect matching polytope.

Theorem 5.1.10. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a partial subhypergraph of $\mathcal{D}_n$. The trivial inequality $x_a \geq 0$ defines a facet for $LP_{PM}(\mathcal{D})$ if and only if there is no constant $k \in [0, 1]$ such that $x_a = k$ is a valid equation for the perfect matching polytope.*

Proof. If $x_a = k$ for all $x \in LP_{PM}(\mathcal{D})$, then $x_a \geq 0$ does not define a facet.

Now let $a \in \mathcal{A}$ such that $x_a = k$ is not valid for the perfect matching polytope. W.l.o.g. we can assume that there is no implicit inequality of the form $x_{a'} \geq 0$ for $a' \in \mathcal{A} \setminus \{a\}$. We look at the subgraph $\mathcal{D}'$ induced by $\mathcal{A} \setminus \{a\}$. The dimension of $LP_{PM}(\mathcal{D})$ and the dimension of $LP_{PM}(\mathcal{D}')$ can be calculated using the incidence matrices A of $\mathcal{D}$ and A' of $\mathcal{D}'$:

$$\dim(LP_{PM}(\mathcal{D})) = |\mathcal{A}| - \operatorname{rank}(A)$$
$$\dim(LP_{PM}(\mathcal{D}')) = |\mathcal{A} \setminus \{a\}| - \operatorname{rank}(A').$$

As there is no valid equation for $LP_{PM}(\mathcal{D})$ of the form $x_a = k$ the rank of A' equals the rank of A. This shows that:

$$\dim(LP_{PM}(\mathcal{D}')) = \dim(LP_{PM}(\mathcal{D})) - 1.$$

So there are exactly $\dim(LP_{PM}(\mathcal{D}'))+1$ affinely independent vectors $z^i \in LP_{PM}(\mathcal{D}')$. Setting

$$z^i_{a'} := \begin{cases} z^i_{a'} & a' \neq a \\ 0 & a' = a \end{cases}$$

gives $\dim(LP_{PM}(\mathcal{D}))$ affinely independent vectors of $LP_{PM}(\mathcal{D})$ satisfying $z^i_a = 0$. As the face $F := \{x \in LP_{PM} | x_a = 0\}$ is not equal to $LP_{PM}(\mathcal{D})$, it must have dimension $\dim(LP_{PM}(\mathcal{D})) - 1$. Hence, F is a facet. □

Theorem 5.1.11. *The trivial inequality $x_a \geq 0$ with $a \in \mathcal{A}_n$ defines a facet for $IP_{PM}(\mathcal{D}_n)$.*

Proof. By the results of the first section in this chapter, the dimension of the polytope $IP_{PM}(\mathcal{D}_n)$ is one bigger than the dimension of the polytope $IP_{PM}(\mathcal{D}_n[\mathcal{A} \setminus \{a\}])$ for every fixed hyperedge $a \in \mathcal{A}$. Thus, the same proof as in Theorem 5.1.4 shows that $x_a \geq 0$ defines a facet for the perfect matching polytope. □

Of course, the odd cycle-inequalities are valid for the perfect matching polytope and Theorem 5.1.9 holds, too.

In the case of graphs, the perfect matching polytope is defined by the trivial inequalities, the equations $x(\delta(v)) = 1$ and the odd set inequalities

$$x(E[U]) \leq \frac{|U|-1}{2}$$

for all odd subsets U of the vertex set. Heismann found a generalization of these inequalities to bipartite hypergraphs. Observe that

$$x(E[U]) = \sum_{e \in E} \left\lfloor \frac{|\{v \in U | e \in \delta(v)\}|}{2} \right\rfloor x_e,$$

as $E[U]$ is the set of edges having both nodes in U. In a bipartite hypergraph the set $\delta(v)$ forms a clique. Substituting $\{\delta(v) | v \in U\}$ by any odd collection of cliques $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_{2k+1}\}$ results in the following inequality

$$\sum_{a \in \mathcal{A}} \left\lfloor \frac{|\{Q \in \mathcal{Q} | a \in Q\}|}{2} \right\rfloor x_a \leq k, \tag{5.7}$$

which is called an odd clique set inequality.

Theorem 5.1.12. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a bipartite hypergraph and $\mathcal{Q}$ be a collection of cliques of size $2k+1$ ($k \in \mathbb{N}$). The inequality (5.7) is valid for $IP_{PM}(\mathcal{D})$.*

Proof. It is enough to show that (5.7) is satisfied by all vectors χ^M, where M is a perfect matching. So let $x := \chi^M$ for a fixed perfect matching M, then

$$\sum_{a \in \mathcal{A}} \left\lfloor \frac{|\{Q \in \mathcal{Q} | a \in Q\}|}{2} \right\rfloor x_a = \sum_{a \in M} \left\lfloor \frac{|\{Q \in \mathcal{Q} | a \in Q\}|}{2} \right\rfloor .$$

For each $Q \in \mathcal{Q}$ there is at most one $a \in M$ with $a \in Q$, implying

$$\sum_{a \in M} |\{Q \in \mathcal{Q} | a \in Q\}| \leq 2k + 1.$$

If equality holds, at least one summand must be odd, so dividing by two and rounding down both sides yields

$$\sum_{a \in M} \left\lfloor \frac{|\{Q \in \mathcal{Q} | a \in Q\}|}{2} \right\rfloor \leq k,$$

which proves the claim. □

Theorem 5.1.13. *Every inequality of the form (5.7) is implied by one for which all $Q \in \mathcal{Q}$ are maximal cliques. Furthermore, we can assume $Q_i \neq Q_j$ for $i \neq j$.*

Proof. For every $Q \in \mathcal{Q}$ let Q' be a maximal clique containing Q. These cliques build a collection $\mathcal{Q}'$ of the same size as $\mathcal{Q}$. Furthermore, for every $a \in \mathcal{A}$

$$|\{Q \in \mathcal{Q} | a \in Q\}| \leq |\{Q \in \mathcal{Q}' | a \in Q\}|$$

holds. So

$$\sum_{a \in \mathcal{A}} \left\lfloor \frac{|\{Q \in \mathcal{Q} | a \in Q\}|}{2} \right\rfloor x_a \leq \sum_{a \in \mathcal{A}} \left\lfloor \frac{|\{Q \in \mathcal{Q}' | a \in Q\}|}{2} \right\rfloor x_a \leq k$$

follows. This implies the first claim.

Now suppose $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_{2k+1}\}$ and $Q_1 = Q_2$. Set $\mathcal{Q}'' = \{Q_3, \ldots, Q_{2k+1}\}$, this is an odd clique set of size $2(k-1)+1$. For a fixed $a \in \mathcal{A}$, $\lfloor |\{Q \in \mathcal{Q} | a \in Q\}/2| \rfloor$ is one bigger than $\lfloor |\{Q \in \mathcal{Q}'' | a \in Q\}|/2 \rfloor$ if $a \in Q_1$, and otherwise both numbers are equal. Adding

$$\sum_{a \in \mathcal{A}} \left\lfloor \frac{|\{Q \in \mathcal{Q}'' | a \in Q\}|}{2} \right\rfloor x_a \leq k - 1$$

and the clique inequality corresponding to $Q_1 = Q_2$, we get

$$\sum_{a \in \mathcal{A}} \left(\left\lfloor \frac{|\{Q \in \mathcal{Q}'' | a \in Q\}|}{2} \right\rfloor + |a \cap Q_1| \right) x_a \leq k,$$

which is equal to inequality (5.7) for $\mathcal{Q}$. □

Remark. A clique inequality $x(Q) \leq 1$ can be seen as an odd clique set inequality. For that, we take 3 copies of Q as $\mathcal{Q}$. We get:

$$1 \geq \sum_{a \in \mathcal{A}} \left\lfloor \frac{|\{Q \in \mathcal{Q} | a \in Q\}|}{2} \right\rfloor x_a = \sum_{a \in Q} x_a.$$

Only in this case it makes sense to take multiple sets into $\mathcal{Q}$, so we look at the clique inequalities separately.

Odd cycle inequalities are a special case of odd clique set inequalities:

If $\{a_1, a_2, \ldots, a_{2k+1}\}$ are the hyperedges of an odd cycle (in the traversed order), then $Q_1 := \{a_1, a_2\}, Q_2 := \{a_2, a_3\}, \ldots, Q_{2k+1} := \{a_{2k+1}, a_1\}$ are cliques and setting $\mathcal{Q} := \{Q_1, \ldots, Q_{2k+1}\}$ yields:

$$k \geq \sum_{a \in \mathcal{A}} \left\lfloor \frac{|\{Q \in \mathcal{Q} | a \in Q\}|}{2} \right\rfloor x_a = \sum_{i=1}^{2k+1} x_{a_i}.$$

So, if we can decide for any odd clique set inequality whether it defines a facet or not, we can characterize when an odd cycle inequality is facet defining. One necessary condition is the following:

Corollary 5.1.14. *Let $C = (v_0, a_1, v_1, a_2, \ldots, v_{2k}, a_{2k+1}, v_0)$ be an odd cycle in a bipartite hypergraph $\mathcal{D} = (V, W, \mathcal{A})$. If there are two cliques Q, Q' with $(Q \cap Q') \setminus C \neq \emptyset$ such that $a_i, a_{i+1} \in Q$ and $a_j, a_{j+1} \in Q'$ for $j \neq i$, i.e., Q and Q' contain each a pair of consecutive hyperedges of C and these two pairs are different, then $x(C) \leq \frac{|C|-1}{2}$ is redundant with respect to $Ax = 1, x \geq 0$ and all odd clique set inequalities.*

Proof. Otherwise, let $a^* \in (Q \cap Q') \setminus C$. Define $\mathcal{Q} := \{Q_1, \ldots, Q_{2k+1}\}$ as above and set $\mathcal{Q}' := \{Q'_1, \ldots, Q'_{2k+1}\}$, where Q'_i is a maximal clique containing Q_i for each $i \in \{1, \ldots 2k+1\}$. It follows that

$$|\{Q' \in \mathcal{Q}' | a \in Q'\}| - |\{Q \in \mathcal{Q} | a \in Q\}| \geq \begin{cases} 2, & \text{if } a = a^* \\ 0, & \text{else.} \end{cases}$$

Thus, the odd clique set inequality of $\mathcal{Q}'$ implies the odd clique set inequality of Q, which is equal to the odd cycle inequality $x(C) \leq \frac{|C|-1}{2}$. $\square$

Remark. If $C = v_0, a_1, v_1, \ldots, v_{2k}, a_{2k+1}, v_{2k+1} = v_0$ is an odd cycle then for every $0 \leq i \leq 2k$ the set $\delta(v_i)$ is a clique containing a pair of adjacent hyperedges of C. So the last corollary implies that if the odd cycle inequality of C defines a facet, then C has no "chord", i.e., a hyperedge $a \notin \{a_1, \ldots, a_{2k+1}\}$ incident to at least two distinct vertices of the cycle.

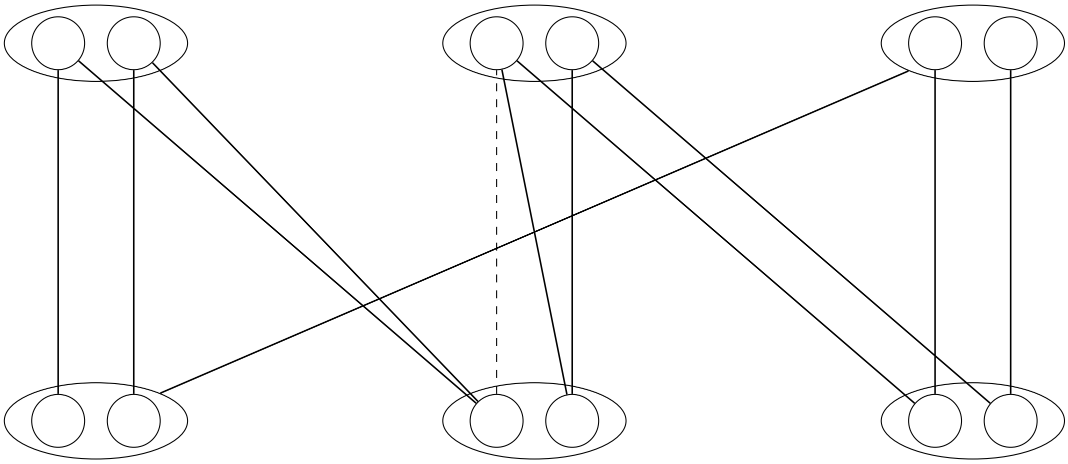


Figure 5.2.: Example with a large integrality gap

5.1.4. Integrality Gap

The integrality gap of LP_{PM} can be arbitrarily large, even if we add all clique inequalities, as the following example shows:

Example. Let $\mathcal{D}$ be the hypergraph drawn in Figure 5.2. $\mathcal{D}$ has one hyperedge which is contained in one strong cycle of length five and in another strong cycle of length seven and it has one extra edge e^* (drawn with a dashed line). e^* is contained in the unique perfect matching of $\mathcal{D}$.

Assigning costs $M > 0$ to e^* and 0 to all other hyperedges gives an integrality gap of M because the cost of the perfect matching is M but the cost of the fractional solution $x \in LP_{PM}(\mathcal{D})$ defined by $x_e = \frac{1}{2}$ for all hyperedges $e \neq e^*$ is zero. M can be chosen arbitrarily large, so the integrality gap of the LP relaxation of the hyperassignment problem is unbounded.

Observe that $\mathcal{D}$ has the Helly property, so it satisfies all clique inequalities.

In contrast to perfect matchings the integrality gap of

$$\begin{aligned} &\max b^t x \\ &\text{subject to } x \in LP_M \end{aligned} \tag{5.8}$$

is bounded for k-uniform hypergraphs. Füredi, Kahn and Seymour show in [FKS93] that it is at most $k - 1 + \frac{1}{k}$. For k-partite hypergraphs the result can be strengthen to $k - 1$. In general, bipartite hypergraphs are not k-partite or k-uniform, however, they can be transformed into those provided they are partitioned:

Theorem 5.1.15. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a partial hypergraph of $\mathcal{D}_n$ for some even $n \in \mathbb{N}$ with weight function $b : \mathcal{A} \to \mathbb{R}_+$.*

There exists a 4-partite hypergraph $\mathcal{D}' = (V' \cup W', \mathcal{A}')$ and a function $b' : \mathcal{A}' \to \mathbb{R}$ such that there is a bijection between the matchings in $\mathcal{D}$ and the matchings in $\mathcal{D}'$ preserving the weights.

Proof. For every edge $a \in \mathcal{A}$, $|a| = 2$, we add two new vertices v_a and w_a, further, we define for every such edge a hyperedge $a' := a \cup \{v_a, w_a\}$. We set $V' := V \bigcup_{a:|a|=2} v_a$, $W' := W \bigcup_{a:|a|=2} w_a$ and $\mathcal{A}' := \{a | a \in \mathcal{A}, |a| = 4\} \cup \{a' | a \in \mathcal{A}, |a| = 2\}$ with weights $b'(a) = b(a)$ and $b'(a') = b(a)$. Every matching M in $\mathcal{D}$ gives a corresponding matching $M' := \{a | a \in M, |a| = 4\} \cup \{a' | a \in M, |a| = 2\}$ in $\mathcal{D}'$ of the same weight and vice versa.

It remains to show that $\mathcal{D}'$ is 4-partite. For every part $P \subseteq V$ we number the two vertices with 1 and 2, for a part $P \subseteq W$ we number them with 3 and 4. Every hyperedge a of size four between two parts contains exactly one vertex of each number. A hyperedge of the form $a' = a \cup \{v_a, w_a\}$ contains two vertices with different numbers and two vertices, namely v_a and w_a, not numbered yet. We give v_a and w_a the two missing numbers, it is not important which vertex gets which number because v_a and w_a are only incident to a' in $\mathcal{D}'$. Now, setting $V_i := \{v \in V' \cup W' | v \text{ has number } i\}$ for $i = 1, 2, 3, 4$, yields a partition of $V' \cup W'$ such that $|a \cap V_i| = 1$ for all $a \in \mathcal{A}'$ and $1 \leq i \leq 4$. This shows that $\mathcal{D}'$ is 4-partite. □

The matching polytope of $\mathcal{D}$ and that of $\mathcal{D}'$ coincide as the inequalities $x(\delta(v)) \leq 1$ are the same for $v \in V \cup W$ and the additional inequalities $x(\delta(v)) = x_a \leq 1$ for $v = v_a$ or $v = w_a$ in $LP_P(\mathcal{D}')$ are redundant. This shows:

Theorem 5.1.16. *The LP (5.8) has integrality gap at most 3 for every partial hypergraph $\mathcal{D}$ of some $\mathcal{D}_n$.*

Proof. The program 5.8 has the same integrality gap for $\mathcal{D}$ as for $\mathcal{D}'$, which is $3 = 4-1$, as $\mathcal{D}'$ is 4-partite. □

Remark. The above ideas can be generalized for partitioned hypergraphs of arbitrary part sizes.

For that, let $\mathcal{D} = (V, W, \mathcal{A})$ be a partitioned hypergraph with parts $\{V_1, \dots, V_k\}$ and $\{W_1, \dots, W_l\}$ and maximum part size $M \in \mathbb{N}$. For every hyperedge $a \in \mathcal{A}$ with $|a \cap V| = |a \cap W| < M$ we add new vertices $v_a^1, \dots, v_a^{M-|a\cap V|}$ to V and $w_a^1, \dots, w^{M-|a\cap W|}$ to W. After that, we replace a by $a \cup \{v_a^1, \dots, v_a^{M-|a\cap V|}, w_a^1, \dots, w^{M-|a\cap W|}\}$.

We number the vertices of a part $V_i \in \{V_1, \dots, V_k\}$ using $|V_i|$ different elements of $\{1, \dots, M\}$ and the vertices of a part $W_j \in \{W_1, \dots, W_l\}$ using $|W_j|$ different elements of $\{M+1, \dots, 2M\}$. Finally, for every hyperedge $a \in \mathcal{A}$ with $|a \cap V| = |a \cap W| < M$ we give the vertices $v_a^1, \dots, v_a^{M-|a\cap V|}$ and $w_a^1, \dots, w^{M-|a\cap W|}$ the numbers for which the hyperedge a does not already contain a vertex of. This can be done arbitrarily. At the end we are left with a $2M$-partite hypergraph $\mathcal{D}'$ such that there is a bijection between the matchings in $\mathcal{D}$ and the matchings in $\mathcal{D}'$ as in Theorem 5.1.15.

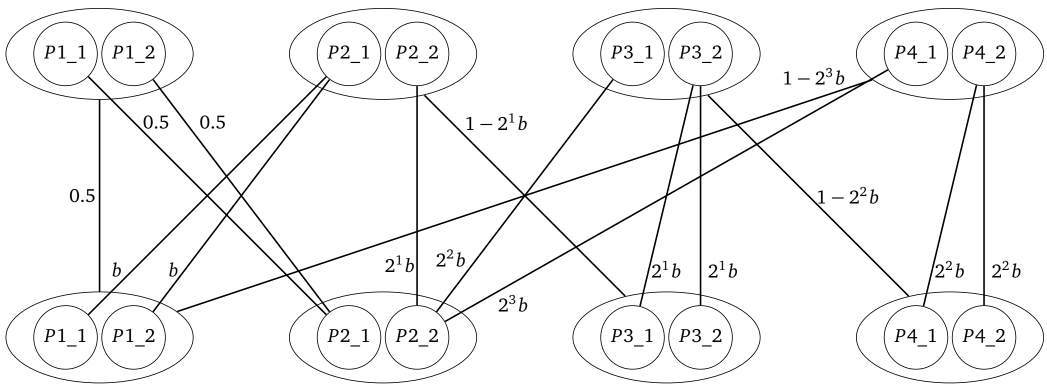


Figure 5.3.: Example

Similar to the case of partitioned hypergraphs with parts of size two, we see that the linear program (5.8) has integrality gap at most $2M - 1$ for any partitioned hypergraph with maximum part size M.

Another interesting fact is that there are vertices of the fractional perfect matching polytope such that the least common multiple of the denominators of all entries is arbitrarily large.

Example. First, look at Figure 5.3. It shows a subhypergraph $\mathcal{D}$ of $\mathcal{D}_8$ containing four strong odd cycles of length three.

In order to find a vector $x \in LP_{PM}(\mathcal{D})$ we first observe that an entry corresponding to one of the three hyperedges leaving the first upper part must be equal to $\frac{1}{2}$. Also the values of the two edges incident to the first lower part must be the same, say they are equal to some $b \in \mathbb{R}$ between zero and one. After that, the x-values of all other hyperedges can be calculated successively because they only depend on b. Now, $x(\delta(v)) = 1$ is satisfied for all vertices except of the lower P1_1, P1_2 and P2_2, for which the corresponding equation holds if and only if

$$\frac{1}{2} + b + 1 - 2^3 b = 1,$$
$$2^1 b + 2^2 b + 2^3 b = 1,$$

which is the case for $b = \frac{1}{2(2^3-1)} = \frac{1}{14}$.

Of course, this example can be generalized by adding k strong odd cycles of length three for an integer $k \geq 2$. Then b has to satisfy the following two equations:

$$\frac{1}{2} + b + 1 - 2^{k-1} b = 1,$$
$$2^1 b + 2^2 b + \ldots + 2^{k-1} b = 1.$$

The first equations has the solution $b = \frac{1}{2(2^{k-1}-1)} = \frac{1}{2^k-2}$ which is also a solution of the second equation as

$$\sum_{i=1}^{k-1} 2^i b = (\sum_{i=0}^{k-1} 2^i - 1)b = (2^k - 1 - 1)b = 1.$$

Because all values of x depend only on b and b is unique, the polytope $LP_{PM}(\mathcal{D})$ contains only one point, therefore x is a vertex of this polytope. For $k \to \infty$ the denominators of the entries of this vertex can become arbitrarily large.

5.2. Computational Results

In this section we give computational results for $\mathcal{D}_n$ for small n and some generalizations for arbitrary sizes.

Even for small n the number of facets and vertices of IP_{PM} and LP_{PM} can be very large. For example, $IP_{PM}(\mathcal{D}_6)$ has $14,049$ facets. Heismann calculated all of them and clustered them using “symmetry”. In this context symmetry is induced by all permutations of the vertices respecting the partition and bipartition, i.e., the n parts of each side V or W can be permuted, the two vertices of each part can be interchanged, and V and W can be swapped. This gives $n! \cdot n! \cdot 2^n \cdot 2^n \cdot 2 = n!^2 \cdot 2^{2n+1}$ bijections of the vertex set.

Each $\pi : V_n \cup W_n \to V_n \cup W_n$ gives a permutation of the set of hyperedges, namely $\pi' : \mathcal{A}_n \to \mathcal{A}_n$ with $\pi'(a) := \{\pi(v) | v \in a\}$. The set $\{\pi(v) | v \in a\}$ is again a hyperedge of $\mathcal{D}_n$, because π maps parts onto parts.

An inequality of the form

$$\sum_{a \in \mathcal{A}_n} d_a x_a \leq e,$$

where $e, d_a \in \mathbb{R}$ for all $a \in \mathcal{A}$, is symmetric to an inequality

$$\sum_{a \in \mathcal{A}_n} d'_a x_a \leq e',$$

where $e', d'_a \in \mathbb{R}$, if there is a permutation $\pi : \mathcal{A}_n \to \mathcal{A}_n$ such that $d'_a = d_{\pi(a)}$ and furthermore $e = e'$ (probably after adding some multiples of equations of the form $x(\delta(v)) = 1$ to the second inequality).

Similarly, two partial hypergraphs $\mathcal{D}^1 = (V_n, W_n, \mathcal{A}^1), \mathcal{D}^2 = (V_n, W_n, \mathcal{A}^2)$ of some $\mathcal{D}_n$ are symmetric if there is a permutation of $\mathcal{A}_n$ mapping $\mathcal{A}^1$ onto $\mathcal{A}^2$.

Exploiting symmetry, the $14,049$ facets of $IP_{PM}(\mathcal{D}_6)$ can be partitioned into 30 classes, where 16 come from odd clique set inequalities.

In contrast to the case of the matching polytope, we do not know when the LP-relaxation of the perfect matching polytope is integral. Of course, if LP_M is integral also LP_{PM} is integral, but not the other way around.

It is not possible to calculate all vertices of $LP_{PM}(\mathcal{D}_6)$ without considering the symmetry, there are too many of them. So it is not possible to use for example PORTA to get all vertices.

To get an idea of the reasons destroying integrality, we calculated (with symmetry) all minimal partial hypergraphs of $\mathcal{D}_6$ with fractional LP-relaxation of the perfect matching polytope, we call them "minimal fractional". Here, minimal means that no proper partial hypergraph has the same property and two symmetric hypergraphs are considered to be equal.

To explain the connection between vertices of $LP_{PM}(\mathcal{D}_6)$ and minimal fractional hypergraphs, the following lemma is needed:

Lemma 5.2.1. *Let $\mathcal{D} = (V, W, \mathcal{A})$ be a partial hypergraph of some $\mathcal{D}_n$ such that $LP_{PM}(\mathcal{D})$ is not integral but $LP_{PM}(\mathcal{D}')$ is integral (or empty) for every proper partial hypergraph $\mathcal{D}'$ of $\mathcal{D}$. Then $\mathcal{D}$ has at most $2n-1$ hyperedges and $LP_{PM}(\mathcal{D})$ contains a unique vertex.*

Proof. Let $x \in LP_{PM}(\mathcal{D})$ be a fractional vertex. If $x_{a^*} = 0$ for some $a^* \in \mathcal{A}$, then projecting x to $\mathbb{R}^{\mathcal{A}\setminus a^*}$ yields a fractional vertex of $LP_{PM}(\mathcal{D}[\mathcal{A} \setminus a^*])$, contradicting that $\mathcal{D}$ is minimal fractional.

By Corollary 2.2.5 and $x > 0$, the $|\mathcal{A}|$ column vectors of the incidence matrix A of $\mathcal{D}$ must be linearly independent. This implies $|\mathcal{A}| \leq \operatorname{rank}(A)$, which is at most $|V_n| + |W_n| - 1 = 2n - 1$ because the sum of all rows corresponding to vertices of V_n equals the sum of the rows corresponding to vertices of W_n.

The size of $\mathcal{A}$ is also an upper bound on the rank of the $2n \times |\mathcal{A}|$-matrix A. So $\operatorname{rank}(A) = |\mathcal{A}|$, hence $Ax = 1$ has at most one solution. This shows that the polytope $LP_{PM}(\mathcal{D})$ contains only one element, which then is the unique vertex of $LP_{PM}(\mathcal{D})$. $\square$

Now, we can show how to get a vertex of $LP_{PM}(\mathcal{D}_n)$ from a minimal fractional hypergraph $\mathcal{D} = (V_n, W_n, \mathcal{A})$ of $\mathcal{D}_n$:

Let $A' \in \mathbb{R}^{2n\times|\mathcal{A}|}$ be the incidence matrix of $\mathcal{D}$ and $A \in \mathbb{R}^{2n\times|\mathcal{A}_n|}$ be the incidence matrix of $\mathcal{D}_n$. Using the unique vertex $x' \in LP_{PM}(\mathcal{D})$, we define a vector $x \in \mathbb{R}^{\mathcal{A}}$ by

$$x_a := \begin{cases} x'_a, & \text{if } a \in \mathcal{A} \\ 0, & \text{else.} \end{cases}$$

x satisfies $Ax = 1$ and $x \geq 0$, so it lies in $LP_{PM}(\mathcal{D}_n)$. The column vectors of A corresponding to entries $x_a > 0$ are exactly the column vectors of A'. These column vectors are linearly independent, thus x is a vertex of the polytope $LP_{PM}(\mathcal{D}_n)$.

On the other hand, let x be a fractional vertex of $LP_{PM}(\mathcal{D}_n)$. If $\mathcal{A}$ is the set of all hyperedges with $x_a > 0$, then by Corollary 2.2.6 the projection of x to $\mathbb{R}^{\mathcal{A}}$ is a fractional vertex of $LP_{PM}(\mathcal{D}_n[\mathcal{A}])$. The rank of the incidence matrix A' of $\mathcal{D}_n[\mathcal{A}]$ is $|\mathcal{A}|$, so $LP_{PM}(\mathcal{D}_n[\mathcal{A}])$ contains only one element, implying that $\mathcal{D}_n[\mathcal{A}]$ is minimal fractional.

This shows in particular that the calculation of all minimal fractional hypergraphs of $\mathcal{D}_6$ is equivalent to the calculation of all fractional vertices of $LP_{PM}(\mathcal{D}_6)$.

In the following, we describe the method for calculating the minimal fractional hypergraphs:

Each partial hypergraph $\mathcal{D} = (V_n, W_n, \mathcal{A})$ of $\mathcal{D}_n$ is represented by a boolean array $b_{\mathcal{D}}$ indexed by the elements of $a \in \mathcal{A}_n$ with "true" at position a if $a \in \mathcal{A}$, and "false" else.

We calculate inductively all symmetry classes of partial hypergraphs having h hyperedges of size four and a edges which have no proper fractional partial hypergraph. For that let $C_{h,a}$ denote the set of representatives for this symmetry classes and let $F_{h,a} \subseteq C_{h,a}$ be the set of representatives corresponding to minimal fractional hypergraphs. We choose the lexicographic smallest representative of a symmetry class to get a unique representative of each class.

Observe that $F_{0,a} = \emptyset$ as $\mathcal{D}$ is just an ordinary bipartite graph if it has no hyperedges. So we only have to calculate $C_{h,a}$ and $F_{h,a}$ for $h \geq 1$. Moreover, by Lemma 5.2.1, we only have to consider $C_{h,a}$ with $h + a \leq 2n - 1$.

$C_{1,0}$ has only one element because all hypergraphs having exactly one hyperedge of size four are symmetric. This hypergraph is not fractional, thus $F_{1,0} = \emptyset$. Now, we can inductively calculate all other $C_{h,a}$:

$$\begin{aligned} C_{h,a} = \{b_{\mathcal{D}} |\ & \mathcal{D} \text{ has } h \text{ hyperedges and } a \text{ edges,} \\ & \nexists\, b_{\mathcal{D}'} \in F_{\tilde{h},\tilde{a}} \text{ with } (\tilde{h}, \tilde{a}) < (h, a) \text{ such that } b_{\mathcal{D}} \geq b_{\mathcal{D}'}\}, \end{aligned}$$

where we define $(\tilde{h}, \tilde{a}) < (h, a)$ to hold if and only if $\tilde{h} \leq h, \tilde{a} \leq a$ and at least one of the inequalities is strict. Furthermore, the boolean arrays $b_{\mathcal{D}}$ and $b_{\mathcal{D}'}$ are interpreted as 0-1 vectors with 0 for "false" and 1 for "true". $b_{\mathcal{D}} \geq b_{\mathcal{D}'}$ holds if and only if every hyperedge of $\mathcal{D}'$ is also an element of $\mathcal{D}$, i.e., $\mathcal{D}'$ is a partial hypergraph of $\mathcal{D}$. So the second condition above guarantees that $\mathcal{D}$ with $b_{\mathcal{D}} \in C_{h,a}$ has no proper fractional partial hypergraph.

Algorithm 1 describes how $F_{h,a}$ is calculated from $C_{h,a}$. To test the integrality of a polytope we use the function LATTICE() from polymake [GJ00] .

Algorithm 1 Calculate $F_{h,a}$ from $C_{h,a}$

```
F_{h,a} ← ∅
for all b_D ∈ C_{h,a} do
    Check if LP_PM(D) is integral.
    if LP_PM(D) fractional then
        F_{h,a} ← F_{h,a} ∪ b_D
    end if
end for
```

It remains to show how to get all $C_{h,a}$ with $h + a \leq 2n - 1$. The set $C_{h,a}$ can be calculated from $C_{h-1,a}$ or $C_{h,a-1}$ (if $a \geq 1$). Algorithm 2 shows the first case, the second one is similar, only $|e| = 4$ has to be replaced by $|e| = 2$.

Algorithm 2 Calculate $C_{h,a}$ from $C_{h-1,a}$

```
C_{h,a} ← ∅
for all b_D ∈ C_{h-1,a} do
    for all hyperedges e ∈ A_n \ A with |e| = 4 do
        D' ← D + e = (V, W, A ∪ {e})
        b ← b_{D'}
        for all permutations π respecting the symmetry do
            permute b_{D'} w.r.t. π to get π(b_{D'})
            if π(b_{D'}) is lexicographic smaller than b then
                b ← π(b_{D'})
            end if
        end for
        if b ∉ C_{h,a} then
            if not (∃ b' ∈ F_{h̃,ã} with (h̃, ã) < (h, a) such that b' ≤ b) then
                C_{h,a} ← C_{h,a} ∪ b
            end if
        end if
    end for
end for
```

Remark. The sets $C_{h,a}$ become very huge for increasing h and a, even though just the representative of a class is stored. For $\mathcal{D}_6$ the calculations were only possible because of 5.2.1.

Every minimal fractional partial hypergraph of $\mathcal{D}_m$ can be extended to one for $\mathcal{D}_n$ with $m < n$ by adding hyperedges which form a perfect matching of the vertices $V_n \setminus V_m, W_n \setminus W_m$. Similarly, two minimal fractional partial hypergraphs, say of $\mathcal{D}_k$ and $\mathcal{D}_m$, give rise to a minimal fractional hypergraph of $\mathcal{D}_{k+m}$. So it is reasonable to start looking at $\mathcal{D}_2$ and $\mathcal{D}_4$ before $\mathcal{D}_6$.

However, in the case $\mathcal{D}_2$ we get no fractional hypergraphs as the bipartite hypergraph $\mathcal{D}_2$ has no strong odd cycle and therefore $LP_{PM}(\mathcal{D}_2)$ is integral.

In the next two subsection the minimal fractional partial hypergraphs of $\mathcal{D}_4$ and $\mathcal{D}_6$ are investigated.

5.2.1. The case $\mathcal{D}_4$

The case $\mathcal{D}_4$ is more interesting than $\mathcal{D}_2$. Here we get three classes, where in two of them there are two vertex disjoint strong odd cycles sharing the same tail-to-tail

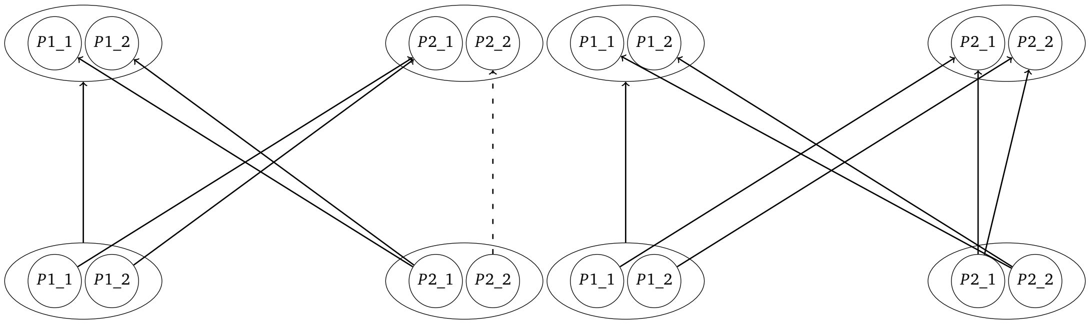


Figure 5.4.: Minimal fractional examples: $F_{1,5}$ and $F_{1,6}$

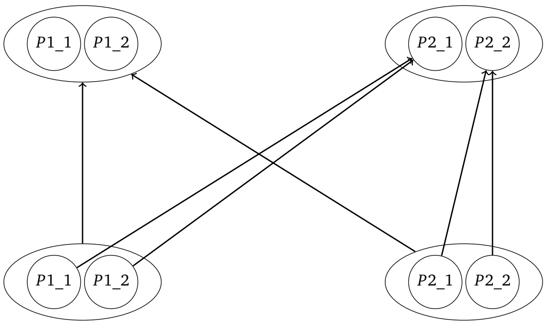


Figure 5.5.: Minimal fractional example: $F_{2,4}$

hyperedge, see Figure 5.4. In the other class there are two vertex disjoint cycles with two different tail-to-tail hyperedges which intersect, see Figure 5.5. In all cases LP_{PM} just contains one fractional point x with $x_e = \frac{1}{2}$ for all hyperedges e drawn solidly and $x_e = 1$ for the hyperedge drawn with a dashed line.

5.2.2. The case $\mathcal{D}_6$

For $\mathcal{D}_6$ the examples cannot be classified as easy as for $\mathcal{D}_4$. We describe the fractional partial hypergraphs which are understood and list the remaining examples in the appendix.

We start with the fractional partial hypergraphs having only one proper hyperedge, i.e. a hyperedge of size four. For them, we know that after adding all odd cycle inequalities the polytope LP_{PM} is integral. This is the case because the deletion of the unique hyperedge of size four gives a balanced hypergraph, i.e., the line graph of such a hypergraph is t-perfect. Furthermore, by Theorem 4.1.1, every fractional partial hypergraph must contain a strong odd cycle. Indeed, all minimal fractional partial hypergraphs having only one proper hyperedge contain two strong odd cycles that are disjoint except for their tail-to-tail hyperedge:

Lemma 5.2.2. *Let $\mathcal{D} = (V_n, W_n, \mathcal{A})$ be a minimal fractional hypergraph with exactly one hyperedge of size four. It's set of hyperedges $\mathcal{A}$ can be partitioned into two strong odd cycles that are disjoint except for the unique proper hyperedge, and possibly a perfect matching of the vertices not covered by the two strong odd cycles.*

Proof. Let $x^* \in \mathbb{R}^{\mathcal{A}}$ be a vertex of $LP_{PM}(\mathcal{D})$. As $\mathcal{D}$ is minimal fractional, this vertex is unique. Now, we delete all $a \in \mathcal{A}$ with $x^*_a = 1$. These hyperedges form a matching and all vertices that are covered by this matching are not incident to any other hyperedges.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be the set of remaining hyperedges and $V \subseteq V_n, W \subseteq W_n$ be the set of vertices not covered by $\mathcal{A} \setminus \mathcal{A}'$. We set $\mathcal{D}' := (V, W, \mathcal{A}')$ which is again a partitioned hypergraph of part size two.

Projecting x^* to $\mathbb{R}^{A'}$ yields a vertex $x' \in \mathbb{R}^{\mathcal{A}'}$ of $LP_{PM}(\mathcal{D}')$ with $0 < x'_a < 1$ for all hyperedges $a \in \mathcal{A}'$. This also shows that the unique hyperedge of size four lies in $\mathcal{A}'$, which will be denoted by $\tilde{a}$ in the remaining of the proof.

As in the proof of Theorem 4.1.1 the size of $\mathcal{A}'$ is at most $|V| + |W| - 1 = 2|V| - 1$. On the other hand, the degree of each vertex $v \in V$ must be at least two. This implies that

$$2|V| \leq \sum_{v \in V} |\delta(v)| = \sum_{v \in V} |\{a \in \mathcal{A}' | v \in a\}| = |\mathcal{A}'| + 1 \leq 2|V|,$$

because each edge is counted exactly once and $\tilde{a}$ is counted twice. So every vertex $v \in V$ has degree two. Similarly, it can be shown that the degree of every vertex $w \in W$ is two.

If we replace $\tilde{a}$ by the two edges $\tilde{a} \cap V$ and $\tilde{a} \cap W$ we get an ordinary graph G. Every vertex in G has degree two, so the edges of G can be partitioned into cycles. If some cycle has even length, then it correspond to a strong even cycle in $\mathcal{D}'$, but then x' can not be a vertex. So G has only odd cycles. Because of the structure of the graph G, exactly one of the two edges $\tilde{a} \cap V$ and $\tilde{a} \cap W$ is contained in an odd cycle of G. Altogether, we see that G must be the disjoint union of two odd cycles.

Transferred to $\mathcal{D}'$ the two odd cycles in G form two strong odd cycles in $\mathcal{D}'$ that are disjoint except for $\tilde{a}$. This shows the claim. $\square$

Remark. Lemma 5.2.2 also shows that the polytope $LP_{PM}(\mathcal{D})$ is half-integral for a minimal fractional hypergraph $\mathcal{D} = (V_n, W_n, \mathcal{A})$ having exactly one proper hyperedge $\tilde{a} \in \mathcal{A}$.

Let $a \in \mathcal{A}$ be a hyperedge with $0 < x_a < 1$ for $x \in LP_{PM}(\mathcal{D})$. Then a lies on a strong odd cycle $C = (v_0, a_1, v_1, \ldots, a_l, v_l = v_0)$ with $a_1 = a$ after some possible renumbering. All x values of the other hyperedges in C are determined by x_a, namely $x_{a_2} = 1 - x_a, x_{a_3} = x_a, \ldots, x_{a_l} = x_a$ as l is odd. This implies that $1 = x_a + x_{a_l} = 2x_a$, thus $x_a = \frac{1}{2}$.

Unfortunately, even for minimal fractional hypergraphs with two proper hyperedges there is no nice analogue of the last lemma. However, some edges in a minimal fractional hypergraph as in Lemma 5.2.2 can be replaced by proper hyperedges. For that we need the notion of parallel edges.

Definition. Let $\mathcal{D} = (V, W, \mathcal{A})$ be a partitioned hypergraph with arbitrary maximum part size. Two edges $a_1, a_2 \in \mathcal{A}$ are called parallel if:

- a_1 and a_2 do not intersect,
- $a_1 \cap V$ and $a_2 \cap V$ lie in the same part of V,
- $a_1 \cap W$ and $a_2 \cap W$ lie in the same part of W.

The next lemma shows that two parallel edges can be replaced by a proper hyperedge.

Lemma 5.2.3. *Let $\mathcal{D} = (V_n, W_n, \mathcal{A})$ be a minimal fractional hypergraph. If $a_1, a_2 \in \mathcal{A}$ are two parallel hyperedges with $x_{a_1} = x_{a_2}$ for the unique element $x \in LP_{PM}(\mathcal{D})$, then replacing a_1 and a_2 by the proper hyperedge $a_1 \cup a_2$ gives a minimal fractional hypergraph.*

Proof. Let $\mathcal{D}' = (V_n, W_n, \mathcal{A}')$ be the bipartite hypergraph obtained from $\mathcal{D}$ by replacing the two edges a_1, a_2 by the hyperedge $a_1 \cup a_2$.

First, we show that $LP_{PM}(\mathcal{D}')$ is fractional. For that, let $x \in LP_{PM}(\mathcal{D})$ and define $\tilde{x} \in \mathcal{A}'$ by

$$\tilde{x}_a := \begin{cases} x_a, & \text{if } a \neq a_1 \cup a_2 \\ x_{a_1}, & \text{if } a = a_1 \cup a_2 \end{cases}.$$

Because of $x_{a_1} = x_{a_2}$ and $a_1 \cap a_2 = \emptyset$, it is clear that $\tilde{x}(\delta(v)) = x(\delta(v)) = 1$ for all $v \in V \cup W$. So $\tilde{x}$ lies in $LP_{PM}(\mathcal{D}')$.

Suppose $\mathcal{D}'$ contains a proper partial hypergraph $\tilde{\mathcal{D}} = (V_n, W_n, \tilde{\mathcal{A}})$ that is fractional. If $a_1 \cup a_2 \notin \tilde{\mathcal{A}}$, then $\tilde{\mathcal{D}}$ is a proper partial hypergraph of $\mathcal{D}$. Otherwise, replacing the hyperedge $a_1 \cup a_2$ by the two edges a_1 and a_2 gives a partial hypergraph of $\mathcal{D}$ that is fractional. So in both cases we get a contradiction to the assumption that $\mathcal{D}$ is minimal fractional, thus also $\mathcal{D}'$ is minimal fractional. □

Example. We start with the fractional hypergraph shown in Figure 5.6 which has $x \equiv \frac{1}{2}$ as the unique element of its perfect matching polytope. This hypergraph has three pairs of parallel edges that can be replaced by a proper hyperedge.

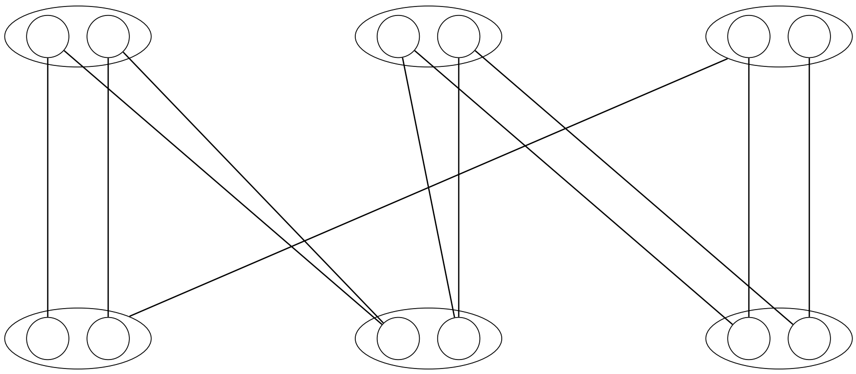


Figure 5.6.: A hypergraph with three pairs of parallel edges.

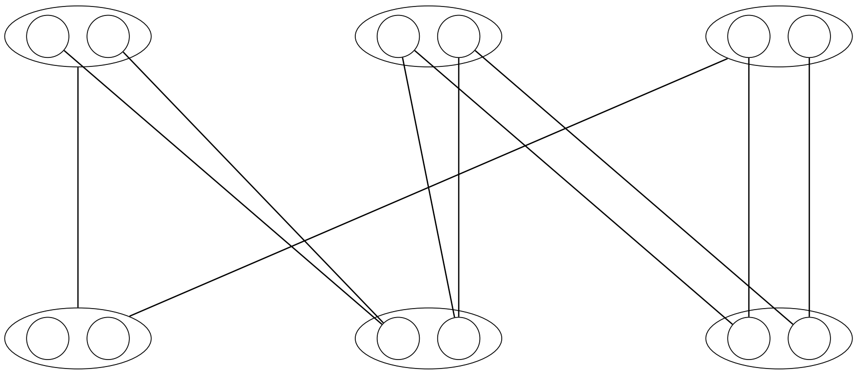


Figure 5.7.: One pair of parallel edges replaced.

In this way we get $2^3 - 1 = 7$ new fractional hypergraphs. However, some of these hypergraphs might be symmetric. This is indeed the case: When replacing two pairs of parallel edges only two non-symmetric hypergraphs occur. All other resulting hypergraphs are not symmetric to each other.

Figure 5.7 and 5.8 depict an example of replacing one or two pairs of parallel edges. Finally, the hypergraph obtained by replacing all three pairs of parallel edges is displayed in Figure 5.9.

As we have no good characterization of the remaining examples, we list them in the appendix. We omit fractional hypergraphs arising from other fractional hypergraphs by replacing parallel edges. Moreover, we do not list the fractional hypergraph arising from the case $k = 3$ in the second example of section 5.1.4, which is the only minimal fractional hypergraph of $\mathcal{D}_6$ for which the vertex of it's fractional perfect matching polytope has an entry equal to $\frac{1}{6}$.

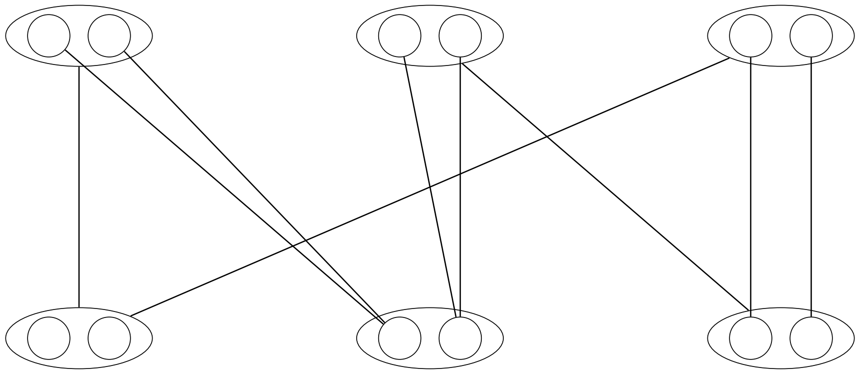


Figure 5.8.: Two pairs of parallel edges replaced

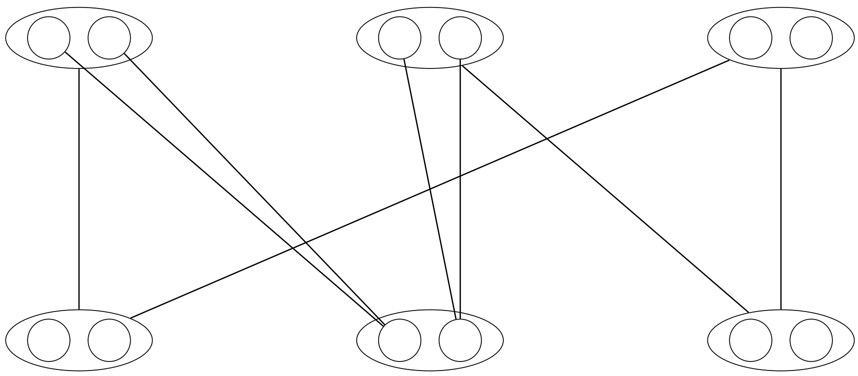


Figure 5.9.: All parallel edges replaced

6. Summary

In this thesis the hyperassignment problem was investigated with a special focus on connections to the theory of hypergraphs, in particular balanced and normal hypergraphs, as well as its relation to the Stable Set Problem. With Theorem 4.2.3 a variant of Hall's Theorem for normal hypergraphs was found.

The main point was the investigation of the matching and perfect matching polytope for partitioned hypergraphs. Therefore, valid inequalities and facets were found, and the dimension of some polytopes could be calculated. It was shown that the trivial LP-relaxation of the Hyperassignment Problem obtained by relaxing $x_i \in \{0,1\}$ by $0 \leq x_i \leq 1$ has an arbitrarily large integrality gap, even after adding all clique inequalities. Whereas the integrality gap of the trivial LP-relaxation of the maximum weight matching problem for partitioned hypergraphs with maximum part size M is at most $2M-1$.

Additionally, computational results for small partitioned hypergraphs of part size two were presented. Using symmetry it was possible to calculate all minimal fractional vertices of the fractional perfect matching polytope of partitioned hypergraphs with part size two having at most twelve vertices.

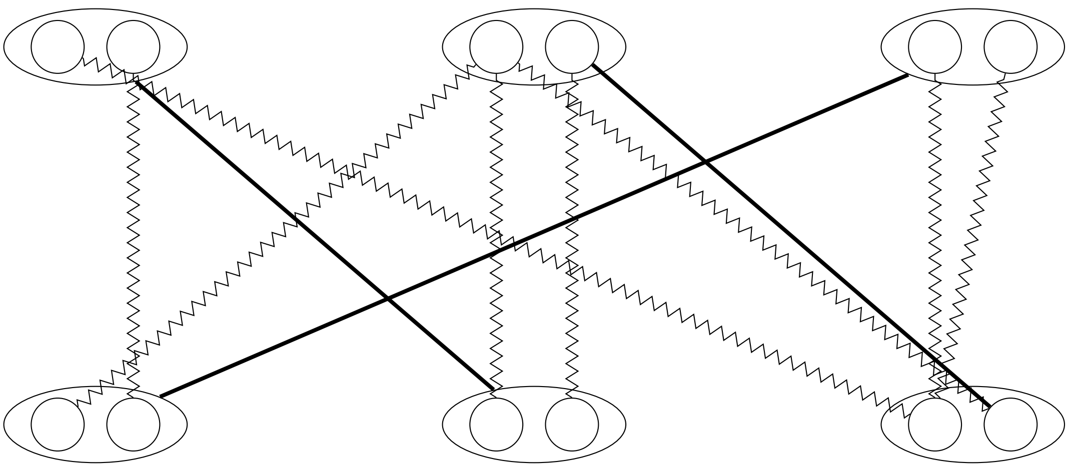
Another open problem is to classify all facets of $\mathcal{D}_6$. Up to symmetry, we know that sixteen classes are induced by odd clique set inequalities. However, we could not classify the inequalities defining the other fourteen classes.

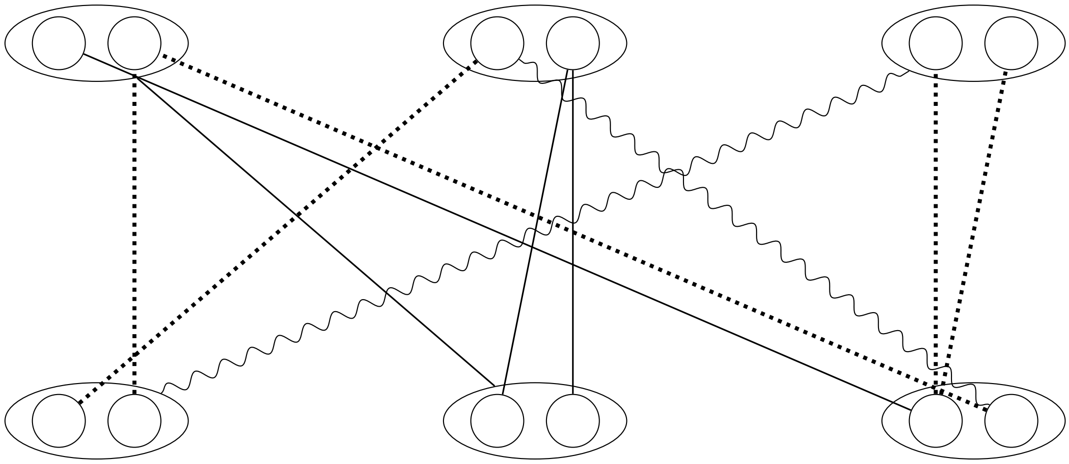
A. Appendix

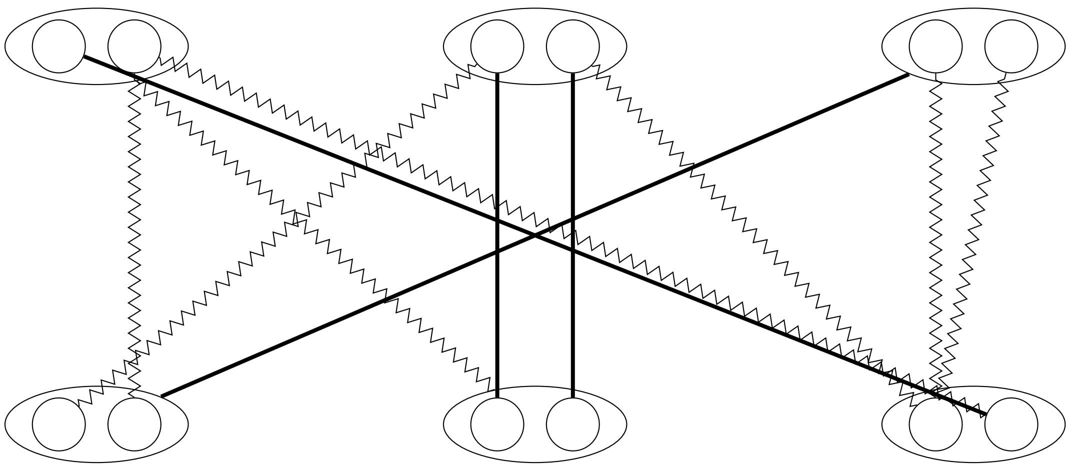
We list all 45 minimal fractional hypergraphs which are not of the forms described in section 5.2. For a listed hypergraph $\mathcal{D} = (V_6, W_6, \mathcal{A})$ the value of the unique vector $x \in LP_{PM}(\mathcal{D})$ at position a is indicated by the line style in which the hyperedge a is drawn:

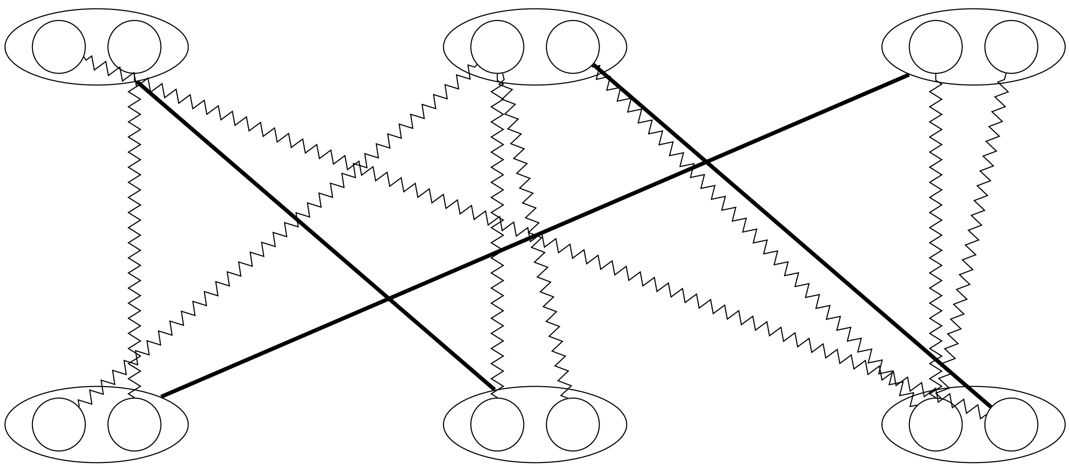
$\simeq 1$ $\simeq \frac{1}{2}$ $\simeq \frac{1}{4}$ $\simeq \frac{3}{4}$ $\simeq \frac{2}{3}$ $\simeq \frac{1}{3}$

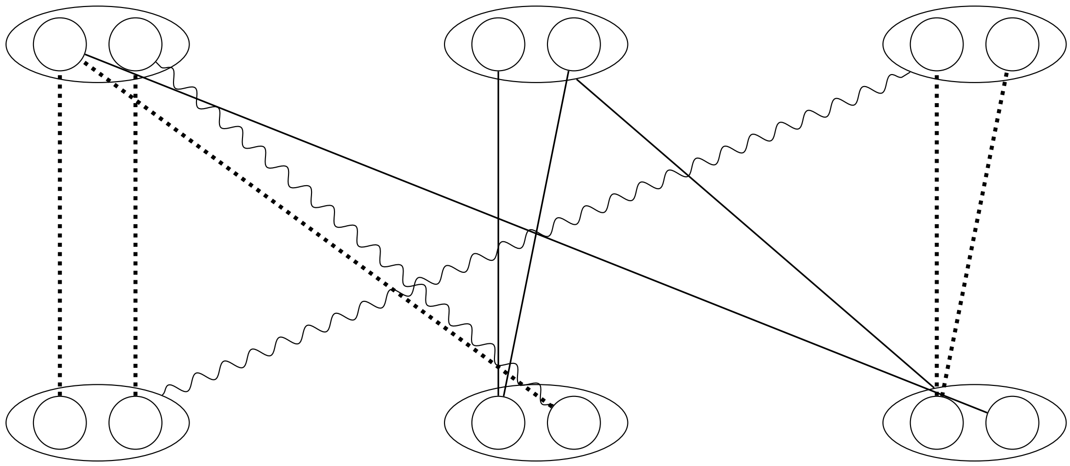
A.1. 2 Hyperedges

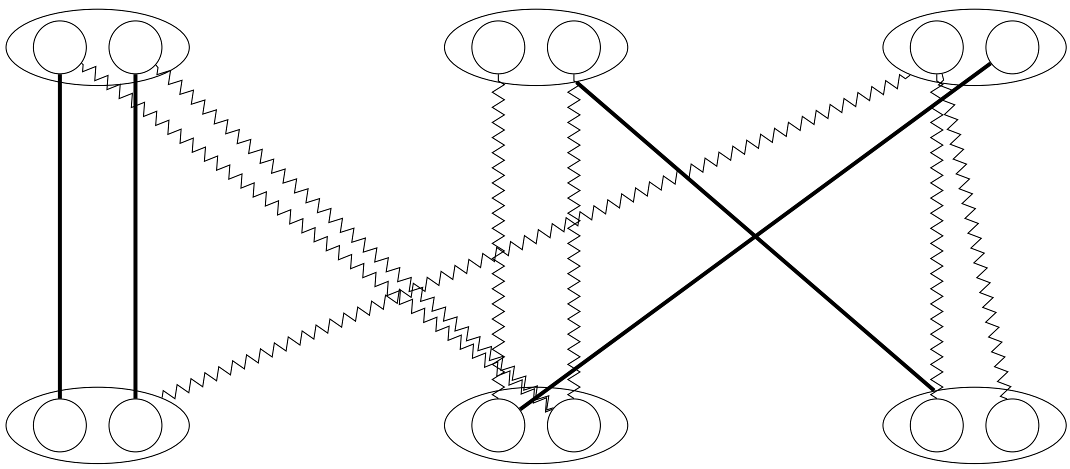


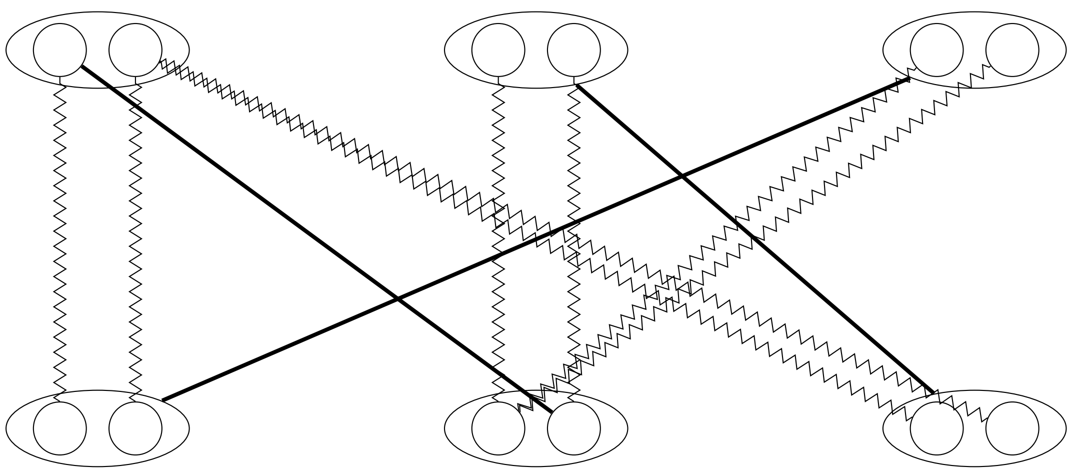


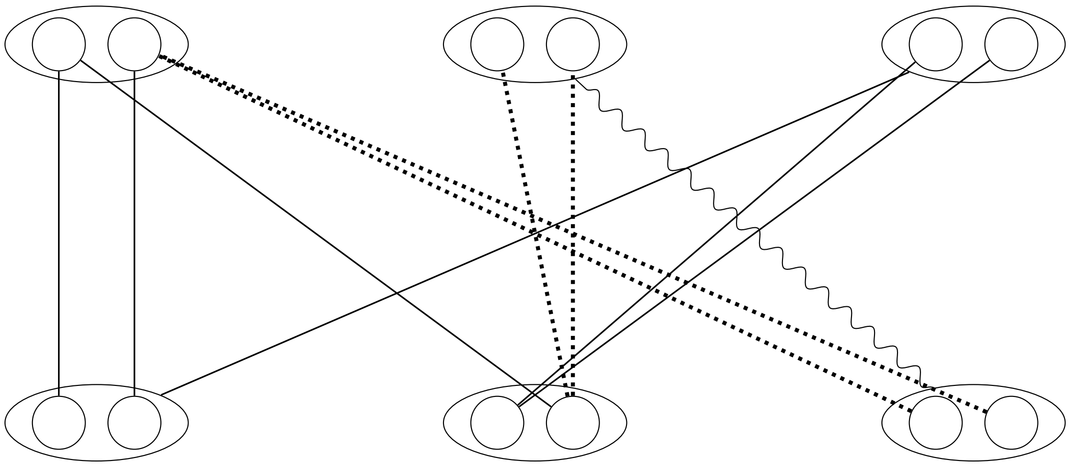


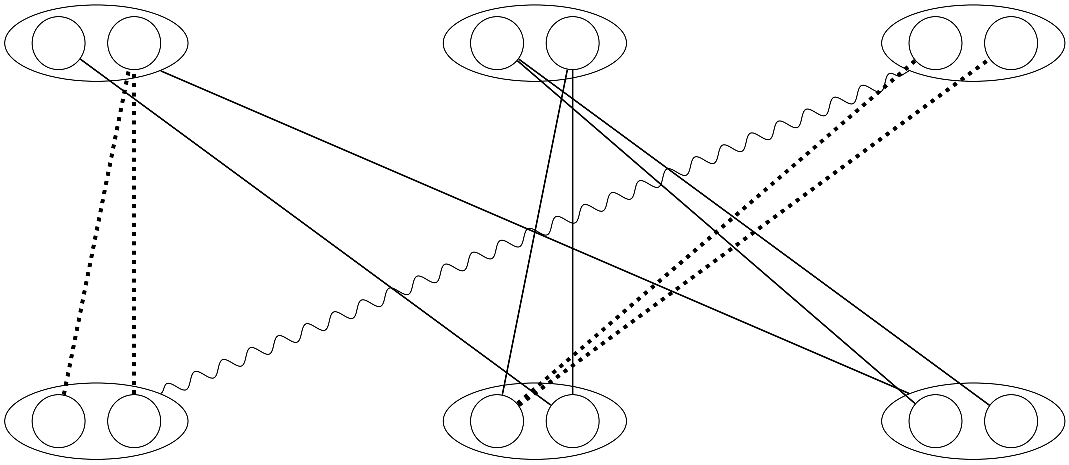


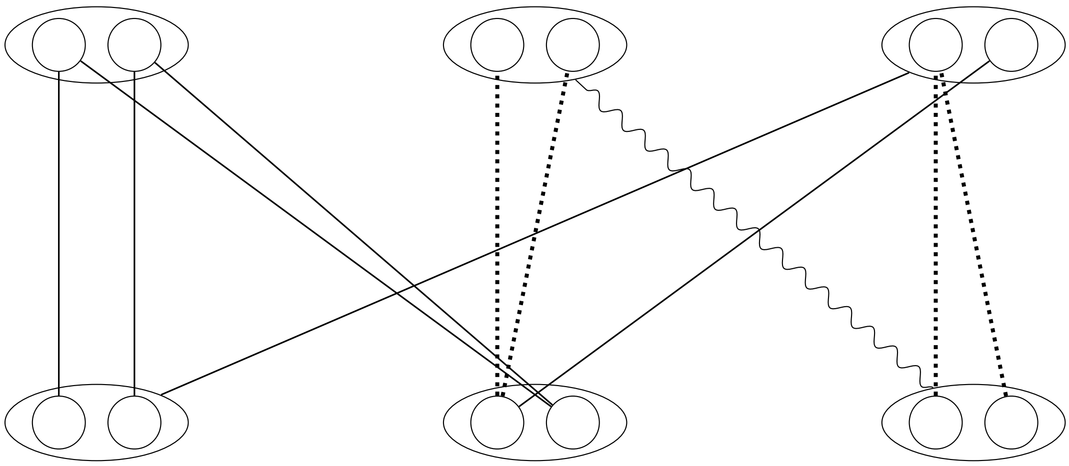


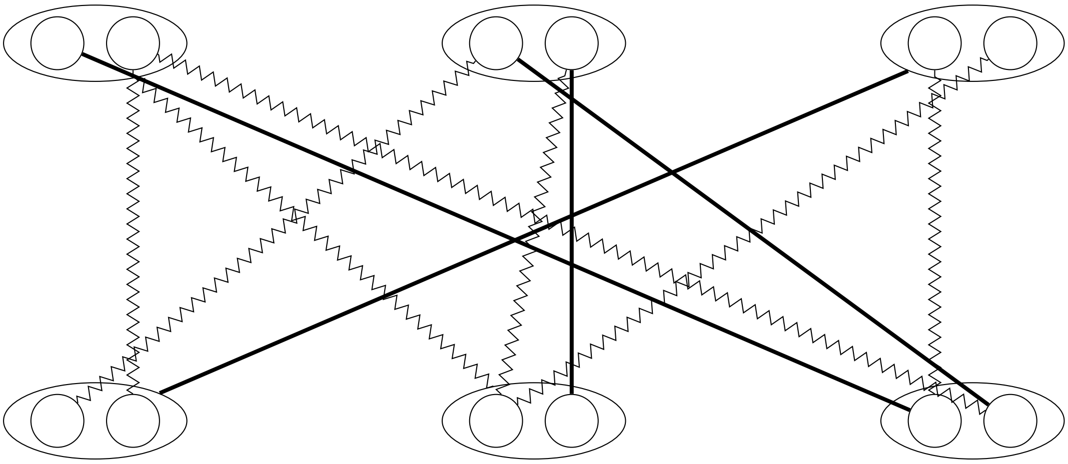


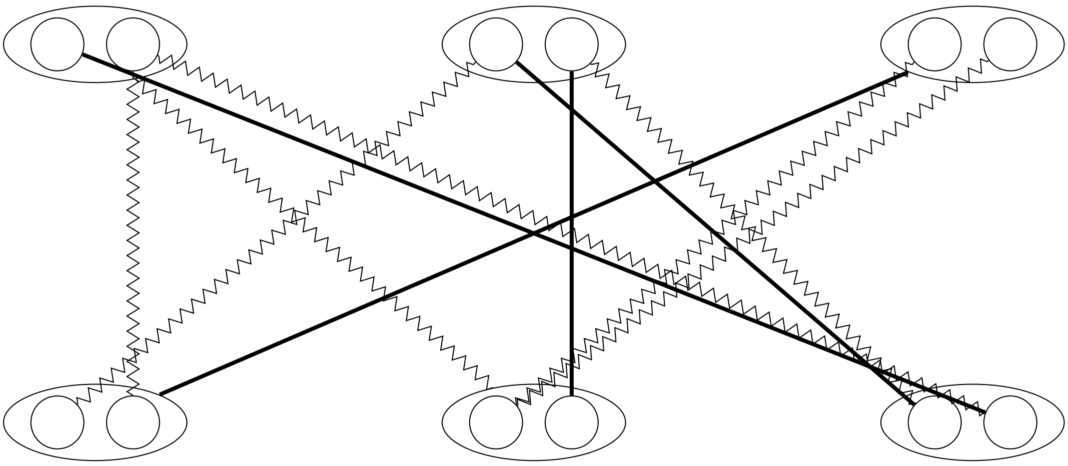


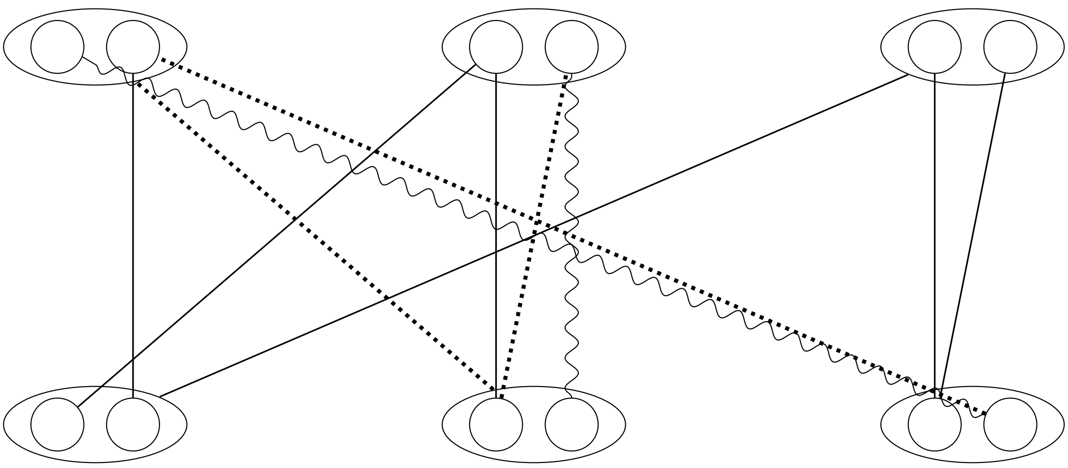


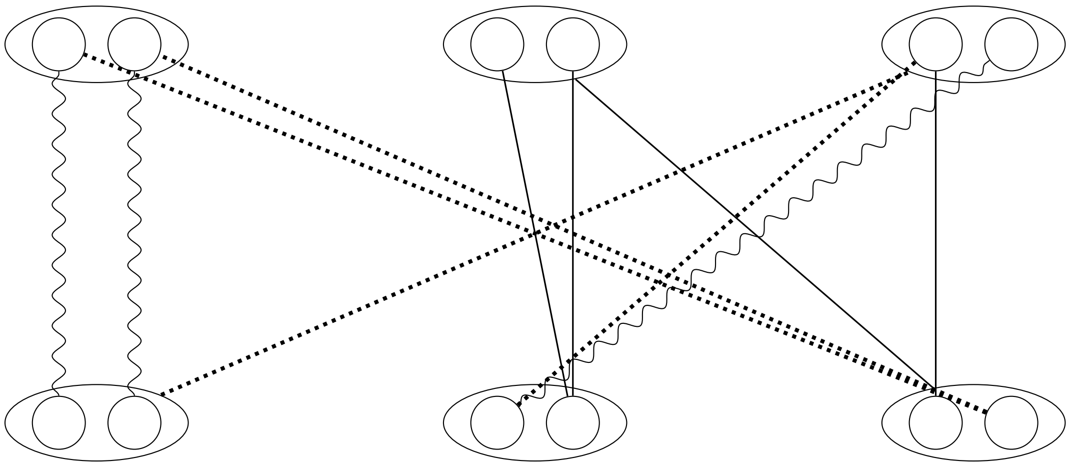


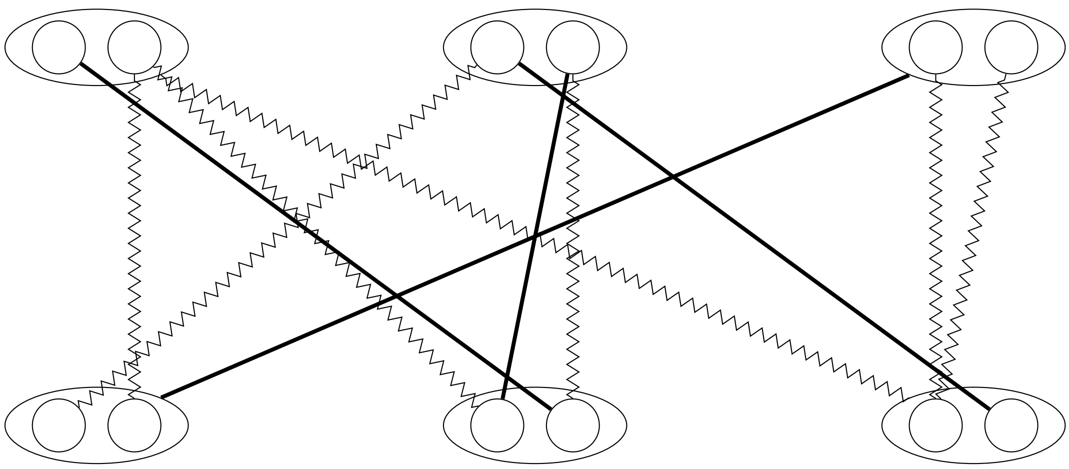


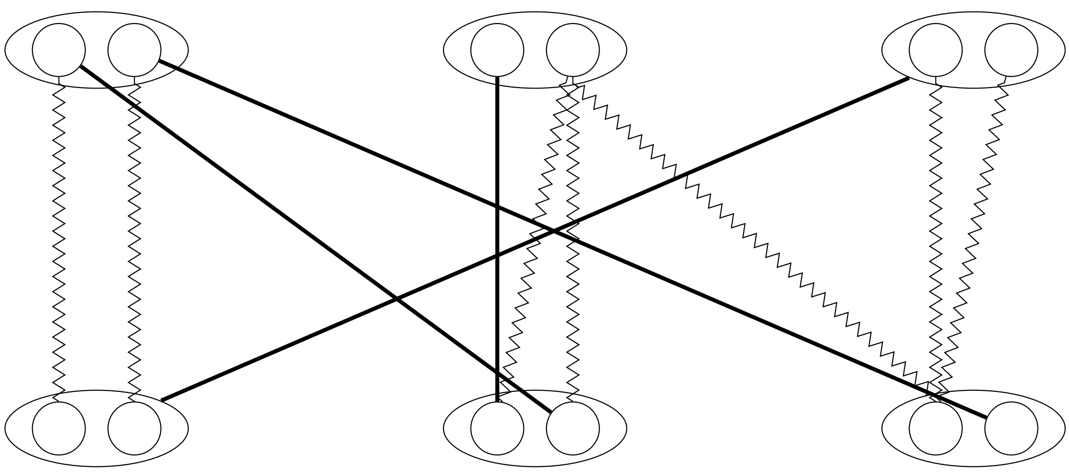


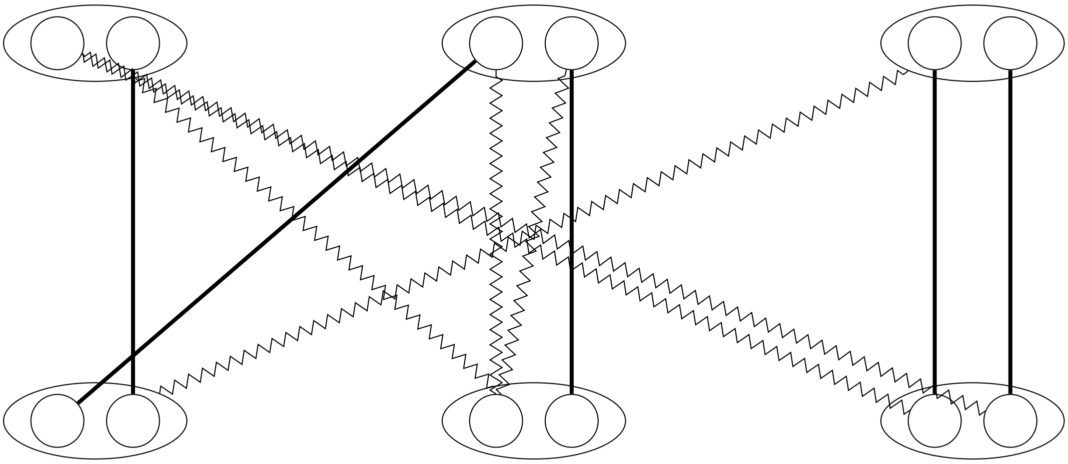


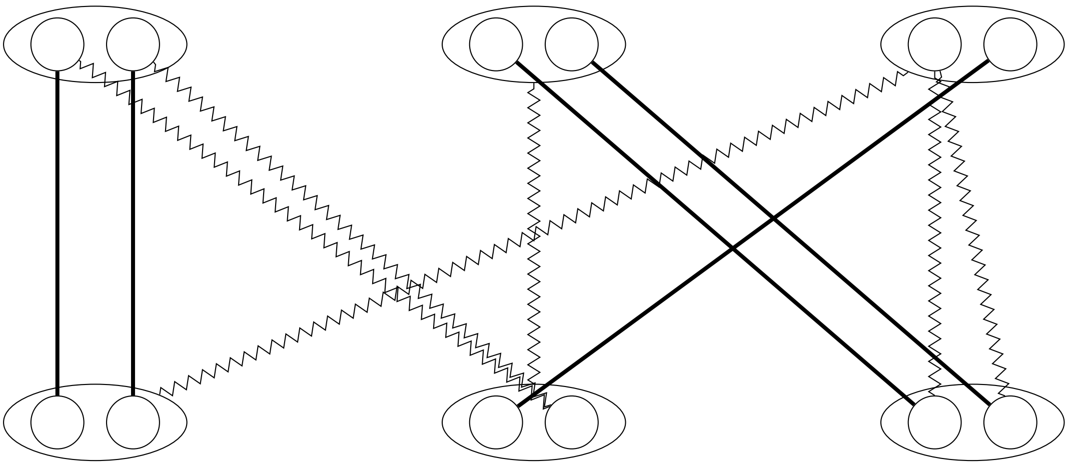


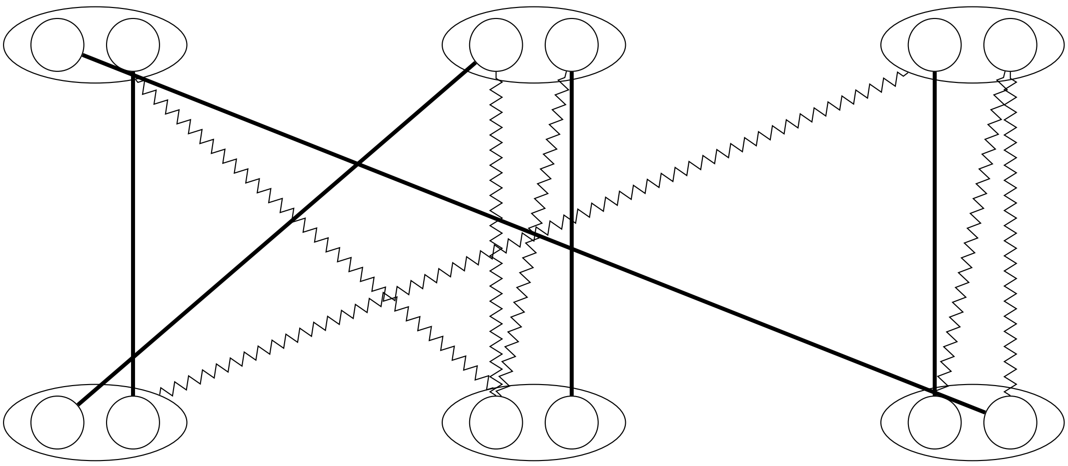


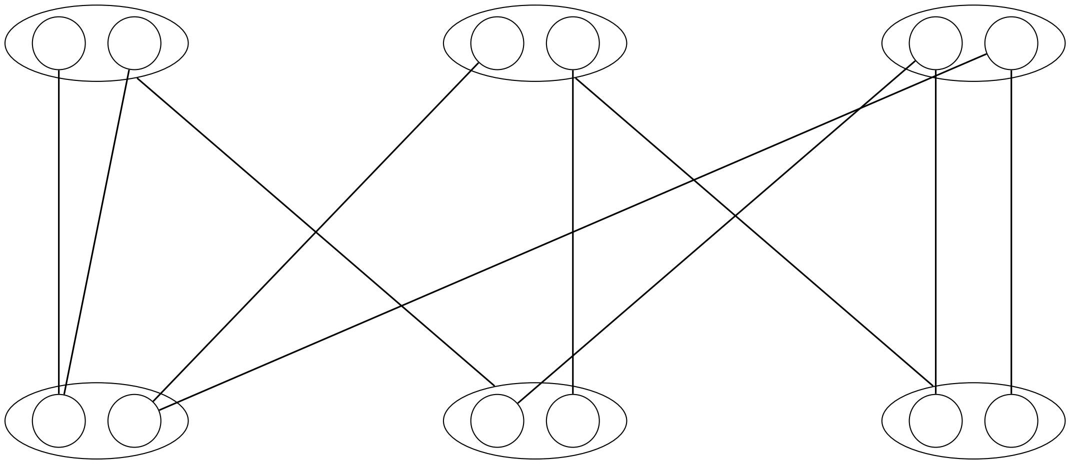


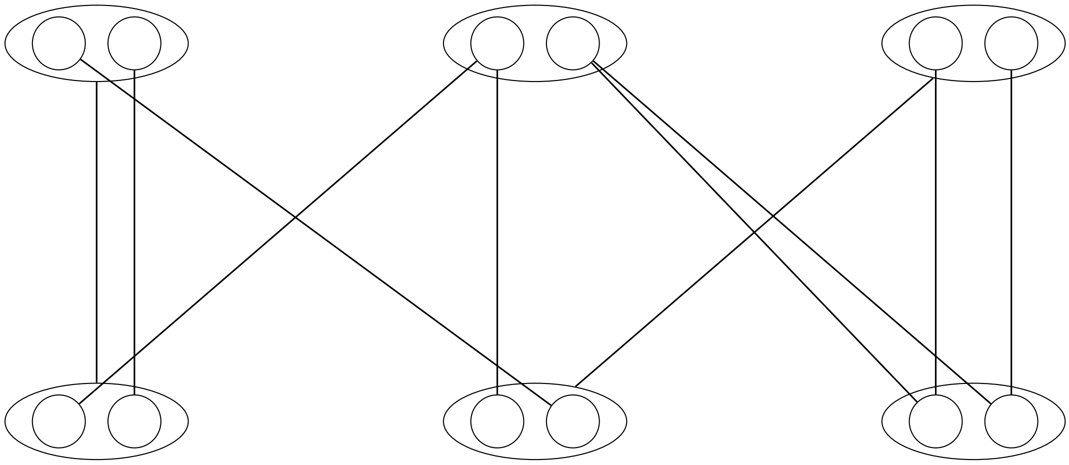


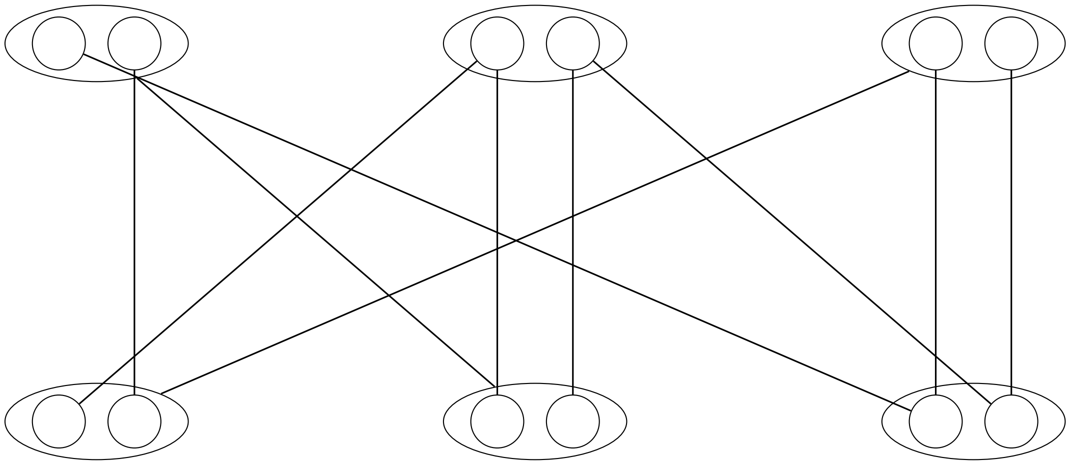


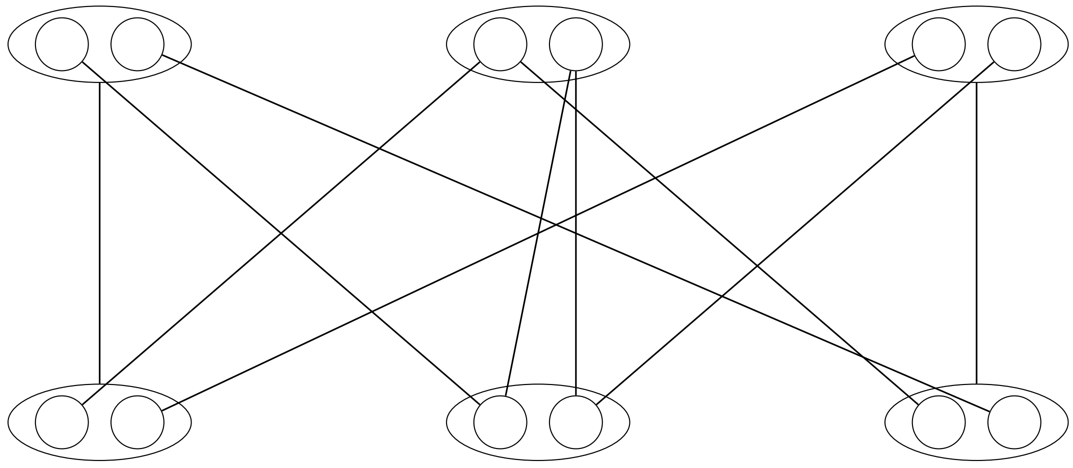


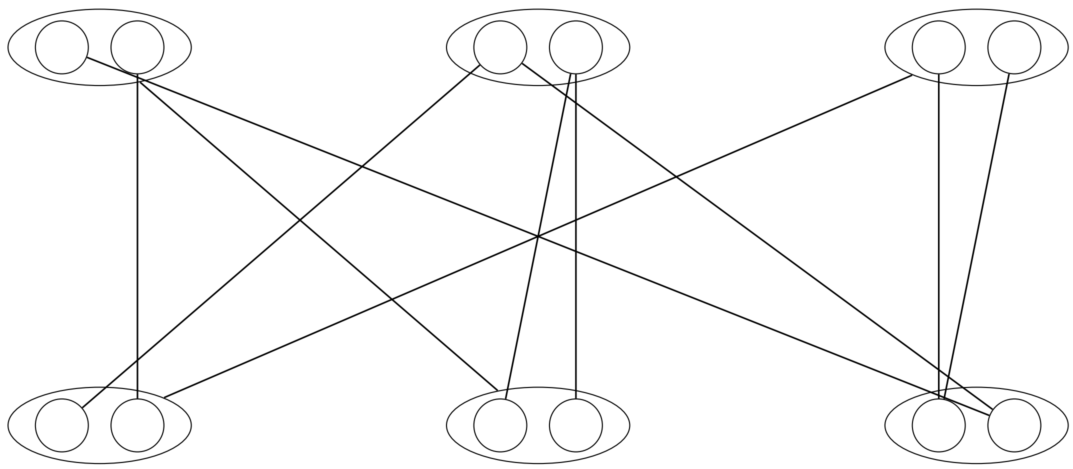


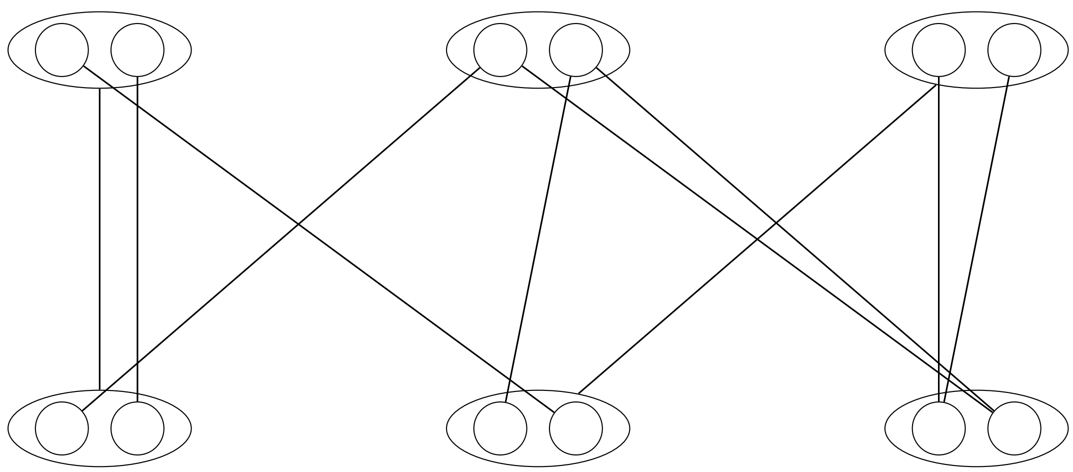


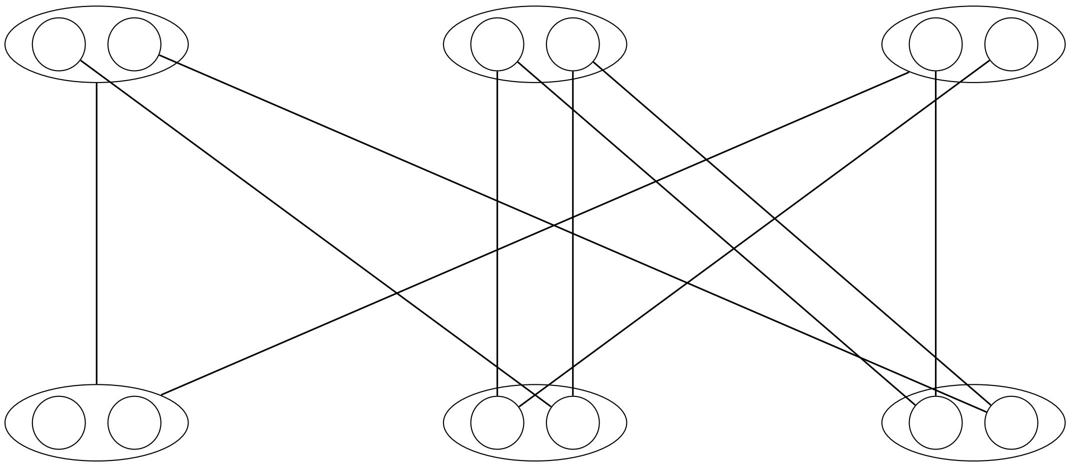


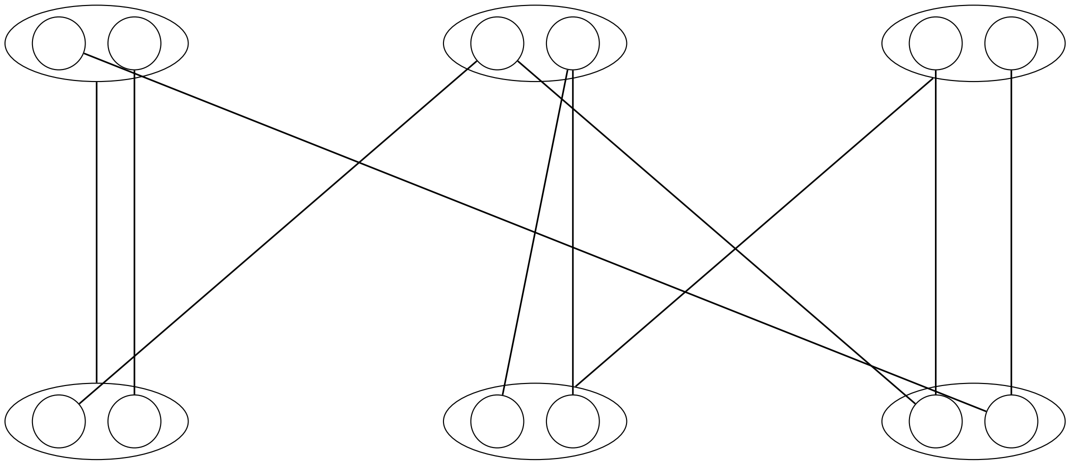


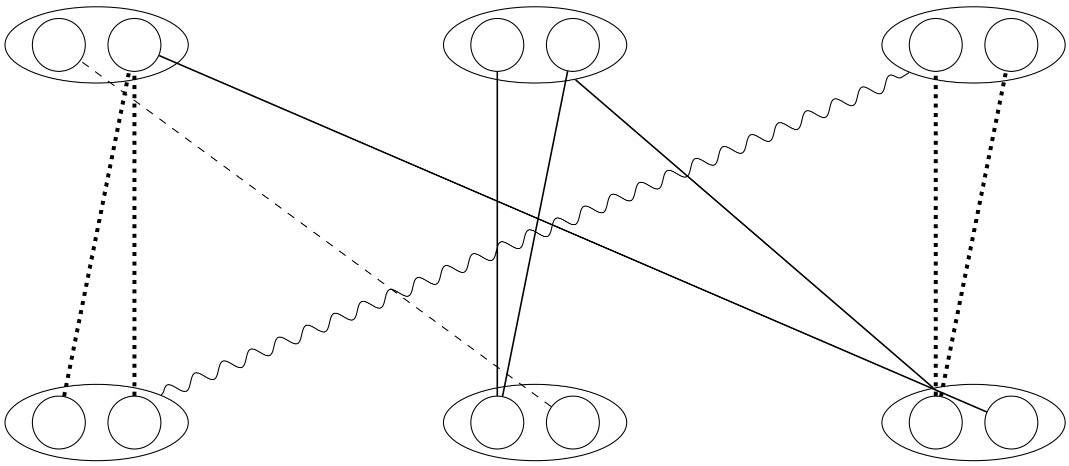


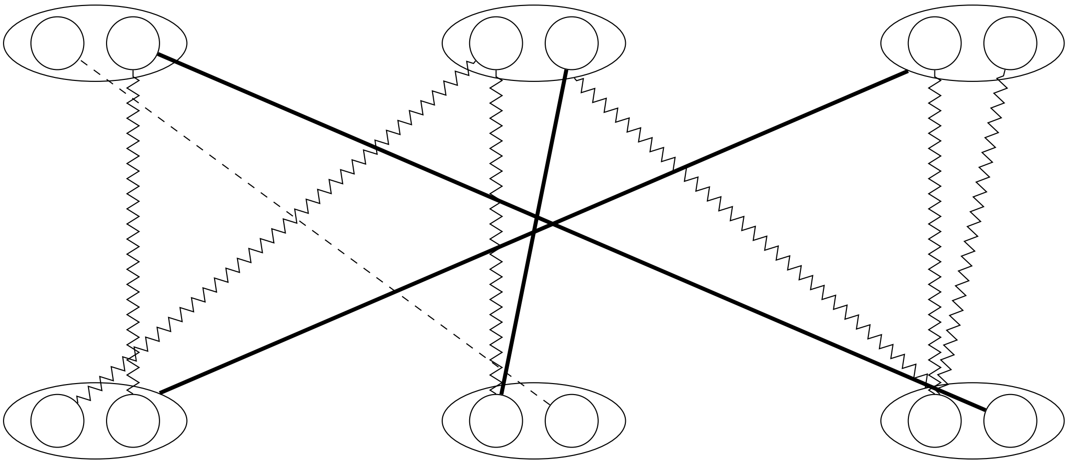


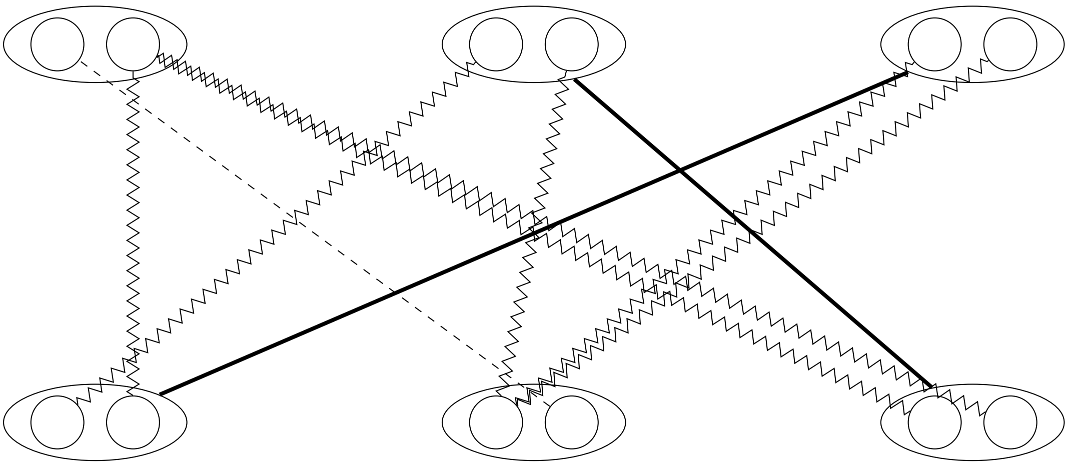


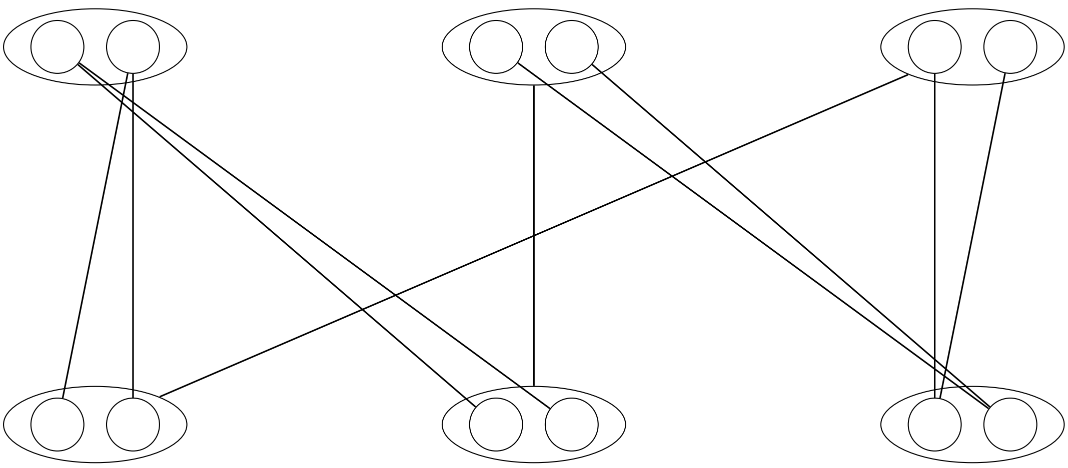


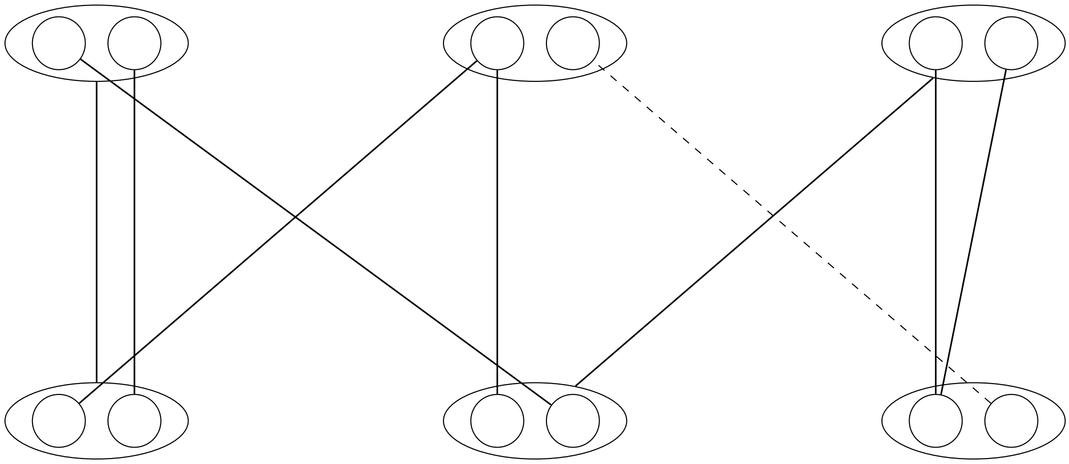


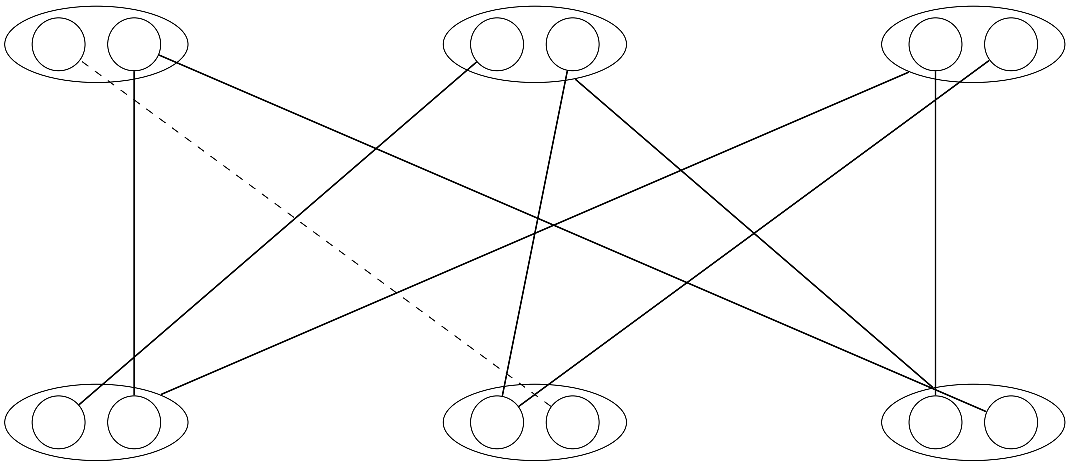




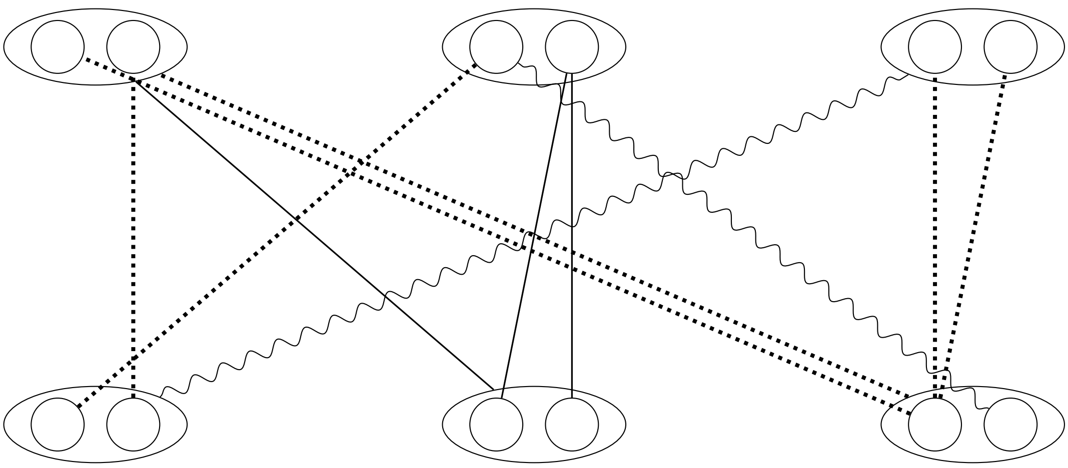


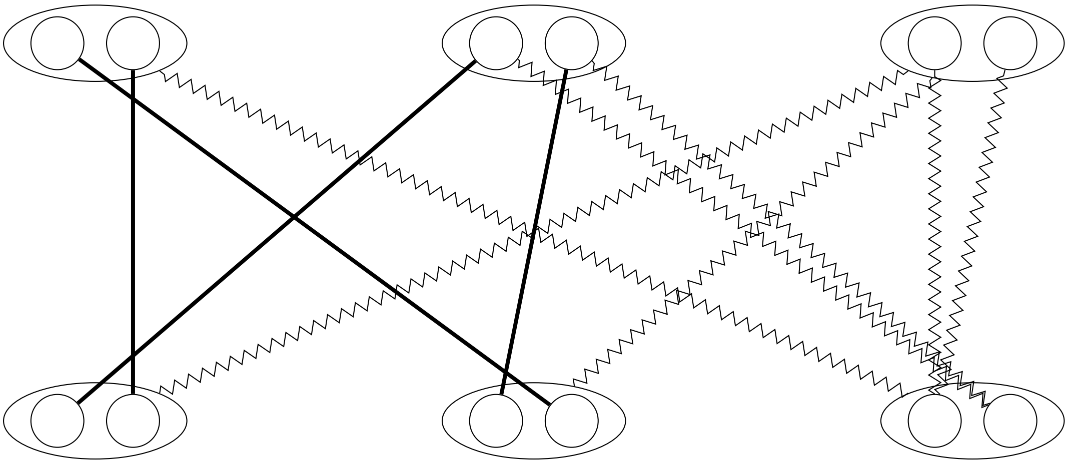


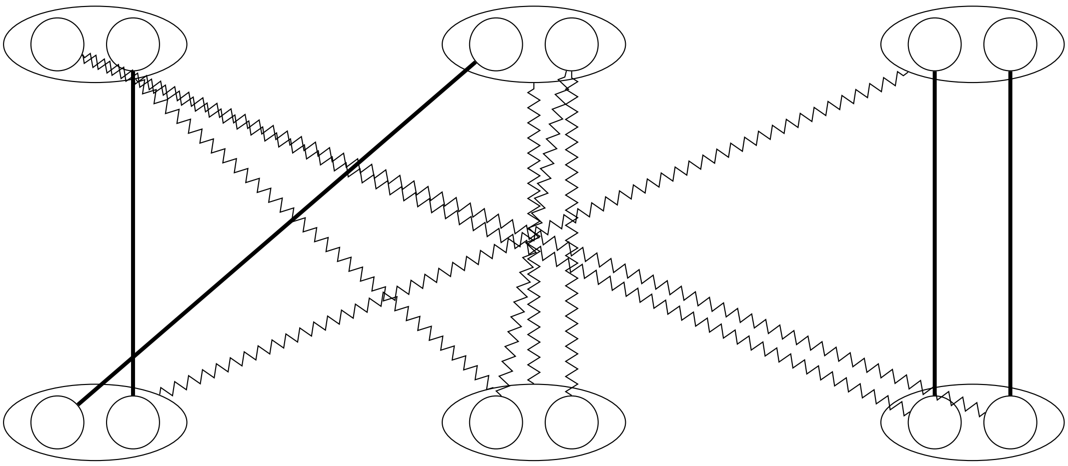


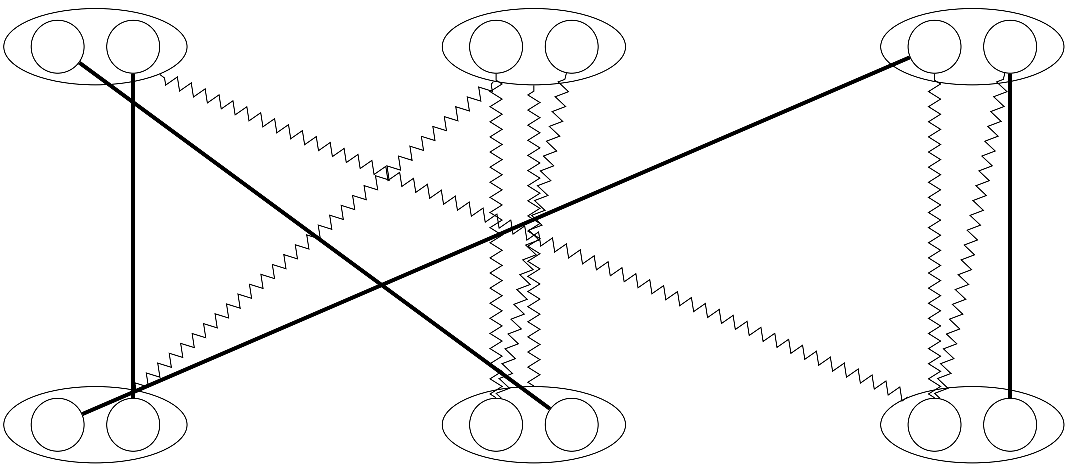


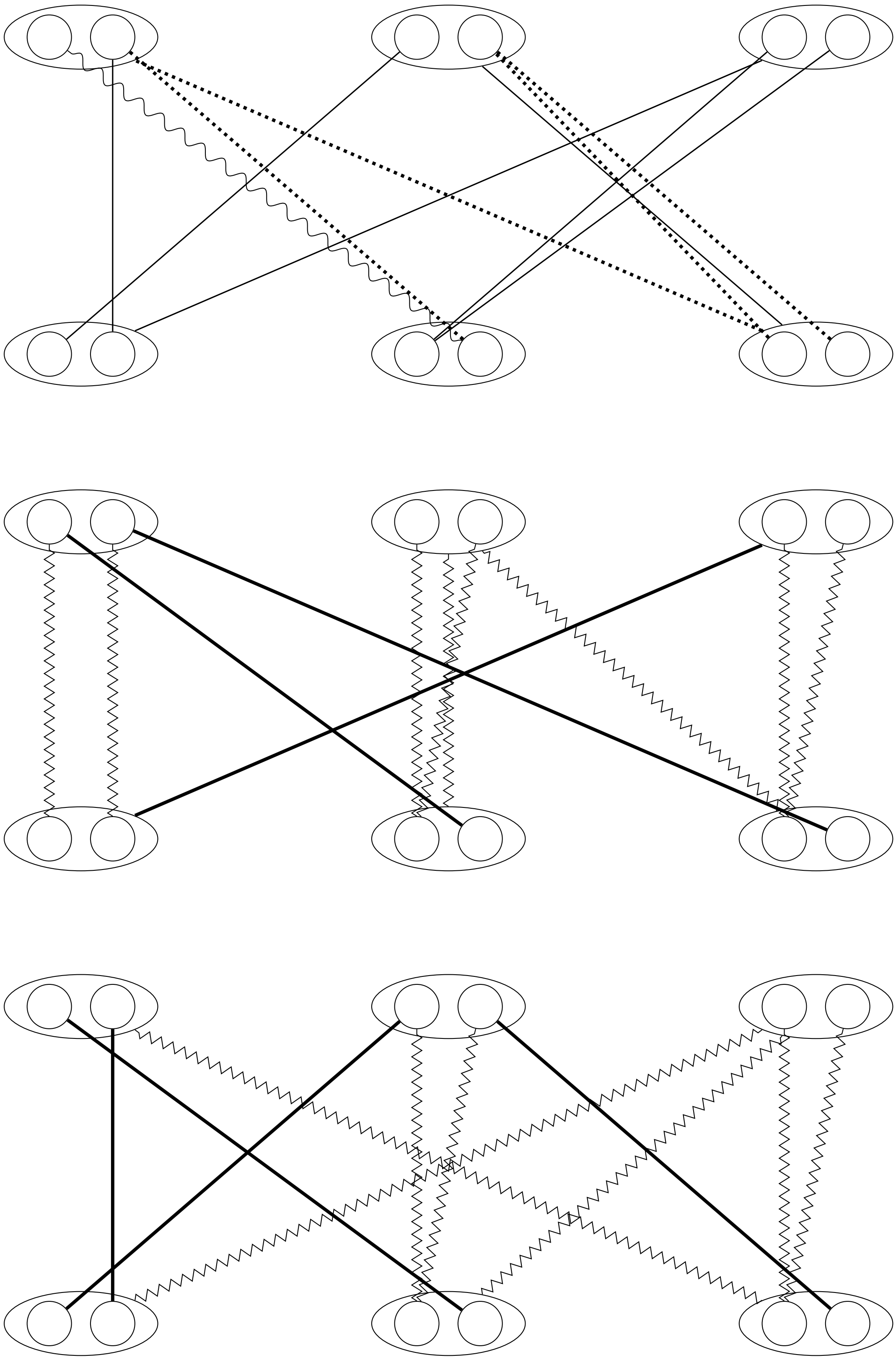
A.2. 3 Hyperedges

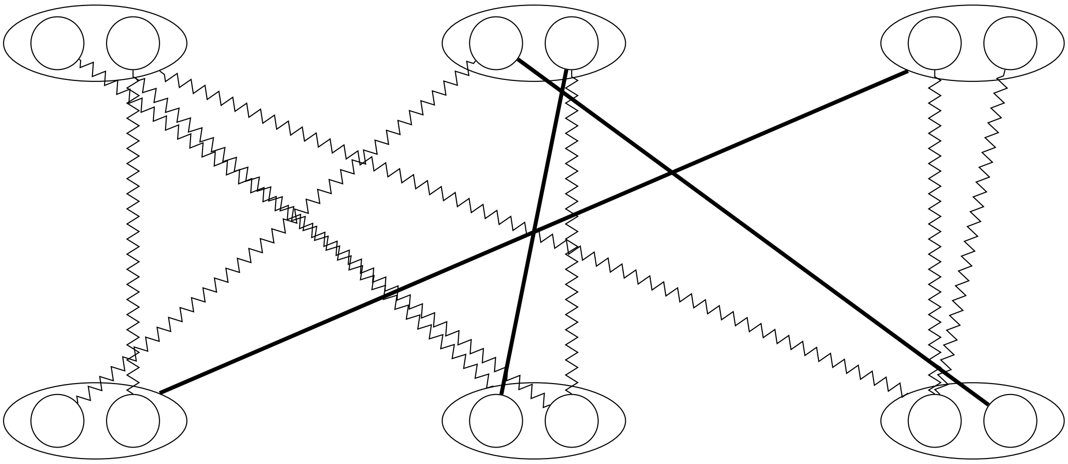


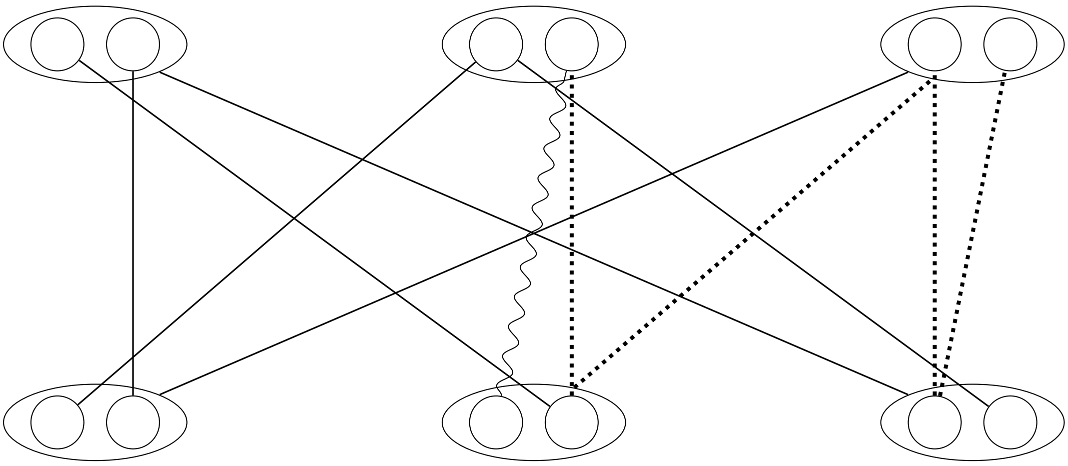


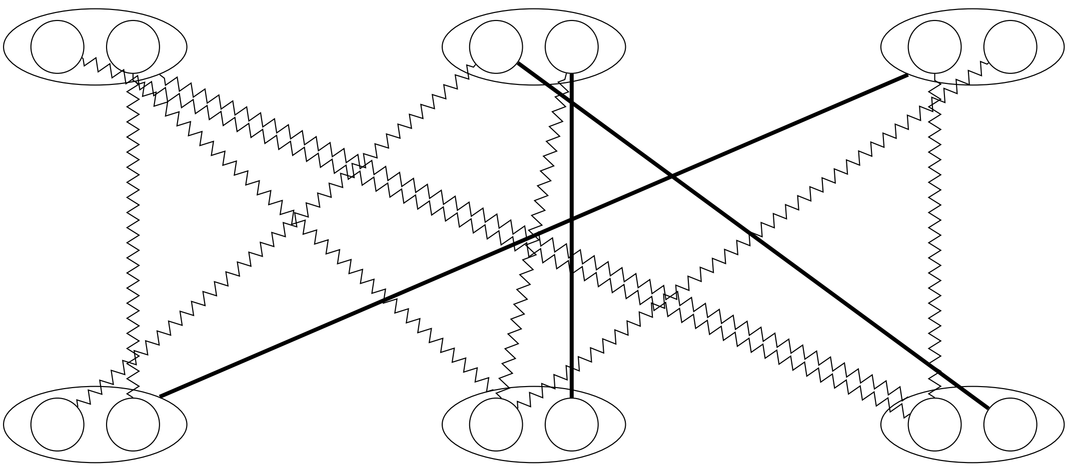


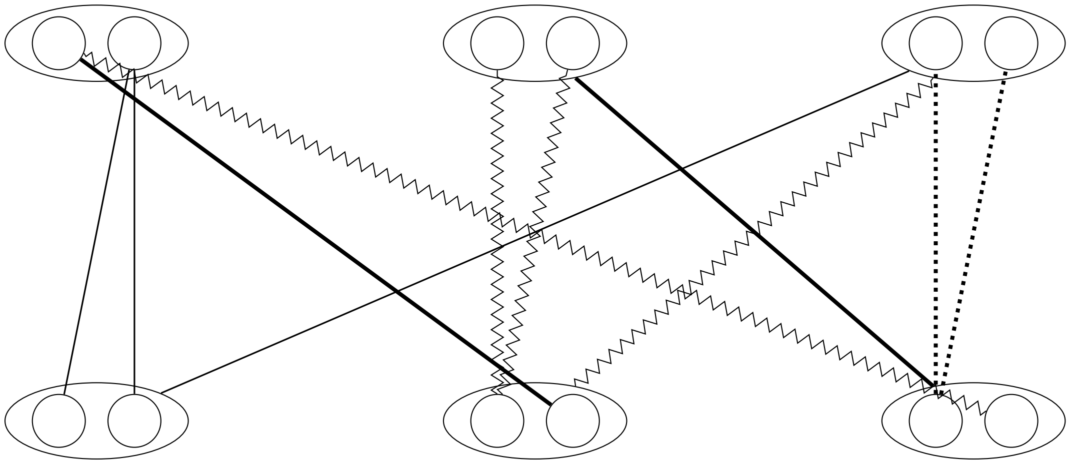




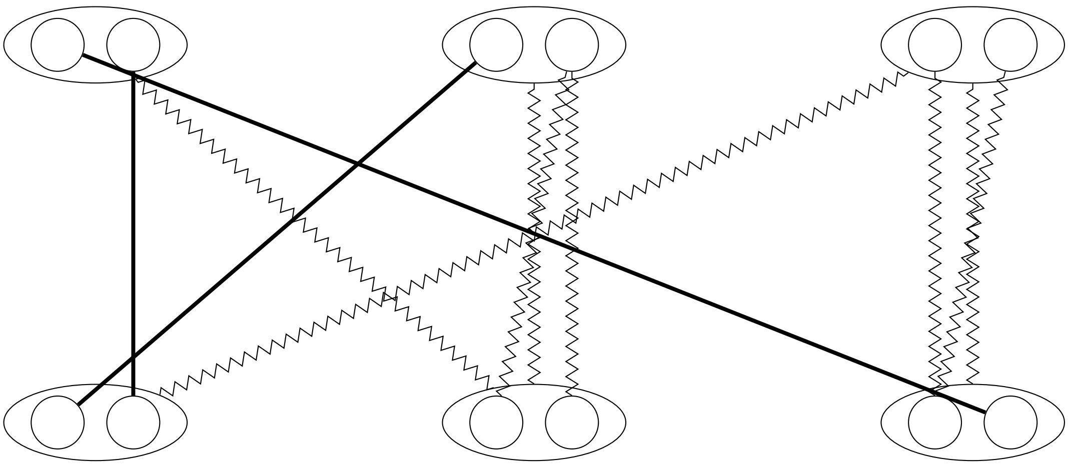








A.3. 4 Hyperedges



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