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Power–Law Type Solutions of Fourth–Order Gravity for Multidimensional Bianchi I Universes

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Contents

Abstract

We study the power-law type solutions of the fourth order field equations derived from a *generic* quadratic Lagrangian density in the case of multidimensional Bianchi I cosmological models. All the solutions of the system of algebraic equations have been found, using computer algebra, from a search of the Groebner bases associated to it. While, in space dimension $d = 3$, the Einsteinian Kasner metric is still the most general power-law type solution, for $d > 3$, no solution, other than the Minkowski space-time, is common to the three systems of equations associated with the three contributions to the Lagrangian density. In the case of a pure Riemann-squared contribution (suggested by a recent calculation of the effective action for the heterotic string), the possibility exists to realize a splitting of the d–dimensional space into a $(d-3)$–dimensional internal space and a physical 3–dimensional space, the latter expanding in time as a power bigger than 2 (about 4.5 when $d = 9$).

1 Introduction

The most promising unified field theories of the four basic interactions must, in general, be formulated in a space-time of high dimensionality (up to 11) and their Lagrangians contain terms which are no longer linear in the scalar curvature.
For example, the low-energy limit field theories which are constructed from superstring theory [1] are formulated in a ten-dimensional space-time and their Lagrangians contain non-linear contributions in the different curvature tensors up to the quartic order.
However, there is no general agreement on the precise form of these contributions. The so-called Lanczos or Gauss-Bonnet terms [2] have been repeatedly suggested as the quadratic contributions to be incorporated in the Lagrangians but a recent calculation [3] of the effective action up to the quartic terms for the heterotic string concludes that a pure Riemann-squared contribution is "more natural" than the Gauss-Bonnet combination [2]. Except for bosonic strings, a cubic contribution seems to be ruled out [4] and the quartic term could also be of pure Riemann type [3].
Therefore, in view of these uncertainties [5], it seems more interesting not to focus on a particular form of the non-linear contributions to the action but rather study the most general non-linear Lagrangian, bearing in mind that most often it will lead to fourth-order field equations. Of course such a task is hopeless unless one suitably restricts its scope.
Here, we limit ourselves to the investigation of the *generic* quadratic contributions and to consider the case of the multidimensional Bianchi I cosmological models. More explicitly, we investigate *power-law type* solutions of the corresponding field equations. The interest of these solutions (which are manifestly not the most general solutions) is due to the key rôle they play in the investigation of the most general solution near a space-like singularity in general relativistic cosmology .
Some of the solutions of these equations have already been found in [6] but the present study is considerably more general. The various systems of algebraic equations we have considered were largely solved using the computer algebra package GROEBNER [7] and some improvement of it which takes into account the symmetry of the equations. Many of the solutions which were found were also checked by hand calculation. The role of the dimensionality of the space has been carefully investigated. The main results we get are :

i. Space dimension $d = 3$ is *special.* The Kasner solution turns out to be the most general power-law type solution to the generic quadratic Lagrangian. This solution does no longer exist for all $d > 3$.

ii. When $d > 3$, the most general solution to the generic Lagrangian is flat Minkowski space-time.

iii. When the quadratic Lagrangian reduces to the Riemann-squared contribution, **all** solutions have been obtained from the Groebner bases calculation done for every dimension up to $d = 10$. We found that some of them, in a pure geometrical way, realize the splitting of the d–dimensional space into a 3–dimensional space and a $(d-3)$–dimensional (internal) space. Particularly interesting is the fact that the exponents corresponding to the coordinates of the 3–dimensional space systematically appear to be **greater than 2**. This behaviour gives rise to a power-law inflation very popular nowadays in the context of the extended inflation proposal [24] which is, here, of purely geometrical origin.

2 The Field Equations for Vacuum Bianchi I Models

We give here the algebraic field equations corresponding to the power-law type solutions of a d-dimensional Bianchi I cosmological model whose metric is given by

$$ds^2 = -dt^2 + \sum_{i=1}^{d} t^{2p_i} (dx^i)^2 \tag{1}$$

where the p_i's are constants to be determined.
Near the cosmological singularity, we choose to consider that the leading terms are the generic quadratic Lagrangian density so that the action S_G gets the expression:

$$S_G = \int (\alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu} + \gamma R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}) \sqrt{-g} d^4x \tag{2}$$

where α,β and γ are constants and where R,$R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ are the scalar curvature, the Ricci and the Riemann tensors of the $(d+1)$–dimensional space-time. The corresponding field equations obtained after a variation with respect to the metric tensor g can be written as a system of algebraic equations for the p_i's. They have been obtained from a combined use of the computer algebra packages EXCALC [9] and COMPACT [11] (see also DERUELLE [6]). Separating the contributions of the three terms in the action we get

$$G_{00}{}^R = -\frac{1}{2}(a_2 + a_1{}^2 - 2a_1)(3a_2 - a_1{}^2 + 6a_1), \tag{3}$$

$$G_{ii}{}^R = (a_2 + a_1{}^2 - 2a_1)(4p_i(3 - a_1) + a_2 + a_1{}^2 - 10a_1 + 24). \tag{3a}$$

$$G_{00}{}^{RIC} = -\frac{3}{2}a_2{}^2 + \frac{1}{2}a_2a_1{}^2 - a_1^3 - a_2a_1 + \frac{3}{2}a_2 + \frac{3}{2}a_1^2, \tag{4a}$$

$$G_{ii}{}^{RIC} = 4p_i(3 - a_1)(a_2 - 1) + a_2^2 + a_2a_1^2 - 2a_1{}^3 - 6a_2a_1 + 7a_2 \\ +11a_1^2 - 12a_1. \tag{4b}$$

$$G_{00}{}^{RIE} = -3a_4 - 3a_2^2 + 4a_3a_1 - 4a_2a_1 + 6a_2, \tag{5a}$$

$$G_{ii}{}^{RIE} = 8p_i(3 - a_1)(p_i^2 - p_ia_1 + a_1 + a_2 - 2) + 2a_4 + 2a_2^2 \\ -8a_3 + 4a_2. \tag{5b}$$

with

$$a_k = \sum_{i=1}^{d} p_i^k \tag{6}$$

3 Solutions of the Polynomial Field Equations

The complete set of solutions of the three preceding systems of polynomial equations has been infered from the calculation of the associated Groebner bases [7]. The solutions corresponding to the pure scalar curvature contribution and those corresponding to the pure Ricci tensor contributions have been partly discussed [6]. The calculation of the Groebner bases as well as further hand calculations have allowed us to complete it. For the pure Riemann-tensor contribution nothing was known up to now. The calculation of the Groebner bases shows that there exit only a *discrete* set of solutions (the ideal is of dimension zero) for all $d > 3$. In that case the technique is particularly helpful because the knowledge of the Groebner bases is sufficient to find the solutions explicitly in a trivial way. First, we present the Groebner bases method, next we discuss the three sets of equations successively [10].

3.1 Groebner bases and Symmetries

The REDUCE package GROEBNER is based on the following considerations. The solutions of a system of polynomial equations $P_j(x) = 0, j = 1,\dots,d, x \in \mathbb{C}^n$ are the zeros of the ideal $\mathcal{A} = (P_1,\dots,P_d)$ spanned by these polynomials. The Buchberger algorithm [14] computes from the given polynomials P_j a special ideal basis, which has the solutions of the given system as common zeros as well. These solutions are easier to find. In particular, if there are only finitely many zeros, they can be computed directly from the Groebner basis [15]. If there is a factorizing polynomial $Q = Q_1Q_2$ in $\mathcal{A}$, then the union of the zeros of the subideals $\mathcal{A}_i = (Q_i, P_1,\dots,P_d)$, $i = 1,2$ gives the zeros of $\mathcal{A}$ [8]. GROEBNER is an implementation of the Buchberger algorithm combined with an algorithm for the detection of factorizing polynomials such that the union of the determined Groebner bases of the subideals contains all solutions of the original system.

In case of symmetries in the given problem, factorizing polynomials are computed easily before using GROEBNER and computations for some subideals are avoided. A permutation of variables in the system (16) is just a permutation of equations. Thus one solution gives by permutation a lot of others. For a given system $P_j(x) = 0, j = 1,\dots,d$ with this property consider the polynomial $Q(x) = (P(x) - P(tx))$, where $P = P_j$ for one j and t is a permutation of two variables x_i, x_k. Then $Q(x) = Q_1(x)Q_2(x)$ is either zero, linear or has a linear factor $(x_i - x_k)$ ([16]). Several factorizing polynomials lead to subideals

$$\mathcal{A}^i = (Q^1_{i_1},\dots,Q^m_{i_m}, P_1,\dots,P_d), \quad i_k \in \{1,2\},$$

such that the union of the zeros of the ideals $\mathcal{A}^i$ gives the zeros of the system. If a permutation of variables in $\mathcal{A}^i$ gives another subideal contained in $\mathcal{A}^j$ (this is the case, if

$$Q^k_{i_k}(tx) \in \operatorname{span}(Q^1_{j_1},\dots,Q^m_{j_m}) \quad \forall k = 1,\dots,m \tag{7}$$

holds), then the zeros of $\mathcal{A}^j$ are a subset of the zeros of $\mathcal{A}^i$ modulo a permutation. Thus the GROEBNER computation for $\mathcal{A}^j$ is not necessary. As the automatic check (7) is time-consuming, only some factorizing polynomials were taken into account. Because the GROEBNER calculation leads to a further splitting of the subsystems, there are still solutions which are equal up to permutations. But these are only few in comparision to those permuted solutions which are avoided by symmetry. Because of the automated process, we are sure to have computed all solutions. The solution of (16) for $d = 6,\dots,10$ couldn't have been done without this approach. A general description and the details of implementation are found in [16].

3.2 Solutions corresponding to a pure R^2 contribution

The solutions can be given independently of the space-time dimension. They correspond of course to real p_i's.

- $p_i = 0$ for all i's. It corresponds to the Minkowski space-time.

- $p_i = 0$ for all i's except one (say $i = i_0$) for which $p_{i0} = 1$. It corresponds again to Minkowski space-time as a very simple change of coordinates shows [12].

- A *continuum* of solutions given by

$$a_1{}^2 - 2a_1 + a_2 = 0 \tag{8}$$

with

$$0 \le a_2 \le 1.$$

A first particular solution is

$$a_1 = 1, \qquad a_2 = 1. \tag{9}$$

This is the well-known Kasner-type solution [13] for the d-dimensional vacuum general relativistic Bianchi I models. It is the "archetype" of *anisotropic* solutions, very important in all issues concerned with the behaviour of a cosmological model near the singularity. A second one is the isotropic solution

$$p_i = p = \frac{2}{d+1} \tag{10}$$

- When $d \neq 3$, another isotropic solution is found

$$p_i = \frac{6}{d-3} \tag{11}$$

3.3 Solutions corresponding to a pure $R^{\mu\nu}R_{\mu\nu}$ contribution

- The same solutions as in the preceding case associated with Minkowski space-time.

- The Kasner solution

$$a_1 = 1, \qquad a_2 = 1 \tag{12}$$

- Isotropic solutions given by

$$p_i = p = \sqrt{d+1}\ \frac{\sqrt{d\,(d+1)} \pm \sqrt{d^2 - 2d + 9}}{\sqrt{d}\,(d-3)} \tag{13}$$

when $d \neq 3$ and

$$p_i = p = -\frac{1}{2} \tag{14}$$

when $d = 3$.

3.4 Solutions corresponding to a pure $R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ contribution

If we substitute the value of a_4 that we get from the expression of G_{00}^{RIE} given above into G_{ii}^{RIE}, we get the system of equations:

$$\frac{8}{3}(3 - a_1)\,[\,3p_i(p_i^2 - p_i a_1 + a_1 + a_2 - 2) + a_2 - a_3\,] = 0 \tag{15}$$

We consider, first of all, the case $a_1 \neq 3$ and solve the system

$$3p_i(p_i^2 - p_i a_1 + a_1 + a_2 - 2) + a_2 - a_3 = 0. \tag{16}$$

with $i = 1 \ldots d$.
All the solutions of this system of equations have been determined with the help of the Groebner package developed in Berlin [7] which is based on the Buchberger algorithm [14], on factorization and on the exploitation of the symmetry properties [16] of the equations. The exploitation of the symmetry properties of the system under cyclic permutations of the p_i's and under the interchange of two of them turned out to be essential to be able to complete the calculations for dimensions up to $d = 10$. Most of the solutions are complex and, therefore must be rejected. The number of physical solutions is small and depends on d . Again, the case $d = 3$ is special. There , there are only three solutions : Minkowski space-time, the Kasner solution ($a_1 = 1 = a_2$) and the isotropic solution $p_i = p = 1/2$. When $d > 3$, the solutions are qualitatively similar. We discuss only the case $d = 9$ in detail because it corresponds to the 10-dimensional space-time characteristic of heterotic superstring theories.

1. Minkowski space-time:

$$p_1 = p_2 = \ldots = p_9 = 0$$

2. Disguised Minkowski space-time:

$$
\begin{aligned}
p_1 = 1, p_2 = \ \dots \ = p_9 = 0 \\
p_1 = 0, p_2 = 1, p_3 = \dots = 0 \\
\vdots \\
p_1 = \ \dots \ = p_8 = 0, p_9 = 1
\end{aligned}
$$

3. Isotropic solutions:

$$
\begin{aligned}
p_1 = \dots = p_9 = 0.1716 \\
p_1 = \dots = p_9 = 5.828
\end{aligned}
$$

In fact, the general expression for these solutions is given by

$$
p = \frac{2d \pm \sqrt{4d^2 - 6d + 18}}{d - 3} \tag{17}
$$

4. All the other solutions are *anisotropic*. We list them

4.a Space dimensions are subdivided in a $8+1$ way and

$$
\begin{aligned}
p_1 = \dots = p_8 = 0.809, \qquad & p_9 = 4.272 \\
p_1 = \dots = p_8 = 5.692, \qquad & p_9 = 1.768
\end{aligned}
$$

4.b Space dimensions are subdivided in a $7+2$ way and

$$
\begin{aligned}
p_1 = \dots = p_7 = 0.952, \qquad & p_8 = p_9 = 4.318 \\
p_1 = \dots = p_7 = 5.429, \qquad & p_8 = p_9 = 1.646
\end{aligned}
$$

4.c Space dimensions are subdivided in a $6+3$ way and

$$
\begin{aligned}
p_1 = \dots = p_6 = 1.102, \qquad & p_7 = p_8 = p_9 = 4.535 \\
p_1 = \dots = p_6 = 5.216, \qquad & p_7 = p_8 = p_9 = 1.518
\end{aligned}
$$

4.d Space dimensions are subdivided in a $5+4$ way and

$$
\begin{aligned}
p_1 = \dots = p_5 = 1.248, \qquad & p_6 = p_7 = p_8 = p_9 = 4.766 \\
p_1 = \dots = p_5 = 4.996, \qquad & p_6 = p_7 = p_8 = p_9 = 1.382
\end{aligned}
$$

The numeric evaluation of the roots has been obtained using the ROOT package of KAMENY [17]. In some cases, because the polynomials have big alternate coefficients and some of the roots were closed to 0, it has been necessary to further check the sign. The Kameny package exploits the Sturm algorithm to give us the number of roots in a given interval and allowed us to verify that all real roots are *positive*. In our case, the knowledge of the sign is crucial for the physical interpretation !
We consider next the case $a_1 = 3$. From the expression of G_{ii}^{RIE} we get

$$a_4 - 4a_3 + a_2(a_2 + 2) = 0 \tag{18}$$

We can show that, in disagreement with solution (45) proposed in [6] there does not exist any set of real p_i's which satisfy the above equation together with $a_1 = 3$.

Some special solutions of the system (16) are easily verified using symmetry reduction. This well-known method works in a special case in the following way. Assume a solution of type

$$(p_1, \ldots, p_1, p_2, \ldots, p_2) \in \mathbb{C}^{d_1+d_2} \quad \text{and} \quad d_1 + d_2 = d\,, \tag{19}$$

where p_1, p_2 occur d_1, d_2 times, respectively. Substitution into (16) shows that two equations are left

$$\begin{aligned} 0 &= 3p_1(p_1^2 - p_1 a_1 + a_1 + a_2 - 2) + a_2 - a_3\,, \\ 0 &= 3p_2(p_2^2 - p_2 a_1 + a_1 + a_2 - 2) + a_2 - a_3\,; \end{aligned}$$

with

$$a_k = d_1 \cdot p_1^k + d_2 \cdot p_2^k\,, \quad k = 1, 2, 3\,.$$

Because these two equations are invariant when interchanging (p_1, d_1) with (p_2, d_2) and since, d_1, d_2 appear in invariant terms only, the difference of both equations factors $(p_1 - p_2)$. So the special solutions of (16) of type (19) with $p_1 \neq p_2$ were verified for arbitrary d.

Note that this reduction method helps to find special solutions of a system of equations with symmetry. But it is not sufficient to compute all arbitrary solutions.

4 The Generic Quadratic Lagrangian

As the previous analysis shows the dimension $d = 3$ is extremely particular. When α, β and γ are ***arbitrary***, three solutions exist:

- Minkowski space-time
- Kasner solution
- the isotropic solution $p_i = p = 1/2$

Note that, in accordance with Gauss-Bonnet theorem, only two of the three contributions are independent.
When $d > 3$, the only solution left is the vacuum Minkowski space-time which is without physical interest in the present cosmological context. In particular, Kasner space-time is *no more a solution of the generic Lagrangian.*

5 Cosmological Discussion of the Power-Law Type Solutions of the Field Equations

It is clear, first of all, that the power-law solutions of the fourth-order gravity equations for Bianchi I universes considered here are particular solutions because they do not posses the required number of degrees of freedom [18]. For instance, in the case of the R^2 theory of gravity, the general solution of the corresponding field equations is not of the power-law type, as shown by BUCHDAL [19] for $d = 3$ and by DERUELLE and SPINDEL [20] for a general value of d ($d \neq 3$). This is, by the way, the easiest theory with a quadratic Lagrangian to study because of its conformal equivalence with general relativity with a scalar field [21]. Most of the cosmological studies concerned with the R^2 theory of gravity deal with the possibility of generating inflation without introducing ad hoc scalar fields [22].
However, the search for power-law solutions is not without interest in the study of the most general approach -chaotic or non–chaotic- to a space-like singularity. The important rôle of the Kasner solutions in Einsteinian gravity, in this context, is well-known [23].
For dimension $d = 3$, the Kasner solution appears as the most general *anisotropic* power-law type solution of the general quadratic theory of gravity for vacuum Bianchi I models. An isotropic solution with exponents $1/2$ is also found, simulating the behaviour of Einsteinian gravity of a Friedmann model filled with pure radiation.
For dimensions $d \neq 3$, the generic behaviour is no more of Kasner type, since the field equations derived from the Riemann-squared part of the Lagrangian density does not admit it any more and since, moreover, the Gauss-Bonnet theorem no longer holds.
The Riemann-squared case is particularly interesting, in view of the conclusion of

a recent calculation of the effective action for the heterotic string that it should be the natural quadratic contribution to consider [3].
Two characteristics of the solutions of the system (5) are worth mentioning :

(i) For each space dimension d, all types of splitting of the exponents p_i into two distinct groups are possible; the values of the exponents are quite distinct and are strictly positive. Moreover, only discrete sets of solutions are possible for each d. Particularly interesting is the splitting into $3 + (d - 3)$, possible for any d with the highest value of the exponents pertaining to the 3-space . For instance, when $d = 9$, as described above, the 3-space gets expanded in time according to the law $t^{4.535}$ while for the six other space coordinates the expansion rate is much slower i.e. as $t^{1.1024}$. The mechanism of spontaneous compactification, which ideally would consist in a splitting of space with an expansion of 3-space and a contraction of $(d - 3)$ space is only marginally realized.

(ii) Recent discussions on theories of inflation have focused attention on the possibility of realizing the " old inflation " program in theories in which a scalar field is implicitly present, like the Brans-Dicke theory [24]. These models do not give rise to a phase of exponential expansion but rather to a " power-law " inflation in the sense that the scale factor of the universe grows as $R(t) = t^p$. A value $p \geq 2$ seems sufficient to solve the cosmological conundrums. It is especially worth noticing that the Riemann-squared contribution to the Lagrangian density gives rise for any $d > 3$ to a value greater than 2 of the exponent of the power-law scale expansion solution for the physical space. However, here, the origin of this value of the exponent is exclusively *geometrical.*

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