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An Approximation Result for Matchings in Partitioned Hypergraphs

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We investigate the matching and perfect matching polytopes of hypergraphs having a special structure, which we call partitioned hypergraphs. We show that the integrality gap of the standard LP-relaxation is at most $2\sqrt{d}$ for partitioned hypergraphs with parts of size $\leq d$. Furthermore, we show that this bound cannot be improved to $\mathscr{O}(d^{0.5-\varepsilon})$.

1 Introduction

The matching problem in hypergraphs is equivalent to the set packing problem, which is well known to be $\mathscr{N}\mathscr{P}$-hard, and which does not admit a constant factor approximation algorithm. There exists a lot of work characterizing classes of hypergraphs for which the matching problem can be solved in polynomial time. The simplest example are bipartite graphs for which the canonical LP-Relaxation of the matching problem is integral. A further class are balanced hypergraphs.

We consider in this article a special class of so-called partitioned hypergraphs, in which the hyperedges have a special structure, see also [2]. The canonical LP-Relaxation of the matching problem need not to be integral for this class of hypergraphs. Nevertheless, we can bound the integrality gap by the square root of the maximum part size.

2 Definitions

In this section we introduce some basic definitions and notations that we use in the remainder. First, we give two definitions that show the close connection between matchings in hypergraphs and the set packing problem, and perfect matching and the set partitioning problem, respectively.

Every hypergraph can be represented by a $0,1$ matrix in the following way:

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Definition. Let $H = (V, E)$ be a hypergraph, The *incidence matrix* of H is the matrix $A = (a_{v,e})_{v \in V, e \in E} \in \{0,1\}^{V \times E}$ defined by

$$a_{v,e} = \begin{cases} 1, \text{ if } v \in e \\ 0, \text{ else.} \end{cases}$$

Now, we define the four polytopes that we investigate in the next section.

Definition. Let $H = (V, E)$ be a hypergraph the *matching polytope*, the *fractional matching polytope*, the *perfect matching polytope*, and the *fractional perfect matching polytope* associated with H are defined by:

$$\begin{aligned} \mathrm{P}_M(H) &= \mathrm{conv}(\{x \in \{0,1\}^E | Ax \leq 1\}) \\ \mathrm{P}_M^{\mathrm{LP}}(H) &= \mathrm{conv}(\{x \in \mathbb{R}^E | Ax \leq 1,\ x \geq 0\}) \\ \mathrm{P}_{PM}(H) &= \mathrm{conv}(\{x \in \{0,1\}^E | Ax = 1\}) \\ \mathrm{P}_{PM}^{\mathrm{LP}}(H) &= \mathrm{conv}(\{x \in \mathbb{R}^E | Ax = 1,\ x \geq 0\}). \end{aligned}$$

The extreme points of $\mathrm{P}_M(H)$ are exactly the incidence vectors of matchings in H. So, finding a maximum weight matching is equivalent to optimizing over $\mathrm{P}_M(H)$ which is hard. However, we can optimize over $\mathrm{P}_M^{\mathrm{LP}}(H)$ to obtain an upper bound. Therefore, if we can bound the integrality gap of $\mathrm{P}_M^{\mathrm{LP}}(H)$ we obtain an approximation result for the maximum weight of a matching in H.

In [2] Borndörfer and Heismann introduced the hypergraph assignment problem which is a generalization of the assignment problem. The hypergraph assignment problem can also be seen as a perfect matching problem in a hypergraph having the following special structure:

Definition. Let $H = (V \cup W, E)$ be a hypergraph with $|V| = |W|$, $V \cap W = \emptyset$, and $|e \cap V| = |e \cap W|$ for all $e \in E$. A nonempty set $P \subseteq V$ or $P \subseteq W$ is called a *part* of H if for all $e \in E$ either $e \cap V \subseteq P$ or $(e \cap V) \cap P = \emptyset$ holds or in the case $P \subseteq W$ either $e \cap W \subseteq P$ or $(e \cap W) \cap P = \emptyset$ holds.

H is a *partitioned hypergraph* with *maximum part size* d if there are disjoint parts $P_1, \ldots, P_r \subseteq V$ and $Q_1, \ldots, Q_s \subseteq W$ that form a partition of V and W, respectively, and $|P_i| \leq d, |Q_j| \leq d$ for $1 \leq i \leq r, 1 \leq j \leq s$.

It is easy to see that the intersection of two parts is empty or again a part. So, there exists a unique finest partition $P_1, \ldots, P_r$ of V and a unique finest partition $Q_1, \ldots, Q_s$ of W into parts. We always assume that we have a partitioned hypergraph with its finest partition into parts. Under this assumption the part size of a partitioned hypergraphs is the maximum size of one of its parts.

For example, Figure 2 shows a partitioned hypergraph with maximum part size three. The finest partition is $\{A, B, C\}$, $\{D, E\}, \{F\}$ and $\{G, H, I\}, \{J, K, L\}$.

We also define the following “complete” partitioned hypergraph with parts of size two:

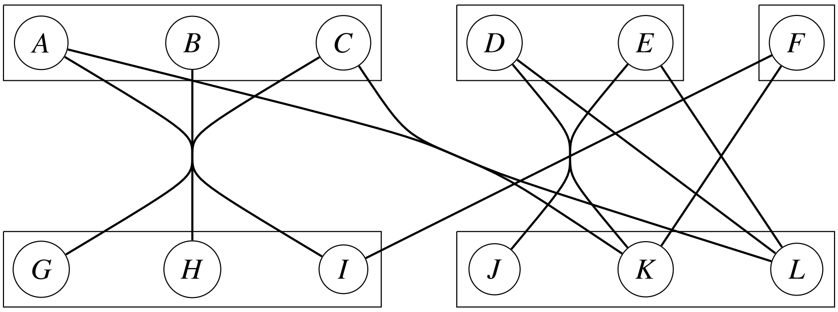

Figure 1: A partitioned hypergraph with maximum part size three

Definition. Let $n \in \mathbb{N}$ be an even number. The partitioned hypergraph D_n consists of two disjoint vertex sets $V_n = \{v_1, \dots, v_n\}$ and $W_n = \{w_1, \dots, w_n\}$. Each of the two vertex sets is partitioned into $\frac{n}{2}$ parts of size two, say $V_n^i = \{v_{2i-1}, v_{2i}\}$, $W_n^i = \{w_{2i-1}, w_{2i}\}$ for all $1 \leq i \leq \frac{n}{2}$. The set of hyperedges E_n of D_n consists of n^2 edges $\{v_i, w_j\}$ for all $1 \leq i, j \leq n$ and $\frac{n^2}{4}$ hyperedges of the form $V_n^i \cup W_n^j$ for all $1 \leq i, j \leq \frac{n}{2}$.

3 Integrality Gap

Füredi, Kahn, and Seymour show in [4] that the integrality gap of

$$\begin{aligned} &\max w^t x \\ &\text{subject to } x \in \mathrm{P}_M^{\mathrm{LP}}(H) \end{aligned} \tag{1}$$

is at most $k - 1 + \frac{1}{k}$ for k-uniform hypergraphs. For k-partite hypergraphs the result can be strengthened to $k - 1$. The proofs of [4] are non-algorithmic, however, in [3] an iterative rounding algorithm with approximation factor $k - 1$ is given for the maximum weight matching problem in k-partite hypergraphs. For the analysis of their algorithm Chan and Lau consider the following linear program for fixed degree bounds $0 \leq B_v \leq 1$:

$$\begin{aligned} &\max w^t x \\ &\text{subject to } x(\delta(v)) \leq B_v \ \forall v \in V(H) \\ &\qquad x_e \geq 0 \ \forall e \in E(H) \end{aligned} \tag{2}$$

Let $N[e] := \{e' : e \cap e' \neq \emptyset\}$ be the set of all hyperedges intersecting e. The crucial point of their proof is that for every extreme point x of (2) with $x > 0$ there exists a hyperedge $e \in E(H)$ with $x(N[e]) \leq k - 1$. The further analysis of the algorithm in [3] does not use the k-partiteness of the hypergraph. If we can show that for every extreme point x with $x > 0$ there exists a hyperedge $e \in E(H)$ with $x(N[e]) \leq \alpha$ for H in some class $\mathscr{C}$ of hypergraphs, then the result of [3] gives an α-approximation algorithm for the weighted matching problem in $\mathscr{C}$.

We can proof the following bound for partitioned hypergraph:

Lemma 1. *Let H be a partitioned hypergraph with maximum part size d and x be an extreme point of (2) with $x_e > 0$ for all $e \in E(H)$. There exists a hyperedge $e^* \in E(H)$ with $x(N[e^*]) \leq 2\sqrt{d}$.*

Proof. 1. Case: There exists a hyperedge e^* of size less than $2\sqrt{d}$. Then

$$\sum_{e \in N[e^*]} x(e) \leq \sum_{v \in e^*} \sum_{e: v \in e} x(e) \leq |e^*| < 2\sqrt{d}. \tag{3}$$

2. Case: $|e| \geq 2\sqrt{d}$ for all $e \in E(H)$. We choose $e^* \in E$ arbitrarily. Let P and P' be the two parts of H such that $e^* \subseteq P \cup P'$. Summing over all inequalities $x(\delta(v)) \leq 1$ for $v \in P$ gives

$$d \geq |P| \geq \sum_{v \in P} \sum_{e: e \in v} x(e) = \sum_{e \in \delta(P)} |e \cap P| x(e) = \sum_{e \in \delta(P)} \frac{|e|}{2} x(e) \geq \sqrt{d} x(e), \tag{4}$$

and the same inequality holds for $e \in \delta(P')$. Thus we get

$$\sum_{e \in N[e^*]} x(e) \leq \sum_{e \in \delta(P)} x(e) + \sum_{e \in \delta(P')} x(e) \leq 2\sqrt{d}. \tag{5}$$

□

Now, we can proof that (1) has an integrality gap $\leq 2\sqrt{d}$ for partitioned hypergraphs with maximum part size d. The proof is based on the ideas used in [3] for the analysis of the k-dimensional matching algorithm.

Theorem 2. *The multiplicative integrality gap of (1) is at most $2\sqrt{d}$ for a partitioned hypergraph H with maximum part size d.*

Proof. Let x be an extreme point of (1). We have to show that there exists a matching M of H such that $w^t x \leq 2\sqrt{d} \times w(M)$.

We use induction on the number of hyperedges $e \in E(H)$ with positive weight. If $w(e) \leq 0$ for all hyperedges $e \in \mathscr{E}$ the claim trivially holds. Otherwise, there exists a hyperedge e^* of positive weight with $x(N[e^*]) \leq 2\sqrt{d}$.

Define a weight function w^1 by $w^1(e) := w(e^*)$ for all $e \in N[e^*]$ and $w^1(e) := 0$ for all other $e \in E(H)$. Furthermore, set $w^2(e) := w(e) - w^1(e)$ for all $e \in E(H)$. The weight function w^2 has fewer hyperedges with positive weight then w. By induction there exists a matching M' of H with $(w^2)^t x \leq 2\sqrt{d} \times w^2(M')$. If $M' \cup \{e^*\}$ is a matching we set $M := M' \cup \{e^*\}$, otherwise we set $M := M'$. In both cases, we have $w^2(M) = w^2(M')$ and $w^1(M) = w(e^*)$, because $w^2(e^*) = 0$ and $N[e^*] \cap M \neq \emptyset$. It follows that:

$$\begin{aligned} 2\sqrt{d} w(M) &= 2\sqrt{d} w^2(M) + 2\sqrt{d} w^1(M) = 2\sqrt{d} w^2(M') + 2\sqrt{d} w(e^*) \\ &\geq (w^2)^t x + w(e^*) x(N[e^*]) = (w^2)^t x + (w^1)^t x = w^t x. \end{aligned}$$

□

For general hypergraphs with hyperedges of size k Hazan, Safra and Schwartz proved in [7] that there is no $\mathscr{O}(\frac{k}{\ln k})$ approximation algorithm for the maximum matching problem unless $\mathscr{P} = \mathscr{N}\mathscr{P}$. If the maximum part size of a partitioned hypergraph is bounded by ck for some constant $c \in \mathbb{Q}_+$ Theorem 2. yields a $\mathscr{O}(\sqrt{k})$-approximation algorithm which is better than $\mathscr{O}(\frac{k}{\ln k})$.

Furthermore, there exists a $2\sqrt{|V(H)|}$-approximation algorithm for the maximum weight matching problem in hypergraphs with hyperedges of unbounded size (see [5]). This approximation factor cannot be improved to $\mathscr{O}(|V(H)|^{\frac{1}{2}-\varepsilon})$ in the unweighted case (see [6]).

Every hypergraph H can be transformed into a partitioned hypergraph H_P with maximum part size $\leq |V(H)|$ by setting $V(H_P) := V(H) \times \{0,1\}$ and $E(H_P) := \{\{(v,0),(v,1) : v \in e\} : e \in E(H)\}$. This shows that (1) cannot have an integrality gap of $\mathscr{O}(d^{\frac{1}{2}-\varepsilon})$. Therefore, the result of Theorem 2. is almost best possible.

Note that a similar result cannot be proved for the perfect matching problem. Even for partitioned hypergraphs with parts of size two the integrality gap is unbounded, see [1].

4 Polyhedral Investigations

We conclude this paper with some general polyhedral results on the matching polytope, the perfect matching polytope, and their fractional variants. We begin with the dimension of these polytopes.

Theorem 3. *$P_M(H)$ and $P_M^{LP}(H)$ have full dimension, i.e. they have dimension $|E_n| = \frac{5}{4}n^2$.*

Proof. $\{\chi_\emptyset\} \cup \{\chi_{\{e\}} | e \in E(H)\}$ is a set of $|E|+1$ affinely independent vectors in $\mathrm{P}_M(H)$ and $\mathrm{P}_M^{\mathrm{LP}}(H)$, so $\mathrm{P}_M(H), \mathrm{P}_M^{\mathrm{LP}}(H)$ have full dimension. □

The dimension of the perfect matching polytope is more difficult to calculate, as it is $\mathscr{N}\mathscr{P}$-hard to decide whether a hypergraph has a perfect matching (i.e. $\mathrm{P}_{PM}(H)$ is non-empty). However, for D_n it is possible to calculate the dimension of the perfect matching polytope and the fractional perfect matching polytope.

Theorem 4. *The dimension of $P_{PM}(D_n)$ and $P_{PM}^{LP}(D_n)$ is $\frac{5}{4}n^2 - 2n + 1$.*

Proof. As every valid equation for $\mathrm{P}_{PM}^{\mathrm{LP}}(D_n)$ is a linear combination of the rows of $Ax = 1$, the dimension of $\mathrm{P}_{PM}^{\mathrm{LP}}(D_n)$ is $|E_n| - \mathrm{rank}(A)$. Let a_e be a column of A corresponding to a hyperedge of the form $V_n^i \cup W_n^j$. Then e is the disjoint union of the two edges $e_1 = \{v_{2i-1}, w_{2i-1}\}$ and $e_2 = \{v_{2i}, w_{2i}\}$ and a_e is the sum of the two column vectors corresponding to e_1 and e_2. So we can delete column a_e from A without changing the rank of A. Doing this for all columns corresponding to hyperedges of size four, shows that the rank of A is the same as the rank of the incidence matrix of $K_{n,n}$ which is $2n-1$. It follows that $\dim(\mathrm{P}_{PM(D_n)}^{\mathrm{LP}}) = |E_n| - 2n + 1 = \frac{5}{4}n^2 - 2n + 1$.

To see that $\dim(\mathrm{P}_{PM}(D_n) = \dim(\mathrm{P}_{PM}^{\mathrm{LP}}(D_n))$, we construct $\frac{5}{4}n^2 - 2n + 2$ affinely independent vectors in $\mathrm{P}_{PM}(D_n)$. First, observe that every fixed hyperedge of size four can be completed to a perfect matching of D_n by adding edges. Clearly, the incidence vectors of these $\frac{n^2}{4}$ perfect matchings are affinely independent. The matching polytope of $K_{n,n}$ has dimension $n^2 - 2n + 1$. Thus, there are $n^2 - 2n + 2$ perfect matchings in $K_{n,n}$ such that their incidence vectors are affinely independent. These vectors can be lifted to vectors in $\mathrm{P}_{PM}(D_n)$ by setting all entries corresponding to hyperedges of size four to 0. The $\frac{n^2}{4}$ first vectors and these $n^2 - 2n + 2$ new vectors are affinely independent. □

Now, we state some results on valid inequalities and facets of the matching polytope and the perfect matching polytope (see [1] for proofs).

Theorem 5. *Every trivial inequality $x_e \geq 0$ defines a facet of $P_M(H)$.*

In the case of the perfect matching polytope it is even difficult to decide when a trivial inequality is facet defining. So we restrict ourselves to the hypergraphs D_n.

Theorem 6. *The trivial inequality $x_e \geq 0$ defines a facet of $P_{PM}(D_n)$*

A clique in a hypergraph is a set $Q \subseteq E$ of hyperedges such that every two elements of Q intersect. Clearly, every matching contains at most one edge from a clique. So $x(Q) \leq 1$ is a valid inequality for $\mathrm{IP}_M(H)$.

Theorem 7. *A clique inequality $x(Q) \leq 1$ defines a facet of $P_M(H)$ if and only if Q is a maximal clique.*

Heismann also generalized the odd set inequalities that are valid for the matching polytope of a graph to valid inequalities for the (perfect) matching polytope of a hypergraph. See [8] and [1] for more details.

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