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REDUCE Package for the Indefinite and Definite Summation

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Abstract

This article describes the REDUCE package ZEILBERG implemented by Gregor Stölting and the author.

The REDUCE package ZEILBERG is a careful implementation of the Gosper¹ and Zeilberger algorithms for indefinite, and definite summation of hypergeometric terms, respectively. An expression a_k is called a hypergeometric term (or closed form), if a_k/a_{k-1} is a rational function with respect to k . Typical hypergeometric terms are ratios of products of powers, factorials, Γ function terms, binomial coefficients, and shifted factorials (Pochhammer symbols) that are integer-linear in their arguments.

1 Gosper Algorithm

The Gosper algorithm [1] is a decision procedure, that decides by algebraic calculations whether a given hypergeometric term a_k has a hypergeometric term antidifference g_k , i. e. $g_k - g_{k-1} = a_k$, and returns g_k if the procedure is successful, in which case we call a_k Gosper-summable. Otherwise no hypergeometric term antidifference exists. Therefore if the Gosper algorithm does not return a closed form solution, it has proved that no such solution exists, an information that may be quite useful and important. The Gosper algorithm is the discrete analogue of the Risch algorithm for integration in terms of elementary functions. Any antidifference is uniquely determined up to a constant, and is denoted by

$$g_k = \sum_k a_k .$$

Finding g_k given a_k is called indefinite summation. The antidifference operator Σ is the inverse of the downward difference operator $\nabla a_k = a_k - a_{k-1}$. There is an analogous summation theory corresponding to the upward difference operator $\Delta a_k = a_{k+1} - a_k$. In case, an antidifference g_k of a_k is known, any sum

$$\sum_{k=m}^n a_k = g_n - g_{m-1}$$

¹The `sum` package contains also a partial implementation of the Gosper algorithm.

can be easily calculated by an evaluation of g at the boundary points like in the integration case. Note, however, that the sum

$$\sum_{k=0}^n \binom{n}{k} \quad (1)$$

e. g. is not of this type as the summand $\binom{n}{k}$ depends on the upper boundary point n explicitly. This is an example of a definite sum that we consider in the next section.

Our package supports the input of powers (`a^k`), factorials (`factorial(k)`), Γ function terms (`gamma(a)`), binomial coefficients (`binomial(n,k)`), shifted factorials (`pochhammer(a,k)=a(a+1)⋯(a+k-1)=Γ(a+k)/Γ(a)`), and partially products (`prod(f,k,k1,k2)`). It takes care of the necessary simplifications, and therefore (in principle) provides you with the solution of the decision problem as long as the memory or time requirements are not too high for the computer used.

2 Zeilberger Algorithm

The (fast) Zeilberger algorithm [9]–[10] deals with the definite summation of hypergeometric terms. Zeilberger's paradigm is to find (and return) a linear homogeneous recurrence equation with polynomial coefficients (called holonomic equation) for an infinite sum

$$s(n) = \sum_{k=-\infty}^{\infty} f(n, k) ,$$

the summation to be understood over all integers k , if $f(n, k)$ is a hypergeometric term with respect to both k and n . The existence of a holonomic recurrence equation for $s(n)$ is then generally guaranteed.

If one is lucky, and the resulting recurrence equation is of first order

$$p(n) s(n-1) + q(n) s(n) = 0 \quad (p, q \text{ polynomials}) ,$$

$s(n)$ turns out to be a hypergeometric term, and a closed form solution can be easily established using a suitable initial value, and is represented by a ratio of Pochhammer or Γ function terms if the polynomials p , and q can be factored.

Zeilberger's algorithm does not guarantee to find the holonomic equation of lowest order, but often it does.

If the resulting recurrence equation has order larger than one, this information can be used for identification purposes: Any other expression satisfying the same recurrence equation, and the same initial values, represents the same function.

Note that a definite sum $\sum_{k=m_1}^{m_2} f(n, k)$ is an infinite sum if $f(n, k) = 0$ for $k < m_1$ and $k > m_2$.

This is often the case, an example of which is the sum (1) considered above, for which the hypergeometric recurrence equation $2s(n-1) - s(n) = 0$ is generated by Zeilberger's algorithm, leading to the closed form solution $s(n) = 2^n$.

Definite summation is trivial if the corresponding indefinite sum is Gosper-summable analogously to the fact that definite integration is trivial as soon as an elementary antiderivative

is known. If this is not the case, the situation is much more difficult, and it is therefore quite remarkable and non-obvious that Zeilberger's method is just a clever application of Gosper's algorithm.

Our implementation is mainly based on [2]. Many more examples can be found in [4], [6], [7], and [8] most of which are contained in the test file `zeilberg.tst`.

3 REDUCE operator GOSPER

The ZEILBERG package must be loaded by:

```
1: load zeilberg;
```

The `gosper` operator is an implementation of the Gosper algorithm.

- `gosper(f,k)` determines a closed form antidifference. If it does not return a closed form solution, then a closed form solution does not exist.
- `gosper(f,k,m,n)` determines

$$\sum_{k=m}^n a_k$$

using Gosper's algorithm. This is only successful if Gosper's algorithm applies.

Example:

```
2: gosper((-1)^(k+1)*(4*k+1)*factorial(2*k)/
  (factorial(k)*4^k*(2*k-1)*factorial(k+1)),k);
```

$$\frac{-(-1)^k \cdot \text{factorial}(2k)}{\text{factorial}(k+1) \cdot \text{factorial}(k)}$$

```
-----
k
4 *factorial(k + 1)*factorial(k)
```

This solves a problem given in SIAM Review ([5], Problem 94-2) where it was asked to determine the infinite sum

$$S = \lim_{n \rightarrow \infty} S_n, \quad S_n = \sum_{k=1}^n \frac{(-1)^{k+1}(4k+1)(2k-1)!!}{2^k(2k-1)(k+1)!},$$

$((2k-1)!! = 1 \cdot 3 \cdots (2k-1) = \frac{(2k)!}{2^k k!})$. The above calculation shows that the summand is Gosper-summable, and the limit $S = 1$ is easily established using Stirling's formula.

The implementation solves further deep and difficult problems some examples of which are:²

²Note that REDUCE Version 3.5 gives the output in terms of Γ functions.

```

3:  gosper(sub(n=n+1,binomial(n,k)^2/binomial(2*n,n))-
        binomial(n,k)^2/binomial(2*n,n),k);

        2
((binomial(n + 1,k) *binomial(2*n,n) - binomial(2*(n + 1),n + 1)*binomial(n,k) )
        2

        *(2*k - 3*n - 1)*(k - n - 1) )/(

        2
        (2*(2*(n + 1) - k)*(2*n + 1)*k - (3*n + 1)*(n + 1) )

        *binomial(2*(n + 1),n + 1)*binomial(2*n,n))

4:  gosper(binomial(k,n),k);

        (k + 1)*binomial(k,n)
        -----
        n + 1
5:  gosper((-25+15*k+18*k^2-2*k^3-k^4)/
        (-23+479*k+613*k^2+137*k^3+53*k^4+5*k^5+k^6),k);

        2
        - (2*k  - 15*k + 8)*k
        -----
        3      2
        23*(k  + 4*k  + 27*k + 23)

```

The Gosper algorithm is not able to give antidifferences depending on the harmonic numbers

$$H_k := \sum_{j=1}^k \frac{1}{j},$$

e. g. $\sum_k H_k = (k+1)(H_{k+1} - 1)$, but, is able to give a proof, instead, for the fact that H_k does not possess a closed form evaluation:

```
6: gosper(1/k,k);
```

***** Gosper algorithm: no closed form solution exists

The following code gives the solution to a summation problem proposed in Gosper's original paper [1]. Let

$$f_k = \prod_{j=1}^k (a + b j + c j^2) \quad \text{and} \quad g_k = \prod_{j=1}^k (e + b j + c j^2).$$

Then a closed form solution for

$$\sum_k \frac{f_{k-1}}{g_k}$$

is found by the definitions

7: operator ff,gg\$

8: let {ff(~k+~m) => ff(k+m-1)*(c*(k+m)^2+b*(k+m)+a) when (fixp(m) and m>0),
ff(~k+~m) => ff(k+m+1)/(c*(k+m+1)^2+b*(k+m+1)+a) when (fixp(m) and m<0)}\$

9: let {gg(~k+~m) => gg(k+m-1)*(c*(k+m)^2+b*(k+m)+e) when (fixp(m) and m>0),
gg(~k+~m) => gg(k+m+1)/(c*(k+m+1)^2+b*(k+m+1)+e) when (fixp(m) and m<0)}\$

and the calculation

10: gosper(ff(k-1)/gg(k),k);

ff(k)

(a - e)*gg(k)

11: clear ff,gg\$

Similarly closed form solutions of $\sum_k \frac{f_{k-m}}{g_k}$ for positive integers m can be obtained, as well as of $\sum_k \frac{f_{k-1}}{g_k}$ for

$$f_k = \prod_{j=1}^k (a + b j + c j^2 + d j^3) \quad \text{and} \quad g_k = \prod_{j=1}^k (e + b j + c j^2 + d j^3)$$

and for analogous expressions of higher degree polynomials.

4 REDUCE operator SUMRECURSION

The **sumrecursion** operator is an implementation of the (fast) Zeilberger algorithm.

- **sumrecursion(f,k,n)** determines a holonomic recurrence equation for $\sum_{k=-\infty}^{\infty} f(n,k)$ with respect to n .
- **sumrecursion(f,k,n,j)** ($j \in \mathbb{N}$) searches only for a holonomic recurrence equation of order j .

A simple example deals with Equation (1)³

12: sumrecursion(binomial(n,k),k,n);

2*sum(n - 1) - sum(n)

The whole hypergeometric database of the Vandermonde, Gauß, Kummer, Saalschütz, Dixon, Clausen and Dougall identities (see [8]), and many more identities, can be obtained using **sumrecursion**. As examples, we consider the difficult cases of Clausen and Dougall:⁴

³Note that with REDUCE Version 3.5 we use the global operator **summ** instead of **sum** to denote the sum.

⁴Note that the latter may need a large amount of computing time.

```

13: summand:=factorial(a+k-1)*factorial(b+k-1)/
    (factorial(k)*factorial(-1/2+a+b+k))*
    factorial(a+n-k-1)*factorial(b+n-k-1)/
    (factorial(n-k)*factorial(-1/2+a+b+n-k))$

14: sumrecursion(summand,k,n);

(2*a + 2*b + 2*n - 1)*(2*a + 2*b + n - 1)*sum(n)*n

- 2*(2*a + n - 1)*(a + b + n - 1)*(2*b + n - 1)*sum(n - 1)

15: summand:=pochhammer(d,k)*pochhammer(1+d/2,k)*
    pochhammer(d+b-a,k)*pochhammer(d+c-a,k)*
    pochhammer(1+a-b-c,k)*pochhammer(n+a,k)*
    pochhammer(-n,k)/(factorial(k)*pochhammer(d/2,k)*
    pochhammer(1+a-b,k)*pochhammer(1+a-c,k)*
    pochhammer(b+c+d-a,k)*pochhammer(1+d-a-n,k)*pochhammer(1+d+n,k))$

16: sumrecursion(summand,k,n);

(2*a - b - c - d + n)*(b + n - 1)*(c + n - 1)*(d + n)*sum(n - 1)

+ (a - b - c - d - n + 1)*(a - b + n)*(a - c + n)*(a - d + n - 1)*sum(n)

```

corresponding to the statements

$${}_4F_3\left(\begin{matrix} a, b, 1/2 - a - b - n, -n \\ 1/2 + a + b, 1 - a - n, 1 - b - n \end{matrix} \middle| 1\right) = \frac{(2a)_n (a+b)_n (2b)_n}{(2a+2b)_n (a)_n (b)_n}$$

and

$${}_7F_6\left(\begin{matrix} d, 1+d/2, d+b-a, d+c-a, 1+a-b-c, n+a, -n \\ d/2, 1+a-b, 1+a-c, b+c+d-a, 1+d-a-n, 1+d+n \end{matrix} \middle| 1\right) \\ = \frac{(d+1)_n (b)_n (c)_n (1+2a-b-c-d)_n}{(a-d)_n (1+a-b)_n (1+a-c)_n (b+c+d-a)_n}$$

(compare next section), respectively.

Other applications of the Zeilberger algorithm are connected with the verification of identities.

To prove the identity

$$\sum_{k=0}^n \binom{n}{k}^3 = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{n},$$

e. g., we may prove that both sums satisfy the same recurrence equation

```

17: sumrecursion(binomial(n,k)^3,k,n);

      2          2          2
8*(n - 1) *sum(n - 2) - sum(n)*n  + (7*n  - 7*n + 2)*sum(n - 1)

```

18: sumrecursion(binomial(n,k)^2*binomial(2*k,n),k,n);

$$8*(n-1)^2 * \text{sum}(n-2) - \text{sum}(n)*n^2 + (7*n^2 - 7*n + 2)*\text{sum}(n-1)$$

and finally check the initial conditions:

19: sum(sub(n=0,binomial(n,k)^3),k,0,0);

1

20: sum(sub(n=0,binomial(n,k)^2*binomial(2*k,n)),k,0,0);

1

21: sum(sub(n=1,binomial(n,k)^3),k,0,1);

2

22: sum(sub(n=1,binomial(n,k)^2*binomial(2*k,n)),k,0,1);

2

5 REDUCE operator HYPERRECURSION

Sums for which the Zeilberger algorithm applies, in general are special cases of the generalized hypergeometric function

$${}_pF_q \left(\begin{matrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{matrix} \middle| x \right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdot (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k k!} x^k$$

with upper parameters $\{a_1, a_2, \dots, a_p\}$, and lower parameters $\{b_1, b_2, \dots, b_q\}$. If a recursion for a generalized hypergeometric function is to be established, you can use the following REDUCE operator:

- **hyperrecursion(upper,lower,x,n)** determines a holonomic recurrence equation with respect to n for ${}_pF_q \left(\begin{matrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{matrix} \middle| x \right)$, where **upper** = $\{a_1, a_2, \dots, a_p\}$ is the list of upper parameters, and **lower** = $\{b_1, b_2, \dots, b_q\}$ is the list of lower parameters depending on n .
- **hyperrecursion(upper,lower,x,n,j)** ($j \in \mathbb{N}$) searches only for a holonomic recurrence equation of order j .

Therefore

23: `hyperrecursion({-n,b},{c},1,n);`

$(n - 1 + c - b) \cdot \text{sum}(n - 1) - (n - 1 + c) \cdot \text{sum}(n)$

establishes the Vandermonde identity

$${}_2F_1\left(\begin{matrix} -n, & b \\ & c \end{matrix} \middle| 1\right) = \frac{(c-b)_n}{(c)_n},$$

whereas

24: `hyperrecursion({d,1+d/2,d+b-a,d+c-a,1+a-b-c,n+a,-n},
 {d/2,1+a-b,1+a-c,b+c+d-a,1+d-a-n,1+d+n},1,n);`

$(2 \cdot a - b - c - d + n) \cdot (b + n - 1) \cdot (c + n - 1) \cdot (d + n) \cdot \text{sum}(n - 1)$
 $+ (a - b - c - d - n + 1) \cdot (a - b + n) \cdot (a - c + n) \cdot (a - d + n - 1) \cdot \text{sum}(n)$

proves Dougall's identity, again.

If a hypergeometric expression is given in hypergeometric notation, then the use of `hyperrecursion` is more natural than the use of `sumrecursion`.

Moreover you may use the REDUCE operator

- `hyperterm(upper,lower,x,k)` that yields the hypergeometric term

$$\frac{(a_1)_k \cdot (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!}$$

with upper parameters `upper= {a1, a2, ..., ap}`, and lower parameters `lower= {b1, b2, ..., bq}`

in connection with hypergeometric terms.

The operator `sumrecursion` can also be used to obtain three-term recurrence equations for systems of orthogonal polynomials with the aid of known hypergeometric representations. By ([3], (2.7.11a)), the discrete Krawtchouk polynomials $k_n^{(p)}(x, N)$ have the hypergeometric representation

$$k_n^{(p)}(x, N) = (-1)^n p^n \binom{N}{n} {}_2F_1\left(\begin{matrix} -n, & -x \\ & -N \end{matrix} \middle| \frac{1}{p}\right),$$

and therefore we declare

25: `krawtchoukterm:=(-1)^n*p^n*binomial(N,n)*hyperterm({-n,-x},{-N},1/p,k)$`

and get the three three-term recurrence equations

```

26: sumrecursion(krawtchoukterm,k,n);

(x + 1 - 2*p - nn*p + (2*p - 1)*n)*sum(n - 1)

- ((n - nn - 2)*(p - 1)*sum(n - 2)*p + sum(n)*n)

27: sumrecursion(krawtchoukterm,k,x);

- ((x - 1 + nn*p - n - 2*(x - 1)*p)*sum(x - 1) + (x - 1 - nn)*sum(x)*p

+ (p - 1)*(x - 1)*sum(x - 2))

28: sumrecursion(krawtchoukterm,k,NN);

(x + 1 + n + (p - 2)*nn)*sum(nn - 1)

- ((x + 1 - nn)*sum(nn - 2) - (n - nn)*(p - 1)*sum(nn))

```

with respect to the parameters n , x , and N respectively.

6 Simplification Operators

For the decision that an expression a_k is a hypergeometric term, it is necessary to find out whether or not a_k/a_{k-1} is a rational function with respect to k . For the purpose to decide whether or not an expression involving powers, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols is a hypergeometric term, the following simplification operators can be used:

- **simplify_gamma(f)** simplifies an expression **f** involving only rational, powers and Γ function terms according to a recursive application of the simplification rule $\Gamma(a+1) = a \Gamma(a)$ to the expression tree. Since all Γ arguments with integer difference are transformed, this gives a decision procedure for rationality for integer-linear Γ term product ratios.
- **simplify_combinatorial(f)** simplifies an expression **f** involving powers, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols by converting factorials, binomial coefficients, and Pochhammer symbols into Γ function terms, and applying **simplify_gamma** to its result. If the output is not rational, it is given in terms of Γ functions. If you prefer factorials you may use
- **gammatofactorial (rule)** converting Γ function terms in factorials using $\Gamma(x) \rightarrow (x-1)!$.

The use of **simplify_combinatorial(f)** is a safe way to decide the rationality for any ratio of products of powers, factorials, Γ function terms, binomial coefficients, and Pochhammer symbols.

Example:

```
29: simplify_combinatorial(sub(k=k+1,krawtchoukterm)/krawtchoukterm);
```

$$\frac{(k - n) \cdot (k - x)}{(k - n) \cdot (k + 1) \cdot p}$$

From this calculation, we see again that the upper parameters of the hypergeometric representation of the Krawtchouk polynomials are given by $\{-n, -x\}$, its lower parameter is $\{-n\}$, and the argument of the hypergeometric function is $1/p$.

Another example is

```
30: simplify_combinatorial(binomial(n,k));
```

$$\frac{\text{gamma}(n + 1)}{\text{gamma}(k + 1) \cdot \text{gamma}(n + 1 - k)}$$

```
31: ws where gammatofactorial;
```

$$\frac{\text{factorial}(n)}{\text{factorial}(- (k - n)) \cdot \text{factorial}(k)}$$

7 Tracing

If you set

```
32: on zb_trace;
```

tracing is enabled, and you get intermediate results, see [2].

Example for the Gosper algorithm:

```
33: gosper(pochhammer(k-n,n),k);
```

$$a(k)/a(k-1) := \frac{k - 1}{k - n - 1}$$

Gosper algorithm applicable

```
p:= 1
```

```
q:= k - 1
```

r:= - (n + 1 - k)

degreebound := 0

$$f := \frac{1}{n+1}$$

Gosper algorithm successful

$$\frac{\text{pochhammer}(k-n, n) \cdot k}{n+1}$$

Example for the Zeilberger algorithm:

34: sumrecursion(binomial(n,k)^2,k,n);

$$F(n,k)/F(n-1,k) := \frac{n^2}{k^2 - 2kn + n^2}$$

$$F(n,k)/F(n,k-1) := \frac{k^2 - 2kn - 2k + n^2 + 2n + 1}{k^2}$$

Zeilberger algorithm applicable

applying Zeilberger algorithm for order:= 1

$$p := \text{zb_sigma}(1) \cdot k^2 - 2 \cdot \text{zb_sigma}(1) \cdot k \cdot n + \text{zb_sigma}(1) \cdot n^2 + n^2$$

$$q := k^2 - 2kn - 2k + n^2 + 2n + 1$$

$$r := k^2$$

degreebound := 1

$$f := \frac{2k^2 - 3n + 2}{n}$$

$$p := \frac{-4k^2n + 2k^2 + 8k^2n - 4k^2n - 3n^3 + 2n^2}{n}$$

Zeilberger algorithm successful

```
4*sum(n - 1)*n - 2*sum(n - 1) - sum(n)*n
```

```
35: off zb_trace;
```

8 Global Variables and Switches

The following global variables and switches can be used in connection with the **ZEILBERG** package:

- **zb_trace**, switch; default setting **on**. Turns tracing on and off.
- **zb_direction**, variable; settings: **down**, **up**; default setting **down**.

In the case of the Gosper algorithm, either a downward or a forward antidifference is calculated, i. e., **gosper** finds g_k with either

$$a_k = g_k - g_{k-1} \quad \text{or} \quad a_k = g_{k+1} - g_k,$$

respectively.

In the case of the Zeilberger algorithm, either a downward or an upward recurrence equation is given. Example:

```
36: zb_direction:=up$
```

```
37: sumrecursion(binomial(n,k)^2,k,n);
```

```
sum(n + 1)*n + sum(n + 1) - 4*sum(n)*n - 2*sum(n)
```

```
38: zb_direction:=down$
```

- **zb_order**, variable; settings: any nonnegative integer; default setting 5. Gives the maximal order for the recurrence equation that **sumrecursion** searches for.
- **zb_factor**, switch; default setting **on**. If **off**, the factorization of the output usually producing nicer results is suppressed.
- **zb_proof**, switch; default setting **off**. If **on**, then several intermediate results are stored in global variables:

- `gosper_representation`, variable; default setting `nil`.

If a `gosper` command is issued, and if the Gosper algorithm is applicable, then the variable `gosper_representation` is set to the list of polynomials (with respect to k) $\{p, q, r, f\}$ corresponding to the representation

$$\frac{a_k}{a_{k-1}} = \frac{p_k}{p_{k-1}} \frac{q_k}{r_k}, \quad g_k = \frac{q_{k+1}}{p_k} f_k a_k,$$

see [1]. Examples:

```
39: on zb_proof;
```

```
40: gosper(k*factorial(k),k);
```

```
(k + 1)*factorial(k)
```

```
41: gosper_representation;
```

```
{k,k,1,1}
```

```
42: gosper(1/(k+1)*binomial(2*k,k)/(n-k+1)*binomial(2*n-2*k,n-k),k);
```

```
(2*k - n + 1)*(2*k + 1)*binomial(- 2*(k - n), - (k - n))*binomial(2*k,k)
-----
(k + 1)*(n + 2)*(n + 1)
```

```
43: gosper_representation;
```

```
{1,
```

```
(2*k - 1)*(k - n - 2),
```

```
(2*k - 2*n - 1)*(k + 1),
```

```

n - 1 - 2*k
-----}
(n + 2)*(n + 1)
```

- `zeilberger_representation`, variable; default setting `nil`.

If a `sumrecursion` command is issued, and if the Zeilberger algorithm is successful, then the variable `zeilberger_representation` is set to the final Gosper representation used, see [2].

9 Messages

The following messages may occur:

- ***** Gosper algorithm: no closed form solution exists
Example input:
`gosper(factorial(k),k).`
- ***** Gosper algorithm not applicable
Example input:
`gosper(factorial(k/2),k).`
The term ratio a_k/a_{k-1} is not rational.
- ***** illegal number of arguments
Example input:
`gosper(k).`
- ***** Zeilberger algorithm fails. Enlarge zb_order
Example input:
`sumrecursion(binomial(n,k)*binomial(6*k,n),k,n)`
For this example a setting `zb_order:=6` is needed.
- ***** Zeilberger algorithm not applicable
Example input:
`sumrecursion(binomial(n/2,k),k,n)`
One of the term ratios $f(n,k)/f(n-1,k)$ or $f(n,k)/f(n,k-1)$ is not rational.

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