Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

RALF BORNDÖRFER, GUILLAUME SAGNOL, STEPHAN SCHWARTZ

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Takustraße 7
D-14195 Berlin-Dahlem

Telefon: 030-84185-0
Telefax: 030-84185-125

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An Extended Network Interdiction Problem for Optimal Toll Control

Ralf Borndörfer, Guillaume Sagnol, Stephan Schwartz

Abstract

We study an extension of the shortest path network interdiction problem and present a novel real-world application in this area. We consider the problem of determining optimal locations for toll control stations on the arcs of a transportation network. We handle the fact that drivers can avoid control stations on parallel secondary roads. The problem is formulated as a mixed integer program and solved using Benders decomposition. We present experimental results for the application of our models to German motorways.

1 Introduction

A distance-based truck toll on German motorways was introduced in 2005 and is enforced by the German Federal Office for Goods Transport (BAG). Controls are conducted by a combination of mobile units and automatic toll control gantries scanning the passing traffic for toll evaders. Within the framework of an ongoing project with the BAG we already studied the optimization of control tours of toll inspectors using game theoretic approaches [3, 4]. In this paper we study the optimal location of toll control gantries from a theoretical point of view. We consider a network optimization problem where we increase the lengths of a limited number of edges. This increase can be interpreted as the detour that potential toll evaders are forced to take in order to avoid the control point (e.g. using a secondary road). The goal is to dissuade as many drivers as possible from trying to avoid the control gantries, by making the required detours prohibitively large.

This paper is organized as follows. In Section 2 we introduce a mathematical model called MAXCOV for the optimal location of control edges. We also present a generalization involving *detour willingness functions*. In particular, the MAXDET problem is defined by penalizing the detour of each driver linearly, and can be seen as a relaxation of the original MAXCOV problem. A Mixed Integer Programming (MIP) approach relying on Benders cuts is presented in Section 3. Finally, we produce numerical results for the application to German motorways in Section 4. In particular, we show that MAXDET is much easier to solve than MAXCOV, while it yields results of a comparable quality.

Related work The present problem is closely related to a class of network interdiction problems which can be seen as Stackelberg games on networks. In the *shortest path interdiction problem* the leader increases the length of certain arcs so as to maximize the shortest path length for the follower. The problem of finding a set of k arcs whose removal maximizes the shortest s-t-path length is also referred to as the *k most vital arcs problem* (MVAP) [2, 6, 8]. The origins of network interdiction problems lie in military and security applications such as interdicting supply networks or disrupting international drug routes, see e.g. [13]. To the best of our knowledge, previous work on shortest path network interdiction is limited to the single-commodity case, while our application requires a multi-commodity formulation.

There is also some literature on the *length-bounded cut problem*, which is essentially the same problem as MVAP, but was given a different name by some authors because its relation with *length bounded flows* was investigated. Several authors contributed to this topic, which has important applications to robust telecommunication networks, see e.g. [1, 12].

There is also a vast literature on *bilevel network pricing problems*, see e.g. [5, 10]. Here, in contrast to network interdiction problems, the decision variables are continuous, and consist of toll prices, which can be set arc-wise or path-wise. Network pricing problems have a *non zero-sum* nature, because setting too high prices incentizes drivers to take alternative, toll-free roads, which leads to a loss of revenue. The problem studied in the present paper is different in that respect; here we solely focus on making all control-free paths excessively long for potential toll evaders.

2 Problem formulation

We consider a directed graph $G = (V, E)$ with edge lengths $\ell_e \geq 0\ \forall e \in E$. We can *interdict* κ edges in G, i.e. increase their length by $c_e \geq 0$. Note that we can restrict the set of interdictable edges by setting $c_e = 0$. We have a set of commodities $\mathcal{K}$ and each commodity $k_i \in \mathcal{K}$ is a triple $k_i = (s_i, t_i, d_i)$. The nodes $s_i, t_i \in V$ are the source and the sink, respectively, of commodity k_i while $d_i \in \mathbb{N}$ is the demand of k_i.

Now let $\mathcal{P}_{s,t}$ denote the set of all s-t-paths in G, let $\ell(P) = \sum_{e \in P} \ell_e$ denote the length of a path P and let $\ell_i = \min_{P \in \mathcal{P}_{s_i,t_i}} \ell(P)$ denote the length of a shortest s_i-t_i-path. Moreover, we are given a maximum path length $L_i > \ell_i$ for each $k_i \in \mathcal{K}$ that no driver is willing to exceed. Finally, we define $\mathcal{C} := \{C \subseteq E : |C| = \kappa\}$. Our goal is to find a set $C \in \mathcal{C}$ that "covers" the most traffic.

Definition 1. *Let $C \subseteq E$. We denote by $\mathcal{K}(C) \subseteq \mathcal{K}$ the set of C-*covered *commodities, where a commodity $k_i \in \mathcal{K}$ is called C-covered if and only if*

$$\forall P \in \mathcal{P}_{s_i,t_i} \ : \quad \ell_C(P) := \sum_{e \in P} \ell_e + \sum_{e \in P \cap C} c_e \ \geq \ L_i.$$

The interpretation for the location of toll control gantries is as follows. A control gantry is placed on every interdicted edge but can be avoided by paying an additional fee c_e. This can be interpreted as a detour on a parallel trunk road or as any other possibility to avoid certain parts of the controlled network at higher costs. Commodity k_i is covered if and only if a shortest s_i-t_i-path that avoids all controls is too long, i.e. the drivers of k_i have to pass at least one control gantry. The MAXCOV problem (for maximum cover) can then be stated as follows.

$$\text{Given } (G, (\ell_e), (c_e), \mathcal{K}, (L_i), \kappa): \qquad \max_{C \in \mathcal{C}} \sum_{\{i \,:\, k_i \in \mathcal{K}(C)\}} d_i.$$

NP-hardness For the complexity of MAXCOV we reconsider the most vital arc problem (MVAP). The corresponding decision problem has an additional input L and asks if there are k arcs whose removal lead to a shortest s-t-path of length at least L. Bar-Noy, Khuller and Schieber [2] prove that this problem is strongly NP-complete.

Proposition 1. *MAXCOV is NP-hard already for* $|\mathcal{K}| = 1$.

Proof. With $c_e = L$ for all $e \in E$ we reduce MVAP to an instance of MAXCOV with a single commodity. □

2.1 A fractional cover

The MAXCOV problem is based on the assumption that all drivers of a commodity k_i have the same detour threshold $L_i - \ell_i$. In reality, however, some drivers might be ready to take longer detours than others. We hence study a slightly more general model. For each commodity $k_i \in \mathcal{K}$ we assume that a monotone *detour willingness function* $\omega_i : [0, \infty) \to [0, 1]$, satisfying $\omega_i(0) = 0$ and $\lim_{\Delta \to \infty} \omega_i(\Delta) = 1$, gives the fraction $\omega_i(L - \ell_i)$ of drivers from commodity k_i who have a detour threshold $\leq L - \ell_i$. Hence, the objective of the MAXCOV problem can be generalized to

$$\max_{C \in \mathcal{C}} \sum_{i=1}^{|\mathcal{K}|} z_i(C) \qquad \text{where } z_i(C) := \min_{P \in \mathcal{P}_{s_i,t_i}} d_i \, \omega_i\big(\ell_C(P) - \ell_i\big). \tag{1}$$

Obviously, the detour willingness functions used for MAXCOV are $\omega_i^{\text{cov}} = \chi_{\{\Delta \,:\, \Delta \geq L_i - \ell_i\}}$ where χ denotes the indicator function. We next examine a particular case, which we refer to as MAXDET (for maximum detour), where the detour willingness functions are of the form $\omega_i^{\text{det}}(\Delta) = \min\left(1, \frac{\Delta}{L_i - \ell_i}\right)$. That is, we assume that the number of covered drivers is proportional to the detour Δ. Note that $\omega_i^{\text{det}} \geq \omega_i^{\text{cov}}$ but $\omega_i^{\text{det}}(\Delta) = 1 \iff \omega_i^{\text{cov}}(\Delta) = 1$. Thus, the decision versions of MAXCOV and MAXDET coincide, and MAXDET is NP-hard. To stress the use of a specific detour willingness ω we also write z_i^{ω}.

Benders decomposition Similar to [8] we use an approach with Benders decomposition to solve problem (1). We consider $\hat{\mathcal{P}}_i \subseteq \mathcal{P}_{s_i,t_i}$ and for notational convenience we define $\hat{z}_i(C) := \min_{P \in \hat{\mathcal{P}}_i} d_i \, \omega_i\big(\ell_C(P) - \ell_i\big)$. To further simplify notation we set $z(C) := \sum_i z_i(C)$ and $\hat{z}(C) := \sum_i \hat{z}_i(C)$.

If we solve the restricted master problem $\max_{C \in \mathcal{C}} \hat{z}(C)$, the solution $\hat{C}$ might not be feasible for (1). Hence, in the subproblem we check for each i whether $z_i(\hat{C}) < \hat{z}_i(\hat{C})$, which is essentially a shortest path problem as ω_i is monotone. If we have equality in the subproblem, $\hat{C}$ is feasible and hence optimal for (1). In the other case we enlarge the sets $\hat{\mathcal{P}}_i$ and iterate.

We conclude that the computational complexity of the problem mainly depends on the ability to solve the restricted master problem, and therefore on the choice of a detour willingness function. We will see in the next section that MAXCOV and MAXDET can be formulated as integer or mixed integer programs, respectively. For piecewise linear ω_i we can derive a MIP formulation similar to MAXDET. Note however that this might require additional binary variables if the ω_i are not concave.

3 IP formulation

Here we present an integer program formulation for MAXCOV. A relaxation of integrality constraints leads to a mixed integer formulation for MAXDET. Both problems are solved using Benders cuts in order to restrict the number of cover constraints. We start with the IP for MAXCOV.

$$
\begin{aligned}
\max_{y,\delta} \quad & \sum_{i=1}^{|\mathcal{K}|} d_i\, \delta_i && && \text{(2a)}\\
\text{subject to} \quad & \sum_{e \in E} y_e \;\le\; \kappa && && \text{(2b)}\\
& \sum_{e \in P} (\ell_e + c_e y_e) \;\ge\; l_i + \delta_i\,(L_i - l_i) && \forall P \in \mathcal{P}_{s_i,t_i},\ \forall i && \text{(2c)}\\
& y_e \;\in\; \{0,1\} && \forall e \in E && \text{(2d)}\\
& \delta_i \;\in\; \{0,1\} && \forall i \in \{1,\ldots,|\mathcal{K}|\} && \text{(2e)}
\end{aligned}
$$

Obviously, y indicates the set of control edges, while $\delta_i = 1$ means that k_i is covered. The objective function (2a) maximizes the amount of covered traffic. Constraint (2b) guarantees that at most κ control gantries are built, and in (2c) we enforce $\delta_i = 0$ if k_i is not covered. Note that we use a slightly different characterization than in Definition 1 which is equivalent as we have integrality for δ_i. These constraints allow us to obtain a formulation for MAXDET by simply relaxing the integrality constraints (2e).

3.1 Solving MAXCOV and MAXDET

As pointed out at the end of Section 2, we employ Benders decomposition in order to reduce the number of cover constraints (2c). Clearly, every optimal solution $\hat{C}$ of the restricted master problem (RMP) gives an upper bound $\hat{z}(\hat{C})$ on the objective value of (2). Furthermore, the solutions $z_i(\hat{C})$ of the subproblem can be used to compute a lower bound $z(\hat{C})$ for the objective of the master problem.

In order to solve MAXCOV or MAXDET to optimality, we have to iterate until the two bounds coincide. After each iteration though, we can derive an optimality gap to monitor the quality of the current solution with respect to the original problem. We can easily adapt this gap to allow α-approximate solutions of RMP, i.e. a solution C' with

$$\max_{C \in \mathcal{C}} \hat{z}(C) \ \le\ \alpha\, \hat{z}(C').$$

This is particularly interesting as the restricted master problem can still be hard to solve to optimality. The following proposition gives a bound for the objective value of the original problem.

Proposition 2. *Let C' be an α-approximate solution of RMP. Then C' is a γ-approximate solution of the original problem with*

$$\gamma \ =\ \alpha\, \frac{\hat{z}(C')}{z(C')}.$$

Proof. We have

$$\max_{C \in \mathcal{C}} z(C) \ \le\ \max_{C \in \mathcal{C}} \hat{z}(C) \ \le\ \alpha\, \hat{z}(C') \ =\ \alpha\, \frac{\hat{z}(C')}{z(C')}\, z(C').$$

□

As we compute $\hat{z}(C')$ and $z_i(C')$ during the iteration, this bound is also easily computed. Furthermore, it is not restricted to MAXCOV and MAXDET but can be applied to arbitrary detour willingness functions.

We like to point out that there is also an explicit formulation of MAXCOV and MAXDET similar to [3]. However, this formulation turned out to be very inefficient in practice.

4 Computational results

We employed the above models to problem instances based on real data from the German motorway network. Our simplified network consists of 405 nodes, 1084 edges and a total of over 130,000 commodities. We reduce the problem size by considering only the top commodities $\mathcal{K}_\xi$ that represent a given fraction ξ of traffic. When a set C of control edges is given, we define the cover rate $r(\omega, \xi) := \frac{\sum_{i\,:\,k_i \in \mathcal{K}_\xi} z_i^\omega(C)}{\sum_{i\,:\,k_i \in \mathcal{K}_\xi} d_i}$, that represents the proportion of covered drivers in K_ξ for the detour willingness ω.

The tests were made on a PC with 8 processors at 3.2 GHz and 16 GB RAM using CPLEX 12.6. We set $c_e = 0.5\,\ell_e + 1$ and $L_i = 1.1\,\ell_i$ and we optimized the location of $\kappa = 302$ control stations in order to compare our solution with the actual location of control gantries. We allowed an optimality gap of 0.5% and a time limit of 10 minutes per IP/MIP iteration as well as a global time limit of 1 hour. In Table 1 we demonstrate some results of our computations. Even though the number of commodities increases from 3226 ($\xi = 1/3$) to 16711 ($\xi = 2/3$), the most obvious difference in computation time is between MAXCOV and MAXDET. For both instances of MAXCOV we observe a

Table 1: Computational results for exemplary instances of MAXCOV and MAXDET. The second and third columns indicate the computing time and the number of iterations before the allowed gap or time limit was reached. The gap in the fourth column corresponds to $\gamma - 1$ (cf. Prop. 2). The last four columns show cover rates in percent for the solution C of the considered problems.

	CPU (s)	it.	gap	$r(\omega^{\text{cov}}, \xi)$	$r(\omega^{\text{cov}}, 1)$	$r(\omega^{\text{det}}, \xi)$	$r(\omega^{\text{det}}, 1)$
MAXCOV ($\xi = 1/3$)	3600	6	8.3%	82.6	77.5	92.6	91.4
MAXCOV ($\xi = 2/3$)	3600	6	12.4%	79.4	78.0	92.3	91.7
MAXDET ($\xi = 1/3$)	19	5	0.5%	71.7	68.5	94.4	92.4
MAXDET ($\xi = 2/3$)	121	5	0.4%	73.0	71.4	93.7	93.0
actual location					4.1		57.3

significant gap between lower and upper bound (cf. Proposition 2) while both computations were quit after 1 hour. In contrast, the computation times and optimality gaps are excellent for the instances of MAXDET. These instances could also be solved to optimality, while this was not possible for the respective MAXCOV instances even with a large time limit. We see that MAXDET gives comparable solutions with respect to r^{cov}, especially in comparison with the actual location of control stations. In Figure 1 the differences between the solutions of MAXCOV and MAXDET for $\xi = 2/3$ are illustrated. Interestingly, 243 of the 302 control edges occur in both solutions ($\approx 80\%$). In Figure 2 we see that we can achieve similar results with less control stations. The computation times correspond to the results in Table 1 even for $\kappa = 50$.

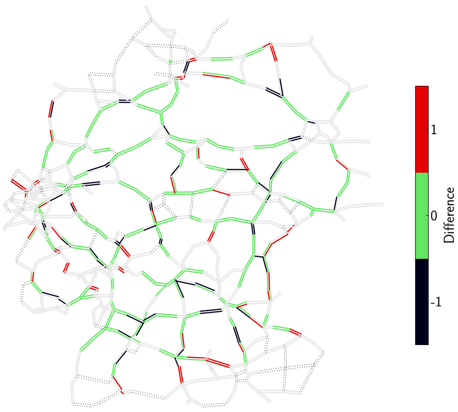

Figure 1: Differences between solutions of MAXCOV and MAXDET for $\xi = 2/3$.

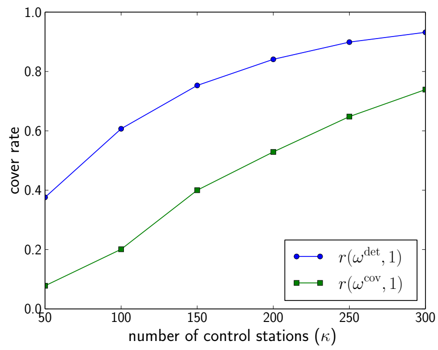

Figure 2: Cover rate in relation to κ. We consider optimal solutions C of MAXDET with $\xi = 0.5$.

5 Conclusions

In this paper we studied the problem of optimizing the locations of automatic toll control stations in a transportation network. Our solution is a multi-commodity shortest path network interdiction problem with thresholds for shortest path lengths. We present a MIP formulation for two variants of this problem and use decomposition methods to solve it efficiently.

Experimental results for the German motorway network show that the two variants give results of comparable quality, while MAXDET surpasses MAXCOV in terms of performance.

Future research should address the uncertainty in demand and detour willingness of the drivers. The development of robust approaches is an important aspect also for practical use.

References

[1] G. Baier, T. Erlebach, A. Hall, E. Köhler, H. Schilling and M. Skutella, *Length-Bounded Cuts and Flows*, Automata, Languages and Programming **4051** (2006), 679–690.

[2] A. Bar-Noy, S. Khuller and B. Schieber, *The Complexity of Finding Most Vital Arcs and Nodes*, University of Maryland, Computer Science Department, Technical Report CS-TR-35-39 (1995).

[3] R. Borndörfer, J. Buwaya, G. Sagnol and E. Swarat, *Optimizing Toll Enforcement in Transportation Networks: a Game-Theoretic Approach*, Proceedings of INOC'2013 **41** (2013), 253–260.

[4] R. Borndörfer, B. Omont, G. Sagnol and E. Swarat, *A Stackelberg Game to Optimize the Distribution of Controls in Transportation Networks*, Game Theory for Networks **105** (2012), 224–235.

[5] L. Brotcorne, M. Labbé, P. Marcotte and G. Savard, *A bilevel model for toll optimization on a multicommodity transportation network*, Transportation Science **35** (2001), 345-358.

[6] H.W. Corley and D.Y. Sha, *Most vital links and nodes in weighted networks*, Operations Research Letters **1** (1982), 157–160.

[7] N. Garg, V. Vazirani and M. Yannakakis, *Multiway cuts in directed and node weighted graphs*, Automata, Languages and Programming **820** (1994), 487–498.

[8] E. Israeli and R. Wood, *Shortest-Path Network Interdiction*, Networks **40** (2002), 97–111.

[9] O. Jahn, R. Möhring and A. Schulz, *Optimal routing of traffic flows with length restrictions in networks with congestion*, Operations Research Proceedings **1999** (2000), 437–442.

[10] M. Labbé and A. Violin, *Bilevel programming and price setting problems*, 4OR **11** (2013), 1–30.

[11] L. Lovász, V. Neumann-Lara and M. Plummer, *Mengerian Theorems for Paths of Bounded Length*, Periodica Mathematica Hungarica **9** (1978), 269–276.

[12] A. Mahjoub and S. McCormick, *Max Flow and Min Cut with bounded-length paths: complexity, algorithms, and approximation*, Mathematical programming **124** (2010), 271–284.

[13] F. Pan, W. Charlton and D. Morton, *A Stochastic Program for Interdicting Smuggled Nuclear Material*, In "Network Interdiction and Stochastic Integer Programming" (2003), 1–19.