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Makespan and Tardiness in Activity Networks with Lognormal Activity Durations

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Abstract

We propose an algorithm to approximate the distribution of the completion time (makespan) and the tardiness costs of a project, when durations are lognormally distributed. This problem arises naturally for the optimization of surgery scheduling, where it is very common to assume lognormal procedure times. We present an analogous of Clark's formulas to compute the moments of the maximum of a set of lognormal variables. Then, we use moment matching formulas to approximate the earliest starting time of each activity of the project by a shifted lognormal variable. This approach can be seen as a lognormal variant of a state-of-the-art method used for the statistical static timing analysis (SSTA) of digital circuits. We carried out numerical experiments with instances based on real data from the application to surgery scheduling. We obtained very promising results, especially for the approximation of the mean overtime in operating rooms, for which our algorithm yields results of a similar quality to Monte-Carlo simulations requiring an amount of computing time several orders of magnitude larger.

1 Introduction

An activity network is a mathematical representation of a project, given by a partial order on a set of activities $\{a_1, \ldots, a_n\}$, such that $a_i < a_j$ indicates that activity a_j cannot start before the end of activity a_i. A crucial question for planners is to estimate the end of a project, when the duration of each uncertainty is uncertain. This question has attracted a considerable attention from the operations research community over the last 60 years. The first techniques to analyze an activity network are known as CPM (*critical path method*, [KJW59]) and PERT (*project evaluation and review technique*, [MRCF59]). These two techniques are simple, but lack of mathematical accuracy, especially when the network contains several uncorrelated paths that have roughly the same length [MR64]. However PERT gained so much popularity that many authors call activity networks *PERT networks* nowadays.

Parametric methods have also been proposed for the analysis of stochastic activity networks: Martin suggested to use polynomials [Mar65], and Sculli used Clark's formulas [Cla61] to construct normal approximations of the earliest starting times of each activity [Scu83]. Also,

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exact expressions for the distribution of the completion time have been derived in the case of exponentially distributed durations, and later extended to the case of phase-type distributions [FG83, KA86, MU15]. However this method requires a state-space model that grows exponentially with the size of the network, so it can be used for very small instances only.

More recently, the problem also attracted much attraction in the literature on digital circuit optimization, where the goal is to approximate the signal delay through a network formed by gates and wires [BCSS08]. It gave rise to a version of Sculli's method in canonical form [VRK+06], which is computationally more efficient. Recently, the technique was also extended by using the family of skew-normal distributions, to take into account the skewness of the completion time [VV14].

Another important question in the field of stochastic activity network analysis is to find out which activity is potentially a bottleneck of the project. To this end, the notion of *criticality* has been developed, and algorithms have been proposed to compute criticality indices [DE85, Mon11]. The notion of criticality has also been linked to that of persistency in robust optimization, and methods based on semidefinite programming have been developed to approximate these indices and to bound completion times over a family of activity distributions [BNT06, NTZ11].

The present article is motivated by an application to surgery scheduling, an area in which it is well known that durations exhibit a lognormal distribution [SMV98], or even a shifted-lognormal distribution [SHDV10]. More generally, the use of lognormal distributions for the analysis of stochastic project networks has also been suggested by the authors of [TB12]. In order to cope with the fat-tail behaviour of lognormals, we introduce a method to estimate the completion time (makespan) of a project and its total tardiness by using shifted lognormal approximations. Our numerical experiments show that our technique provide reasonable estimations in a much shorter time than Monte-Carlo simulations. Since the state-of-the-art in stochastic resource-constrained scheduling is to use metaheuristics combined with Monte-Carlo simulations for the evaluation of planning policies [Bal07, RCL16], we think that our method has the potential to drastically improve the efficiency of algorithms for the planning of surgical resources.

Notation We use boldface letters to denote vectors. Capital letters are reserved for matrices or random variables, depending on the context. Hence boldface capital letters refer to a random vector. The symbol $\boldsymbol{e}_k$ is used for the kth vector of the canonical basis of $\mathbb{R}^n$, where the dimension n should be clear from the context. In other words, $\boldsymbol{e}_k$ is the kth column of $\boldsymbol{I}_n$, the identity matrix of dimension n. We use the notation $\mathbb{E}[\boldsymbol{X}]$ for the expected value of $\boldsymbol{X}$, and $\mathbb{V}[\boldsymbol{X}]$ for its variance-covariance matrix. We write $\mathrm{cov}(X, Y)$ for the covariance between two scalar random variables X and Y. In particular, we have $\mathrm{cov}(X, X) = \mathbb{V}[X]$. The probability of a random event E is denoted by $\mathbb{P}[E]$.

Throughout this paper, the symbols Φ and φ represent respectively the cumulative distribution function (CDF) and probability distribution function (PDF) of the standard normal distribution.

$$\forall x \in \mathbb{R},\ \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \Phi(x) = \int_{s=-\infty}^{x} \varphi(s)\ ds.$$

We use the standard notation $\boldsymbol{X} \sim \mathcal{N}_k(\boldsymbol{\mu}, \Sigma)$ to say that $\boldsymbol{X}$ follows a (multivariate) normal distribution with mean vector $\boldsymbol{\mu} \in \mathbb{R}^k$ and variance-covariance matrix $\Sigma \in \mathbb{S}_+^k$ (where $\mathbb{S}_+^k$ represents the set of all $k \times k$-symmetric positive semidefinite matrices), or simply $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$ when the dimension is clear from the context. Then, the probability density of X at $\boldsymbol{x}$ is denoted by

$$\varphi_k(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) := \frac{1}{\sqrt{(2\pi)^k |\Sigma|}} \, \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}} \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right),$$

where $|\Sigma|$ is the determinant of Σ. The multivariate normal CDF at $\boldsymbol{x}$ is the probability that $\boldsymbol{X} \leq \boldsymbol{x}$, where the inequality is elementwise:

$$\Phi_k(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) := \int_{z_1=-\infty}^{x_1} \cdots \int_{z_k=-\infty}^{x_k} \varphi_k(\boldsymbol{z}; \boldsymbol{\mu}, \Sigma) \, d^k \boldsymbol{z}.$$

In particular, for $k = 1$, we have the identities $\varphi(x) = \varphi_1(x; 0, 1)$ and $\Phi(x) = \Phi_1(x; 0, 1)$.

2 Sum and Maximum of correlated lognormal variables

2.1 Maximum of correlated normal variables

Let $\boldsymbol{X}$ follow a multivariate normal distribution with mean vector $\boldsymbol{\mu} \in \mathbb{R}^n$ and variance-covariance matrix Σ, and let $Z = \max(X_1, X_2)$. Then, the first four moments of Z have known analytical expressions that were given by Clark in 1961 [Cla61]. The first two of them are:

$$\mathbb{E}[Z] = \mu_1 Q + \mu_2 (1 - Q) + \theta P \tag{1}$$

$$\mathbb{E}[Z^2] = (\mu_1^2 + \Sigma_{1,1}) Q + (\mu_2^2 + \Sigma_{2,2})(1 - Q) + (\mu_1 + \mu_2) \theta P, \tag{2}$$

were $\theta = \sqrt{\mathbb{V}[X_1 - X_2]} = \sqrt{\Sigma_{1,1} + \Sigma_{2,2} - 2\Sigma_{1,2}}$, $P = \varphi(\frac{\mu_1 - \mu_2}{\theta})$ and $Q = \Phi(\frac{\mu_1 - \mu_2}{\theta})$. In particular, Q coincides with the probability that $X_1 \geq X2$, or equivalently that $Z = X_1$. Moreover, Clark showed that for all $i \in \{3, \ldots, n\}$, we have

$$\operatorname{cov}(Z, X_i) = Q \Sigma_{1,i} + (1 - Q) \Sigma_{2,i}. \tag{3}$$

This suggests a recursive algorithm to estimate the density of $X_{\max} := \max(X_1, \ldots, X_n)$, by rewriting $X_{\max} = \max(Z, X_3, \ldots, X_N)$ and by approximating the law of the vector $(Z, X_3, \ldots, X_N)$ by a multivariate normal distribution of dimension $n-1$, with mean and covariance matrix computed using (1)–(3).

2.2 Maximum of correlated lognormal variables

Now, we recall that a random vector $\boldsymbol{X}$ of dimension n is said to have a log-normal distribution of parameters $\boldsymbol{\mu}$ and Σ if the vector $\log \boldsymbol{X} := (\log X_1, \ldots, \log X_n)$ follow a multivariate normal law with the same parameters:

$$\boldsymbol{X} \sim LN(\boldsymbol{\mu}, \Sigma) \iff \log(\boldsymbol{X}) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma).$$

The recursive algorithm of Clark can immediately be adapted for lognormal variables. Indeed, we have the relation

$$X_{\max} := \max(X_1, \ldots, X_n) = e^{\max(\log X_1, \ldots, \log X_n)}.$$

So we can use Clark's algorithm to approximate $\max(\log X_1, \ldots, \log X_n)$ by a normal distribution $\mathcal{N}(\mu_{\max}, \sigma^2_{\max})$ for some parameters $\mu_{\max}$ and $\sigma_{\max}$. But then, this suggests that $\log X_{\max}$ can be approximated by a normal distribution, and so we can use the approximation $X_{\max} \overset{\text{approx}}{\sim} LN(\mu_{\max}, \sigma^2_{\max})$.

Our tests suggest that this lognormal approximation provides a reasonable estimate $e^{\mu_{\max} + \frac{1}{2}\sigma^2_{\max}}$ for the mean of $X_{\max}$, but the variance of $X_{\max}$ is poorly approximated. To get a better approximation of the law of $X_{\max}$, we suggest to use the family of shifted lognormal distributions, which have one additional location parameter and thus allow more flexibility. In particular, we can match the first three moments of any right-skewed distribution with a shifted lognormal, as shown in Lemma 2.1.

We point out that other authors have proposed to use the family of log-skew-normal distributions to approximate the sum of lognormal variables [HB15]. We chose to work with shifted lognormals instead, because they allow to use simple closed-form expressions for moment-matching, which is very important to achieve computational efficiency. Another argument in favour of shifted lognormals is that they can be used to approximate normal variables (by letting the shift parameter go to $-\infty$), which could be useful to estimate the sum of a large numbers of lognormal variables (by the central limit theorem).

2.3 Maximum of shifted lognormal variables

A scalar variable X is said to be have a shifted lognormal distribution with parameters c, μ, and σ^2 if $\log(X - c)$ follows a normal law of parameters μ and σ^2. In this paper, we consider the following multivariate extension of the shifted lognormal: $\boldsymbol{X} \in \mathbb{R}^n$ is said to have a ($n-$variate) shifted lognormal distribution with parameters $\boldsymbol{c}, \boldsymbol{\mu}$ and Σ if $\log(\boldsymbol{X} - \boldsymbol{c}) := (\log(X_1 - c_1), \ldots, \log(X_1 - c_1))$ has a $n-$variate normal distribution with parameters $\boldsymbol{\mu}$ and Σ.

$$\boldsymbol{X} \sim SLN(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma) \iff \log(\boldsymbol{X} - \boldsymbol{c}) \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma).$$

Simple calculus allows us to derive the first central moments of $\boldsymbol{X} \sim SLN(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$:

$$\mathbb{E}[X_i] = c_i + e^{\mu_i + \frac{1}{2}\Sigma_{i,i}}, \tag{4}$$

$$\operatorname{cov}(X_i, X_j) = e^{\mu_i + \frac{1}{2}\Sigma_{i,i}}\ e^{\mu_j + \frac{1}{2}\Sigma_{j,j}}\ (e^{\Sigma_{i,j}} - 1), \tag{5}$$

$$\operatorname{skew}(X_i) := \frac{\mathbb{E}\big[X_i - \mathbb{E}[X_i]\big]^3}{\mathbb{V}(X_i)^{\frac{3}{2}}} = (e^{\Sigma_{i,i}} + 2)\ (e^{\Sigma_{i,i}} - 1)^{\frac{1}{2}}. \tag{6}$$

The introduction of the shift parameter allows one to invert the above relations in order to match the first three moments of any right-skewed variable:

Lemma 2.1. *Consider a random variable Y with mean m and variance v, and assume that* $\mathrm{skew}(Y) = \gamma > 0$. *Then, Y has the same first three moments as $X \sim SLN(c, \mu, \sigma^2)$, where the parameters c, μ and σ^2 are given by the following formulas:*

$$\sigma^2 = \log\left[\left(\frac{2}{2+\gamma^2+(4\gamma^2+\gamma^4)^{\frac{1}{2}}}\right)^{\frac{1}{3}} + \left(\frac{2+\gamma^2+(4\gamma^2+\gamma^4)^{\frac{1}{2}}}{2}\right)^{\frac{1}{3}} - 1\right] \tag{7}$$

$$\mu = \frac{1}{2}\log\left(\frac{v}{e^{\sigma^2}(e^{\sigma^2}-1)}\right) \tag{8}$$

$$c = m - e^{\mu+\frac{1}{2}\sigma^2} \tag{9}$$

Proof. It is straightforward (though lengthy) to verify that the expression in (7) is the unique solution to the equation $(e^{\sigma^2}+2)\,(e^{\sigma^2}-1)^{\frac{1}{2}} = \gamma$, and is well defined as long as $\gamma > 0$. Then, the expressions of μ and c are simply found by inverting (5) and (4). $\square$

We will show how to to compute the moments of the maximum of (correlated) shifted lognormals. Then, we propose to approximate the law of the maximum by a shifted lognormal using Lemma 2.1 to match its first three moments. We start with the maximum of 2 shifted lognormal variables. Analogous to the case of normal variables, the general case with n variables can be handled recursively, see Section 2.4.

So, let $\boldsymbol{X} \sim SLN(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma) \in \mathbb{R}^2$. By definition of a shifted lognormal, we have $X_1 = c_1 + e^{Y_1}$, $X_2 = c_2 + e^{Y_2}$ for some random vector $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

Now, assume without loss of generality that $c_1 \geq c_2$ and define $\delta := \log(c_1 - c_2)$, or $\delta = -\infty$ if $c_1 = c_2$. Denote by $\mathcal{D}_\delta$ the subset of all vectors $\boldsymbol{y} \in \mathbb{R}^2$ satisfying $y_2 \geq \log(e^\delta + e^{y_1})$ (in the limit we set $\mathcal{D}_{-\infty} := \{\boldsymbol{y} \in \mathbb{R}^2 : y_2 \geq y_1\}$). This region is plotted in Figure 1, together with the asymptotes. $y_2 = \delta$ and $y_2 = y_1$. It is straightforward that $Z := \max(X_1, X_2)$ takes the value of X_2 if and only if $\boldsymbol{Y} \in \mathcal{D}_\delta$. So the kth moment of Z around c_2 can be expressed as the integral

$$\mathbb{E}[(Z-c_2)^k] = \int_{\boldsymbol{y}\in\mathbb{R}^2\setminus\mathcal{D}_\delta} (e^\delta + e^{y_1})^k\ \varphi_2(\boldsymbol{y};\boldsymbol{\mu},\Sigma)\,d^2\boldsymbol{y} + \int_{\boldsymbol{y}\in\mathcal{D}_\delta} (e^{y_2})^k\ \varphi_2(\boldsymbol{y};\boldsymbol{\mu},\Sigma)\,d^2\boldsymbol{y}.$$

Now, observe that the relation $e^{\boldsymbol{v}^T\boldsymbol{x}}\,\varphi_n(\boldsymbol{x};\boldsymbol{\mu},\Sigma) = e^{\boldsymbol{\mu}^T\boldsymbol{v}+\frac{1}{2}\boldsymbol{v}^T\Sigma\boldsymbol{v}}\,\varphi_n(\boldsymbol{x};\boldsymbol{\mu}+\Sigma\boldsymbol{v},\Sigma)$ holds for all $\boldsymbol{v} \in \mathbb{R}^n$. We apply this relation for vectors of the form $\boldsymbol{v} = \alpha\boldsymbol{e}_i$ $(i = 1, 2)$ and we expand the previous formula. After some simplifications, we find

$$\begin{aligned}\mathbb{E}[(Z-c_2)^k] = \sum_{j=0}^{k}\binom{k}{j}\, e^{\delta(k-j)+j\mu_1+\frac{j^2}{2}\Sigma_{1,1}}\left[1 - P_\delta\left(\left[\begin{array}{c}\mu_1 + j\Sigma_{1,1}\\ \mu_2 + j\Sigma_{1,2}\end{array}\right],\Sigma\right)\right]\\ +\quad e^{k\mu_2+\frac{k^2}{2}\Sigma_{2,2}} P_\delta\left(\left[\begin{array}{c}\mu_1 + k\Sigma_{1,2}\\ \mu_2 + k\Sigma_{2,2}\end{array}\right],\Sigma\right),\end{aligned} \tag{10}$$

where $P_\delta(\boldsymbol{m}, S)$ represents the probability that a random variable $\boldsymbol{W} \sim \mathcal{N}_2(\boldsymbol{m}, S)$ belongs to $\mathcal{D}_\delta$.

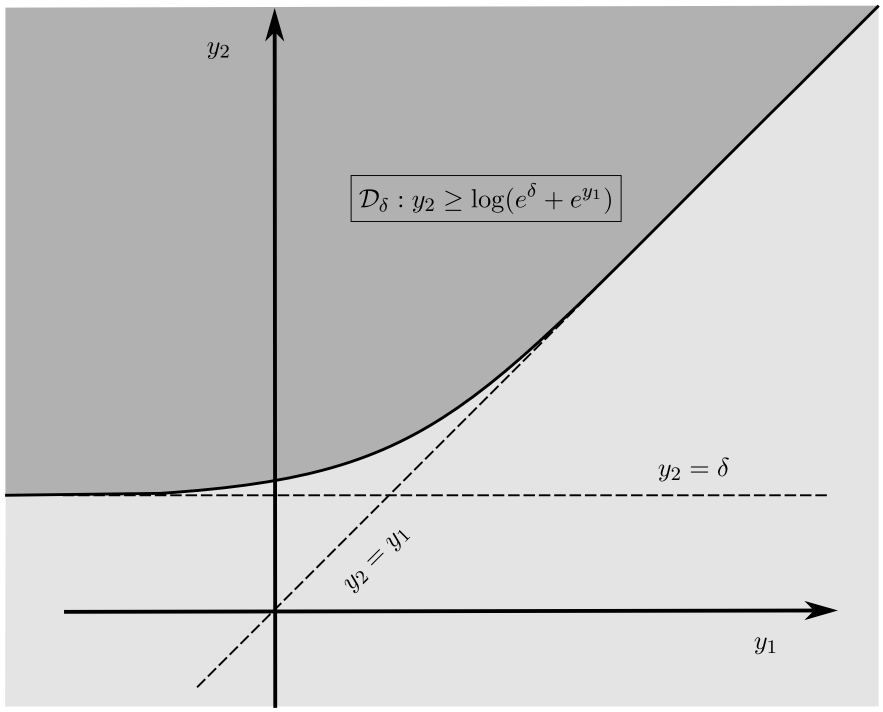

Figure 1: Region $\mathcal{D}_\delta \subseteq \mathbb{R}^2$.

Hence, we can determine $\mathbb{E}[Z^k]$ by evaluating probabilities of the form $P_\delta(\boldsymbol{m}, S)$. We first observe that the case $\delta = -\infty$ can be handled easily. Indeed, we have

$$P_{-\infty}(\boldsymbol{\mu}, \Sigma) = \mathbb{P}(Y_2 \geq Y_1) = \Phi(\frac{\mu_2 - \mu_1}{\theta}), \tag{11}$$

where $\theta = \sqrt{\mathbb{V}[Y_1 - Y_2]} = \sqrt{\Sigma_{1,1} + \Sigma_{2,2} - 2\Sigma_{1,2}}$.

To avoid numerical integration over a two-dimensional space we present a change of variable that allows to integrate in one dimension only in Appendix A. Then, one can obtain quick approximations of $P_\delta(\boldsymbol{m}, S)$ using a Gaussian quadrature formula; see Proposition A.1.

The proposed approach to approximate the maximum of two shifted lognormal distribution is summarized in Algorithm 1 (`MAX_BIVARIATE_SLN`)

2.4 Handling more than two variables

To allow for a recursive computation of the maximum of $n \geq 2$ variables, we also need to update the covariances of $M := \max(X_{n-1}, X_n)$ with X_i $(i = 1, \ldots, n-2)$. Rather than performing an exact computation –which would require to compute an integral over a tri-dimensional region, we use the following heuristic procedure: Assume that

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}, \tag{12}$$

Algorithm 1 (MAX_BIVARIATE_SLN)
Approximates $M := \max(X_1, X_2)$, where $\boldsymbol{X} \sim \mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$.

Input: $\boldsymbol{c} \in \mathbb{R}^2$ with $c_1 \geq c_2$, $\boldsymbol{\mu} \in \mathbb{R}^2, \Sigma \in \mathbb{S}_+^2$;
Output: (c_0, μ_0, σ_0^2), such that $\mathit{SLN}(c_0, \mu_0, \sigma_0^2)$ approximates M,
and $P_0 = \mathbb{P}[X_1 \geq X_2]$.

$\delta \leftarrow \log(c_1 - c_2) \in \mathbb{R} \cup \{-\infty\}$
for all $j \in \{0, 1, 2, 3\}$ **do**
  Compute $P_j \leftarrow 1 - P_\delta(\boldsymbol{\mu} + j\Sigma\boldsymbol{e}_1, \Sigma)$
  Compute $Q_j \leftarrow P_\delta(\boldsymbol{\mu} + j\Sigma\boldsymbol{e}_2, \Sigma)$ ▷ use Formula (11) or integral in Appendix A
end for
for all $k \in \{1, 2, 3\}$ **do**
  $M_k \leftarrow \sum_{j=0}^{k} \binom{k}{j} e^{\delta(k-j)+j\mu_1+\frac{j^2}{2}\Sigma_{1,1}} P_j + e^{k\mu_2+\frac{k^2}{2}\Sigma_{2,2}} Q_k$
end for
$m \leftarrow M_1 + c_2$
$v \leftarrow M_2 - M_1^2$
$\gamma \leftarrow (M_3 - 3M_2M_1 + 2M_1^3)v^{-3/2}$
Compute (c_0, μ_0, σ_0) from (m, v, γ), with the moment matching formulas (7)–(9).
return $(c_0, \mu_0, \sigma_0^2, P_0)$

where the blocks Σ_{11}, Σ_{12} and Σ_{22} are of size $(n-2)\times(n-2)$, $(n-2)\times 2$, and 2×2, respectively. Let $X_0 = \mathit{SLN}(c_0, \mu_0, \sigma_0^2)$ be the approximation of $M = \max(X_{n-1}, X_n)$ returned by Algorithm 1, and let $P_0 := \mathbb{P}(X_n \leq X_{n-1})$, $Q_0 = 1 - P_0$ (Note that P_0 and Q_0 can also be returned by Algorithm 1). Then, we propose to approximate the distribution of $(X_1, \ldots, X_{n-2}, M)$ by a random variable of the form $\mathit{SLN}(\boldsymbol{c}^+, \boldsymbol{\mu}^+, \Sigma^+)$, where $\boldsymbol{c}^+ = [c_1, \ldots, c_{n-2}, c_0]^T \in \mathbb{R}^{n-1}$, $\boldsymbol{\mu}^+ = [\mu_1, \ldots, \mu_{n-2}, \mu_0]^T \in \mathbb{R}^{n-1}$,

$$\Sigma^+ = \begin{bmatrix} \Sigma_{11} & \boldsymbol{u} \\ \boldsymbol{u}^T & \sigma_0^2 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{u} = \Sigma_{12}\begin{bmatrix} P_0 \\ Q_0 \end{bmatrix} \in \mathbb{R}^{n-2}.$$

A heuristic justification for this approximation is that it both works in the situation where $c_n - c_{n-1} \to +\infty$ and $c_n - c_{n-1} \to 0$. In the first case, M tends to X_n, so Q_0 tends to 1 and Σ^+ tends to the principal submatrix of Σ corresponding to the variables $(X_1, \ldots, X_{n-2}, X_n)$. In the latter case, M tends to $c_n + \exp(\max(Y_{n-1}, Y_n))$, and Clark's formula (3) holds for the covariance of the maximum of the two normal variables Y_{n-1}, Y_n. Moreover, we observed numerically that the matrix Σ^+ is positive semidefinite when the diagonal elements of Σ are small enough, for a wide range of input parameters: the following proposition gives a sufficient condition for Σ^+ to be positive semidefinite that only depends on the parameters of the distribution of (X_{n-1}, X_n); Figure 2 shows that this condition seems to hold with a high probability when $\max_i(\Sigma_{i,i})$ is small. In particular, during our tests the condition was never violated when $\max_i(\Sigma_{i,i}) < 0.6$, which is often true in practice. Indeed, for a log-normal variable $X \sim \mathit{LN}(\mu, \sigma^2)$, $\sigma^2 = 0.6$ already allows huge deviations from the nominal scenario: 95%-confidence interval is $[0.22m, 4.56m]$, where $m := e^\mu$ is the median of X.

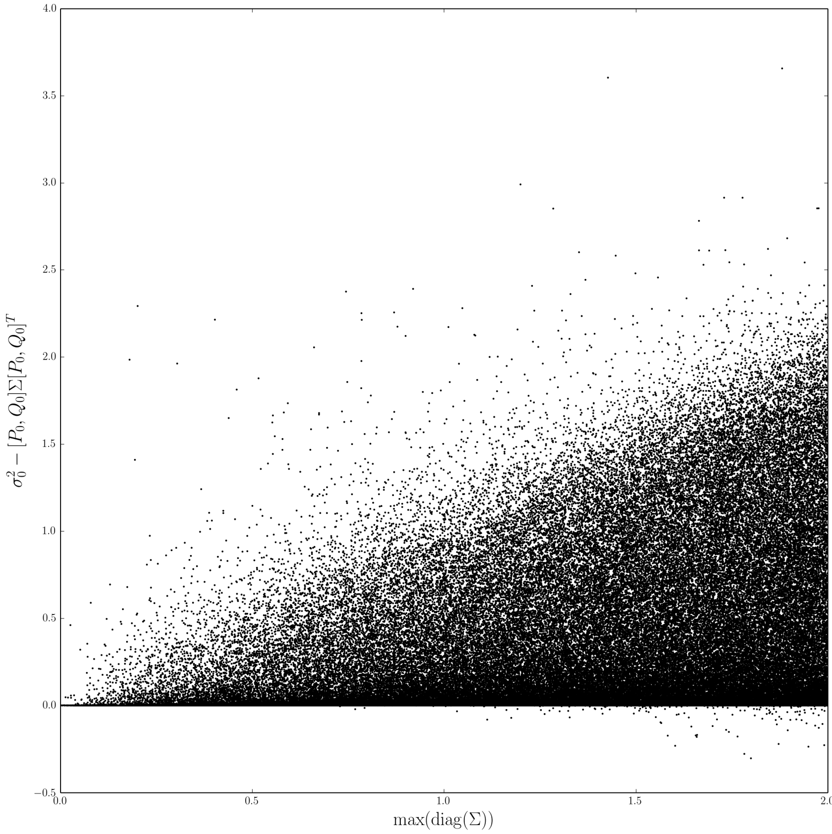

Figure 2: Scatter plot showing whether the sufficient condition of Proposition 2.2 for Σ^+ to be positive semidefinite holds, for $N = 10^5$ pairs of random variables $(X_1, X_2) \sim SLN(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$. The parameters of (X_1, X_2) were generated with $c_1 \sim LN(0, 2^2)$, $c_2 = 0$, $\mu_1(4, 2^2)$, $\mu_2(4, 2^2)$, $\Sigma_{1,1} \sim$ Uniform([0, 2]), $\Sigma_{2,2} \sim$ Uniform([0, 2]), and $\Sigma_{1,2} = \rho(\Sigma_{1,1}\Sigma_{2,2})^{1/2}$, where $\rho \sim$ Uniform([−1, 1]).

Proposition 2.2. *If $\sigma_0^2 \geq [P_0, Q_0]\Sigma_{22}[P_0, Q_0]^T$, then Σ^+ is positive semidefinite.*

Proof. We know that Σ is positive semidefinite, so by the Schur complement lemma, we have $\Sigma_{22} \succeq \Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12}$. Hence, the inequality of the proposition implies

$$\sigma_0^2 \geq [P_0, Q_0]\Sigma_{12}^T \Sigma_{11}^{-1} \Sigma_{12}[P_0, Q_0]^T = \boldsymbol{u}^T \Sigma_{11}^{-1} \boldsymbol{u},$$

which, by the Schur complement lemma again, implies that Σ^+ is positive semidefinite. □

The approach proposed above is summarized in Algorithm 2 (`MAX_2_SLN`). Note that the algorithm also addresses the case where the condition of Proposition 2.2 is violated, by scaling down the vector $\boldsymbol{u}$. Then, the maximum of n SLN variables can be approximated by applying `MAX_2_SLN` $(n-1)$ times, thanks to the relation

$$\max(X_1, \dots, X_n) = \max(X_1, \dots, X_{n-2}, \max(X_{n-1}, X_n)).$$

Algorithm 2 (`MAX_2_SLN`)
Approximates $M := (X_1, \dots, X_{n-2}, \max(X_{n-1}, X_n))$, where $\boldsymbol{X} \sim \mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$.

Input: $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{\mu} \in \mathbb{R}^n, \Sigma \in \mathbb{S}_+^n$, with $n \geq 2$;
Output: $(\boldsymbol{c}^+, \boldsymbol{\mu}^+, \Sigma^+)$, such that $\mathit{SLN}(\boldsymbol{c}^+, \boldsymbol{\mu}^+, \Sigma^+)$ approximates M,
and $Q_0 \in [0, 1]$ such that $\mathbb{P}[X_n \geq X_{n-1}] = Q_0$.

$(\Sigma_{11}, \Sigma_{12}, \Sigma_{22}) \leftarrow$ Block-decomposition of Σ, see (12)
$(c_0, \mu_0, \sigma_0^2, P_0) \leftarrow$ `MAX_BIVARIATE_SLN`$([c_{n-1}, c_n]^T, [\mu_{n-1}, \mu_n]^T, \Sigma_{22})$
$Q_0 \leftarrow 1 - P_0$
$\boldsymbol{c}^+ \leftarrow [c_1, \dots, c_{n-2}, c_0]^T \in \mathbb{R}^{n-1}$
$\boldsymbol{\mu}^+ \leftarrow [\mu_1, \dots, \mu_{n-2}, \mu_0]^T \in \mathbb{R}^{n-1}$
$\boldsymbol{u} \leftarrow \Sigma_{12}[P_0, Q_0]^T \in \mathbb{R}^{n-2}$
if $\sigma_0^2 < [P_0, Q_0]\Sigma_{22}[P_0, Q_0]^T$ **then**
  $\alpha \leftarrow \sigma_0 \left([P_0, Q_0]\Sigma_{22}[P_0, Q_0]^T\right)^{-1/2}$
  $\boldsymbol{u} \leftarrow \alpha \boldsymbol{u}$ ▷ correction factor to ensure $\Sigma^+ \succeq 0$
end if
$\Sigma^+ \leftarrow \begin{bmatrix} \Sigma_{11} & \boldsymbol{u} \\ \boldsymbol{u}^T & \sigma_0^2 \end{bmatrix} \in \mathbb{S}_+^{n-1}$
return $(\boldsymbol{c}^+, \boldsymbol{\mu}^+, \Sigma^+, Q_0)$

We can also use this algorithm to compute recursively an approximation of the probabilities $p_i = \mathbb{P}[\max(X_1, \dots, X_n) = X_i]$. This is what we do in Algorithm 3 (`MAX_K_SLN`), where an additional parameter $k \in \{1, \dots, n\}$ allows us to take the maximum of the k last variables only.

We point out that a recent paper [SZS06] studies the order in which variables are considered within the maximum operation (for the case of normal variables). The authors present a greedy approach to optimize the binary tree used to take pairwise maximums. Similar ideas could also be applied to the case of (shifted) lognormal variables, but we have not tried this so far.

Algorithm 3 (`MAX_K_SLN`)
Approximates $M := (X_1, \dots, X_{n-k}, \max(X_{n-k+1}, \dots, X_n))$, where $\boldsymbol{X} \sim \mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$.

Input: $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{\mu} \in \mathbb{R}^n, \Sigma \in \mathbb{S}_+^n$, and $k \in \{1, \dots, n\}$;
Output: $(\boldsymbol{c}^+, \boldsymbol{\mu}^+, \Sigma^+)$, such that $\mathit{SLN}(\boldsymbol{c}^+, \boldsymbol{\mu}^+, \Sigma^+)$ approximates M,
and $\boldsymbol{p} \in \mathbb{R}^k$ such that $\mathbb{P}[\max(X_{n-k+1}, \dots, X_n) = X_{n-k+i}] = p_i$.

$\boldsymbol{p} \leftarrow [1] \in \mathbb{R}^1$
for $i \in \{1, \dots, k-1\}$ **do**
  $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma, Q_0) \leftarrow$`MAX_2_SLN`$(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$
  $\boldsymbol{p} \leftarrow [1 - Q_0, Q_0 \boldsymbol{p}^T]^T$
end for
return $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma, \boldsymbol{p})$

2.5 Sum of shifted lognormal variables

Let $\boldsymbol{Y} \sim \mathit{SLN}(\boldsymbol{c}^Y, \boldsymbol{\mu}^Y, \Sigma) \in \mathbb{R}^n$, and let $\boldsymbol{X} \sim \mathit{SLN}(\boldsymbol{c}^X, \boldsymbol{\mu}^X, \mathrm{Diag}(\boldsymbol{\sigma})^2) \in \mathbb{R}^n$ be independent from $\boldsymbol{Y}$. In this section, we propose to approximate the law of $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y}$ by a (multivariate) shifted lognormal distribution, using the method of moments. To do so, we use the following relations: for all $i \in \{1, \dots, n\}$,

$$\begin{aligned}\mathbb{E}[S_i] &= \mathbb{E}[X_i] + \mathbb{E}[Y_i]\\ \mathbb{V}[S_i] &= \mathbb{V}[X_i] + \mathbb{V}[Y_i]\\ \mathrm{skew}[S_i] &= \frac{\mathbb{V}[X_i]^{\frac{3}{2}}\,\mathrm{skew}[X_i] + \mathbb{V}[Y_i]^{\frac{3}{2}}\,\mathrm{skew}[Y_i]}{(\mathbb{V}[X_i] + \mathbb{V}[Y_i])^{\frac{3}{2}}}.\end{aligned}$$

Then, we can use the moment matching formulas (7)–(9) to approximate S_i by a shifted lognormal variable with parameters c_i, μ_i, σ_i^2. This suggests to use a bivariate distribution of the form $\mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma')$, where the diagonal elements of Σ' are $\Sigma'_{i,i} = \sigma_i^2$, and the off-diagonal elements can be set by equating $\mathrm{cov}(S_i, S_j)$ and $\mathrm{cov}(Y_i, Y_j)$:

$$\Sigma'_{i,j} = \log\left[1 + \frac{(\mathbb{E}[Y_i] - c_i^Y)(\mathbb{E}[Y_j] - c_j^Y)}{(\mathbb{E}[S_i] - c_i)(\mathbb{E}[S_j] - c_j)}(e^{\Sigma_{i,j}} - 1)\right].$$

Again, we conjecture that the approximate variance-covariance matrix Σ' is positive semidefinite for practical values of Σ. At least the situation $\Sigma' \not\succeq 0$ never occurred during our numerical experiments, cf. Section 4. Should this situation arise, a possible fix would be to take a projection of Σ' over the cone $\mathbb{S}_+^n$, for which an explicit expression is given by the eigenvalue decomposition [SA79].

This approach is summarized in Algorithm 4.

3 Makespan and tardiness in an activity network

Consider a directed acyclic graph $G = (V, E)$ with random variables $X_e \sim \mathit{SLN}(c_e, \mu_e, \sigma_e^2)$ attached to each arc $e \in E \subset V \times V$. Each arc of the graph corresponds to an activity, and no activity can start before all its predecessors are finished. We assume that each activity starts as early as possible (we say that the schedule is *semi-active*; see [SKD95]). Therefore, all activities corresponding to outgoing arcs of a particular vertex $v \in V$ have the same starting time, which we denote by Y_v. The completion time of activity $e = (u, v) \in E$ is denoted by $Z_e = Y_u + Z_{(u,v)}$. In addition we assume that no activity starts before $t = 0$. The semi-active property of the schedule can be expressed as follows:

$$\forall v \in V, \qquad Y_v = \max_{u \in \mathrm{pred}(v)} \left(Y_u + X_{(u,v)}\right) = \max_{e \in \delta^-(v)} Z_e,$$

where $\mathrm{pred}(v) \subset V$ denotes the set of predecessors of v in G, $\delta^-(v) \subset E$ is the set of incoming arcs of v, and the maximum of an empty sequence is set to 0, by convention.

Algorithm 4 (SUM_SLN)
Approximates $\boldsymbol{S} = \boldsymbol{X} + \boldsymbol{Y}$, where $\boldsymbol{X} \sim \mathit{SLN}(\boldsymbol{c}^X, \boldsymbol{\mu}^X, \mathrm{Diag}(\boldsymbol{\sigma})^2) \in \mathbb{R}^n$ and $\boldsymbol{Y} \sim \mathit{SLN}(\boldsymbol{c}^Y, \boldsymbol{\mu}^Y, \Sigma) \in \mathbb{R}^n$ are mutually independent.

Input: $\boldsymbol{c}^X, \boldsymbol{\mu}^X, \boldsymbol{\sigma}^2, \boldsymbol{c}^Y, \boldsymbol{\mu}^Y \in \mathbb{R}^n, \Sigma \in \mathbb{S}_+^n$;
Output: $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma')$, such that $\mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma')$ approximates the law of $\boldsymbol{S}$.
  for all $i \in \{1, \dots, n\}$ **do**
    $m_i^X \leftarrow c_i^X + \exp(\mu_i^X + \frac{1}{2}\sigma_i^2)$; $m_i^Y \leftarrow c_i^Y + \exp(\mu_i^Y + \frac{1}{2}\Sigma_{i,i})$
    $m_i \leftarrow m_i^X + m_i^Y$
    $v_i^X \leftarrow (m_i^X)^2\,(e^{\sigma_i^2} - 1)$; $v_i^Y \leftarrow (m_i^Y)^2\,(e^{\Sigma_{i,i}} - 1)$
    $v_i \leftarrow v_i^X + v_i^Y$
    $\gamma_i^X \leftarrow (e^{\sigma_i^2} + 2)(e^{\sigma_i^2} - 1)^{1/2}$; $\gamma_i^Y \leftarrow (e^{\Sigma_{i,i}} + 2)(e^{\Sigma_{i,i}} - 1)^{1/2}$
    $\gamma_i \leftarrow v_i^{-3/2}\big((v_i^X)^{3/2}\gamma_i^X + (v_i^Y)^{3/2}\gamma_i^Y\big)$
    Compute $(c_i, \mu_i, \Sigma'_{i,i})$ from (m_i, v_i, γ_i), with the formulas (7)–(9).
  end for
  for all $(i,j) \in \{1, \dots, n\}^2$, $i \neq j$ **do**
    $\Sigma'_{i,j} \leftarrow \log[1 + \frac{m_i^Y - c_i^Y}{m_i - c_i}\frac{m_j^Y - c_j^Y}{m_j - c_j}(e^{\Sigma_{i,j}} - 1)]$
  end for
  return $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma')$

The *makespan* of the activity network corresponds to the time at which all activities are finished:

$$M \;=\; \max_{v \in V} Y_v \;=\; \max_{e \in E} Z_e.$$

If we are given a *due date* d_v and a per unit tardiness cost $\kappa_v \geq 0$ for the starting time of activities $\{e \in \delta^+(v)\}$ that start from vertex $v \in V$, we can define the total tardiness by

$$T = \sum_{v \in V} \kappa_v (Y_v - d_v)^+,$$

where $(x)^+ := \max(x, 0)$ is the nonnegative part of x. Note that we could define the tardiness analogously for ending times of activities, by defining the tardiness costs arc-wise (rather than vertex-wise).

3.1 Distribution of the makespan

From now on, we assume without loss of generality that V contains 2 dummy vertices $\mathtt{s}$ and $\mathtt{f}$, corresponding to the start and the end of the schedule, respectively. We can always reduce to this case, by adding a dummy arc with zero duration from $\mathtt{s}$ to every vertex in G that has no predecessors. Similarly, every vertex in G without any successors is connected to $\mathtt{f}$ by a dummy arc. Note that the makespan M corresponds to the duration on the longest $(\mathtt{s}, \mathtt{f})-$path of G. This longest path is called *critical path* in the literature on project scheduling, and it is unique with probability one under mild conditions on the activity durations [KBY+07].

The notion of critical path can be extended to the case with stochastic activity durations. The criticality index of activity is defined as the probability that it belongs to the critical path; see [DE85]. For all $v \in V \setminus \{\mathtt{s}\}$ and $e \in E$, we define $\boldsymbol{f}_e^v$ as the probability that arc e belongs to the longest path from $\mathtt{s}$ to v. It is not difficult to see that $\boldsymbol{f}^v$ must be an $(\mathtt{s}, v)-$flow of unit value, that is, it satisfies the flow conservation equations

$$\forall u \in V, \sum_{e \in \delta^-(u)} f_e^v - \sum_{e \in \delta^+(u)} f_e^v = \begin{cases} -1 & \text{if u=}\mathtt{s}; \\ 1 & \text{if u=v;} \\ 0 & \text{otherwise.} \end{cases}$$

Given a topological sort of the graph $\mathtt{s} = v_1, v_2, \ldots, v_n = \mathtt{f}$, we can compute $M = Y_{v_n}$ recursively. We initialize the recursion with $Y_{v_1} = 0$. Then, for all $i = 2, \ldots, n$, we compute

$$\begin{cases} \forall u \in \text{pred}(v_i), \ Z_{(u,v_i)} = Y_u + X_{(u,v_i)}; \\ Y_{v_i} = \max_{e \in \delta^-(v_i)}(Z_e). \end{cases} \tag{13}$$

Our approach is to approximate the result of each of these operations by a shifted lognormal variable; it is summarized in Algorithm 5 (`MAKESPAN_SLN`). At line 5 of the algorithm, we use Algorithm 4 (`SUM_SLN`) to approximate the joint law of $\boldsymbol{Y}_i = [Y_{v_0}, \ldots, Y_{v_{i-1}}]^T$ and the variables Z_e ($\forall e \in \delta^-(v_i)$). This is possible because $\boldsymbol{Y}_i$ and the $Z_e's$ are mutually independent, and because for $\boldsymbol{Y} \sim \mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$ we can use relations of the form:

$$\begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Y} + \boldsymbol{X} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Y} \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{X} \end{bmatrix}, \text{ where } \begin{bmatrix} \boldsymbol{Y} \\ \boldsymbol{Y} \end{bmatrix} \sim \mathit{SLN}\left(\begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{c} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu} \end{bmatrix}, \begin{bmatrix} \Sigma & \Sigma \\ \Sigma & \Sigma \end{bmatrix} \right). \tag{14}$$

Then, we use Algorithm 3 (`MAX_K_SLN`) to replace the Z_e's by their maximum. So at the end of the ith iteration $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$ are the parameters of a SLN approximation of $[Y_{v_0}, \ldots, Y_{v_i}]^T$. Finally, at lines 7-8 we use the vector of probabilities $\boldsymbol{p}$ to compute an approximation of the critical flow $\boldsymbol{f}^{v_i}$ from $\mathtt{s}$ to v_i.

3.2 Distribution of the tardiness

Recall that the tardiness is defined by $T = \sum_{v \in V} \kappa_v (Y_v - d_v)^+$, where d_v and κ_v are parameters specifying the due date and tardiness cost per unit of time for vertex v. We can use Algorithm 5 to approximate the joint distribution of the Y_v's by a shifted lognormal. In what follows, we assume that $\boldsymbol{Y} = [Y_{v_1}, \ldots, Y_{v_n}]^T \sim \mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$ (with respect to some topological sort $v_1, \ldots, v_n$ of the graph).

Denote by $D_v = (Y_v - d_v)^+$ the tardiness of vertex v. If we are solely interested in the expected value of T, we only need to approximate the expected values of the D_v's. For $X \sim \mathcal{N}(\mu, \sigma^2)$, $a > 0$ and $k \in \mathbb{N}$, it holds that

$$H_k(a, \mu, \sigma^2) := \mathbb{E}\left[\left(\max(a, e^X) \right)^k \right] = \int_{x=-\infty}^{\log a} a^k \varphi_1\left(x; \mu, \sigma^2\right) \, dx + \int_{x=\log a}^{\infty} e^{kx} \varphi_1\left(x; \mu, \sigma^2\right) \, dx$$

Algorithm 5 (`MAKESPAN_SLN`)

Estimate the distribution of the makespan of the activity graph G

Input: Directed acyclic graph G with topological order $\mathtt{s} = v_1, \ldots, v_n = \mathtt{f}$,
Distribution $X_e \sim \mathit{SLN}(c_e, \mu_e, \sigma_e^2)$, $\forall e \in E$;

Output: $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$, such that $\mathit{SLN}(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma)$ approximates $(Y_{v_1}, \ldots, Y_{v_n})$,
and approximate critical flows $\boldsymbol{f}^v$, $\forall v \in V$.

Initialization: $\boldsymbol{c} \leftarrow [0]$; $\boldsymbol{\mu} \leftarrow [-\infty]$; $\Sigma_{1,1} \leftarrow [1]$ ▷ i.e., $Y_{v_0} \leftarrow 0$
$f_e^{v_1} \leftarrow 0 \qquad (\forall e \in E)$ ▷ critical flow from `s` to `s`
for all $i \in \{2, \ldots, n\}$ **do**
  Let $u_1, \ldots, u_k \in V$ be the predecessors of v_i
  Use `SUM_SLN()` to compute $(\boldsymbol{c}_i, \boldsymbol{\mu}_i, \Sigma_i)$ such that the joint law of $(\boldsymbol{Y}_i, \boldsymbol{Z}_i) \in \mathbb{R}^{i+k}$ can be approximated by $\mathit{SLN}(\boldsymbol{c}_i, \boldsymbol{\mu}_i, \Sigma_i)$, where $\boldsymbol{Y}_i = [Y_{v_0}, \ldots, Y_{v_{i-1}}]^T$ and $\boldsymbol{Z}_i = [Y_{u_1} + X_{(u_1,v_i)}, \ldots, Y_{u_k} + X_{(u_k,v_i)}]^T$; see Eq. (14).
  $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma, \boldsymbol{p}) \leftarrow$ `MAX_K_SLN`$(\boldsymbol{c}_i, \boldsymbol{\mu}_i, \Sigma_i, k)$
  $f_e^{v_i} \leftarrow \sum_{j=1}^k p_j f_e^{u_j} \qquad (\forall e \in E \setminus \delta^-(v_i))$
  $f_{(u_j,v_i)}^{v_i} \leftarrow p_j \qquad (\forall j \in \{1, \ldots, k\})$
end for
return $(\boldsymbol{c}, \boldsymbol{\mu}, \Sigma, \boldsymbol{f}^{v_1}, \ldots, \boldsymbol{f}^{v_n})$

By using the relation $e^{kx}\varphi_1\left(x; \mu, \sigma^2\right) = e^{k\mu + \frac{1}{2}k^2\sigma^2}\varphi_1\left(x; \mu + k\sigma^2, \sigma^2\right)$, we find that for all $a \in \mathbb{R}$,

$$H_k(a, \mu, \sigma^2) = \begin{cases} a^k \Phi(\frac{\log(a)-\mu}{\sigma}) + e^{k\mu+\frac{1}{2}k^2\sigma^2}\Phi(\frac{\mu - log(a)}{\sigma} + k\sigma) & \text{if } a > 0 \\ e^{k\mu+\frac{1}{2}k^2\sigma^2} & \text{otherwise.} \end{cases}$$

Then, we have $\mathbb{E}[T] = \sum_{v\in V} \kappa_v \, \mathbb{E}[D_v]$, where $\mathbb{E}[D_v] = H_1\big(d_v - c_v, \mu_v, \Sigma_{v,v}\big) + c_v - d_v$.

The variance of T can be expressed as a function of the covariances between the D_v's: $\mathbb{V}[T] = \sum_{u,v\in V\times V} \kappa_u \, \kappa_v \operatorname{cov}[D_u, D_v]$. By writing $\mathbb{V}[D_v] = \mathbb{E}[D_v^2] - \mathbb{E}[D_v]^2$, we obtain the expression

$$\mathbb{V}[D_v] = H_2\big(d_v - c_v, \mu_v, \Sigma_{v,v}\big) - H_1\big(d_v - c_v, \mu_v, \Sigma_{v,v}\big)^2.$$

By using the same technique as above, we can also express the cross-covariances in terms of the bivariate normal CDF Φ_2. Indeed, we provide in Appendix B formulas to compute $G(a, b, \boldsymbol{\mu}, \Sigma) := \mathbb{E}\left[\max(a, e^X)\max(b, e^Y)\right]$, where $(X, Y) \sim \mathcal{N}_2(\boldsymbol{\mu}, \Sigma)$. Then, we have:

$$\begin{aligned} \operatorname{cov}[D_u, D_v] =& G\left(d_u - c_u, d_v - c_v, \begin{bmatrix} \mu_u \\ \mu_v \end{bmatrix}, \begin{bmatrix} \Sigma_{u,u} & \Sigma_{u,v} \\ \Sigma_{u,v} & \Sigma_{v,v} \end{bmatrix}\right) \\ &- H_1\big(d_u - c_u, \mu_u, \Sigma_{u,u}\big) H_1\big(d_v - c_v, \mu_v, \Sigma_{v,v}\big). \end{aligned} \tag{15}$$

If we need an approximation for the distribution of T, note that we need to use a family of distributions that can give a strictly positive weight to the $T = 0$. Indeed, if $\boldsymbol{c} < \boldsymbol{d}$, T is not

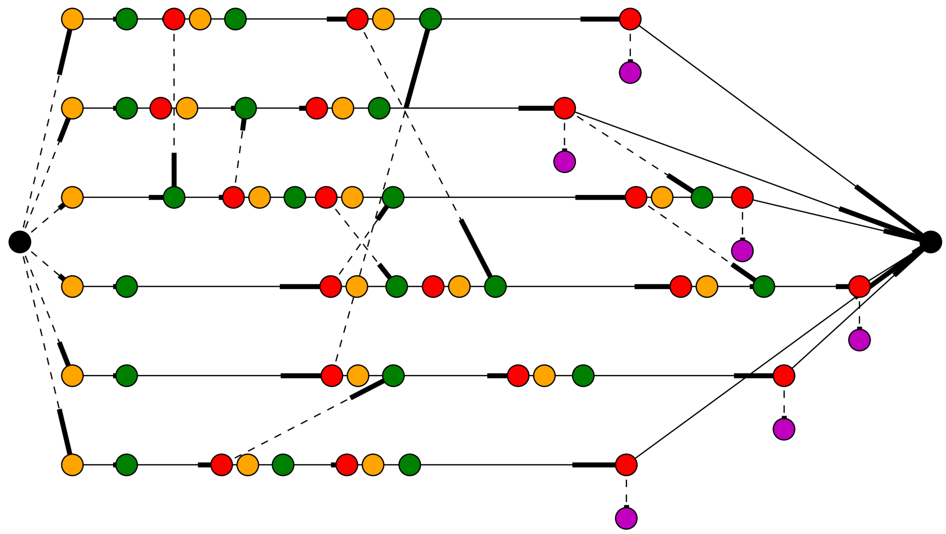

Figure 3: Activity graph of one OR scheduling instance

absolutely continuous with respect to Lebesgue measure in 0, and the probability that $T = 0$ can be expressed thanks to the multivariate normal CDF:

$$P_0 := \mathbb{P}[T = 0] = \mathbb{P}[\forall k = 1, \ldots, m, \quad Z_{e_k} \leq d_{e_k}] = \Phi_m(\log(\boldsymbol{d} - \boldsymbol{c}); \boldsymbol{\mu}, \Sigma).$$

Our natural candidate is to approximate the distribution of T by that of $\max(X, 0)$, where $X \sim \mathit{SLN}(c_0, \mu_0, \sigma_0^2)$. We propose to set the parameters (c_0, μ_0, σ_0^2) so that T matches the first two moments of X^+, and $\mathbb{P}[X \leq 0] = P_0$.

4 Numerical Experiments

We run numerical experiments for $N = 516$ instances from an application to surgery scheduling, based on real data from the Charité hospital in Berlin. On a short term perspective, that is, when the goal is to schedule resources for the next day of operation (a problem often termed as *operational planning* in the literature on operating room (OR) management, see e.g. [GG11]), the most relevant criterion is the *overtime* [DT02, MDE06], i.e. the time used by surgical procedures outside of the regular block time. This criterion can be expressed as the total tardiness of dummy vertices that indicate the end of surgical procedures in each OR.

An exemplary activity network from one of these instances is depicted in Figure 3. Here, the two black nodes on both sides of the graph represent the start and the end of the schedule. Each *row* corresponds to activities taking place in the same operating room: yellow, green, and red vertices correspond to the start of the setup, surgical procedures, and clean-up of an operation. The horizontal position of a node indicates the expected starting time of an activity. Purple nodes indicate the end of the last surgical procedure in a given OR, so that the overtime can be expressed as the total tardiness of all purple nodes (with respect to due dates indicating the end of the regular block time in each room). Finally, dashed arcs are dummy activities with zero-duration (nodes linked by dashed arcs could have been merged, but we left the dummy activities on the figure for the sake of visualization). For example, a dummy arc linking a red node R to a green node G indicates that the surgical procedure starting at G must wait for the end of the procedure ending at R, because they use a common resource (in this situation, the same surgeon).

We used maximum likelihood estimators to fit the parameters of a lognormal model for the durations of $N = 20.849$ surgical procedures, and for the time required to prepare and clean-up the OR before and after each operation. Our model is similar to [SHDV10], and relies on characteristics of the patient, operation, and surgical team.

In a follow-up work, we want to use metaheuristics to optimize the allocation of resources in the operating rooms (OR), and use the parametric method presented in this article to quickly evaluate tentative schedules. In this article, we simply evaluate the quality of the approximation of the makespan and the tardiness of a given schedule, in terms of mean and standard deviation. For the sake of planning and risk management, another relevant statistic are upper quantiles of the makespan and overtime. This measure is also known as *value at risk* (VaR) in the literature on portfolio management:

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}[X \geq x] \leq \alpha\}.$$

For each schedule, we computed these statistics by means of a Monte-Carlo simulation with 10^5 runs. All relative errors are given with respect to the values computed in this Monte-Carlo simulation, which we consider as the true values. We computed three approximations of the distributions of the Y variables (in particular, recall that the makespan is $M = Y_{v_n}$):

- The normal approximation of Sculli [Scu83]: each duration is first approximated by a normal variable by using the method of moments. Then, we iteratively approximate the Y_{v_i}'s by a normal variable using the method of moments, using Clark's formulas (1)–(3) for the max-operation in Equation (13).

- A lognormal approximation based a straightforward variant of the above method, in which all Y_{v_i} are approximated by a lognormal. The sum of two lognormals is approximated by another lognormal variable using the method of moments. For the maximum of log-normal variables, $X_{\max} := \max(X_1, \ldots, \log X_n)$, we used Clark's method to construct an approximation $\max(\log X_1, \ldots, \log X_n) \overset{\text{approx}}{\sim} \mathcal{N}(\mu_0, \sigma_0^2)$, which suggests the approximation $X_{\max} \overset{\text{approx}}{\sim} LN(\mu_0, \sigma_0^2)$.

- The shifted lognormal approximation proposed in the present article (Algorithm 5).

Then, for each of these three approximations we compute an estimated of the mean and the variance of the overtime T, by following the lines of Section 3.2. To estimate values-at-risk of the overtime, we fit a shifted lognormal variable X so that $\mathbb{E}[\max(X, 0)], \mathbb{V}[\max(X, 0)]$ and $\mathbb{P}[X \leq 0]$ match with our estimates of $\mathbb{E}[T], \mathbb{V}[T]$, and $\mathbb{P}[T = 0]$. Thereafter we used the estimates $\mathrm{VaR}\,\alpha(T) \simeq \mathrm{VaR}_\alpha(\max(X, 0))$.

Finally, we also computed estimates of the statistics of interest based on a Monte-Carlo approximations with $N = 40$ runs. The number $N = 40$ was chosen such that the CPU time

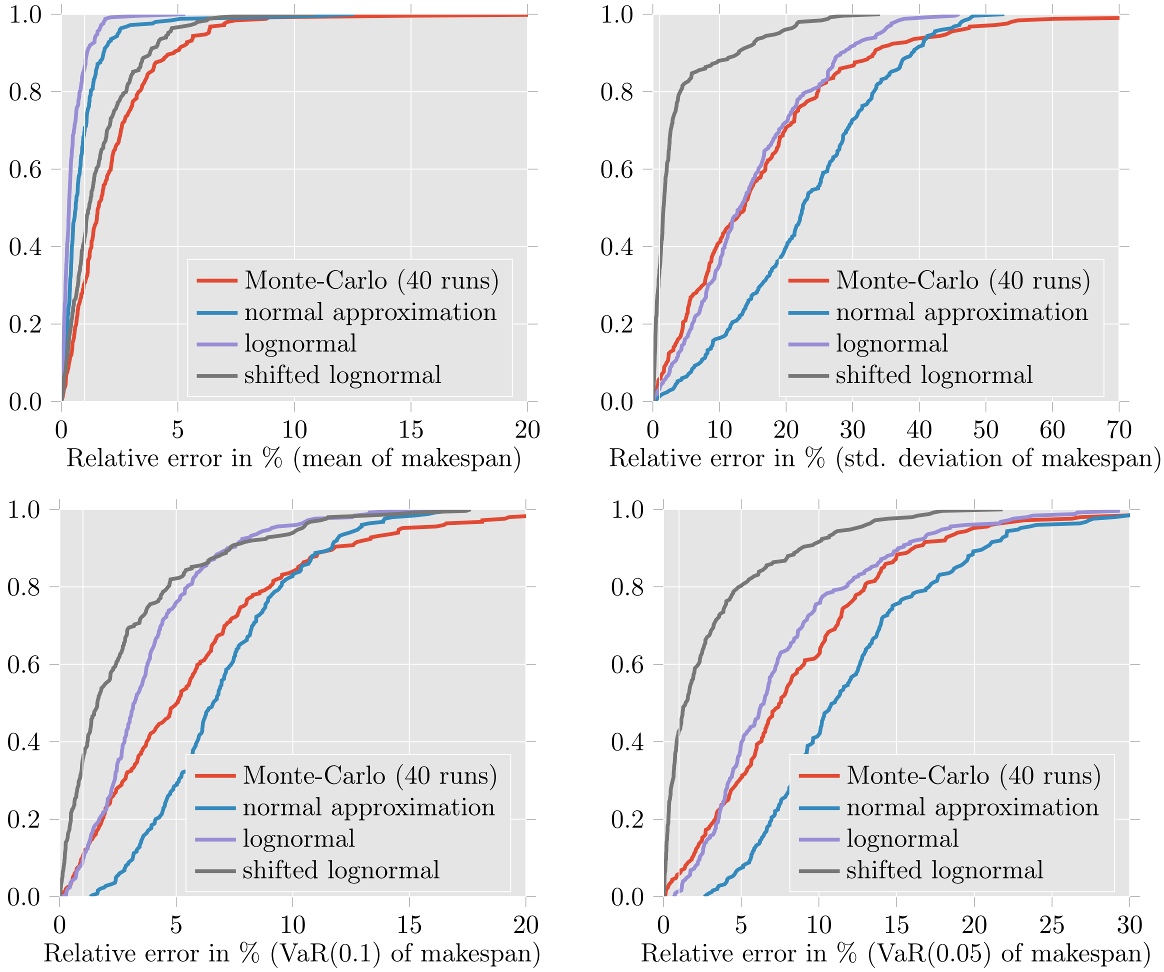

Figure 4: Distribution of the error for the approximations of the mean, standard deviation and values-at-risk of the makespan, over $N = 516$ instances.

spent by the Monte-Carlo simulation and our shifted lognormal approximation procedure are approximately equal.

Figure 4 shows the distribution of the relative error among the $N = 516$ instances for the estimates of the mean, standard deviation, and values at risk of the makespan. Interestingly, we see that the normal and lognormal approximations actually provide better estimates of the mean, but the shifted lognormal is far better in terms of standard deviation and upper quantiles.

The distribution of the relative error for the four estimates of the mean, standard deviation, and value-at-risk 10% are depicted in Figure 5. In addition, we have also plotted the distribution of the absolute error for the estimates of the probability of zero-overtime (relative error is not appropriate because this probability is often very small). The estimates based on the shifted lognormal approximation are the clear winners for the mean and the standard deviation, and

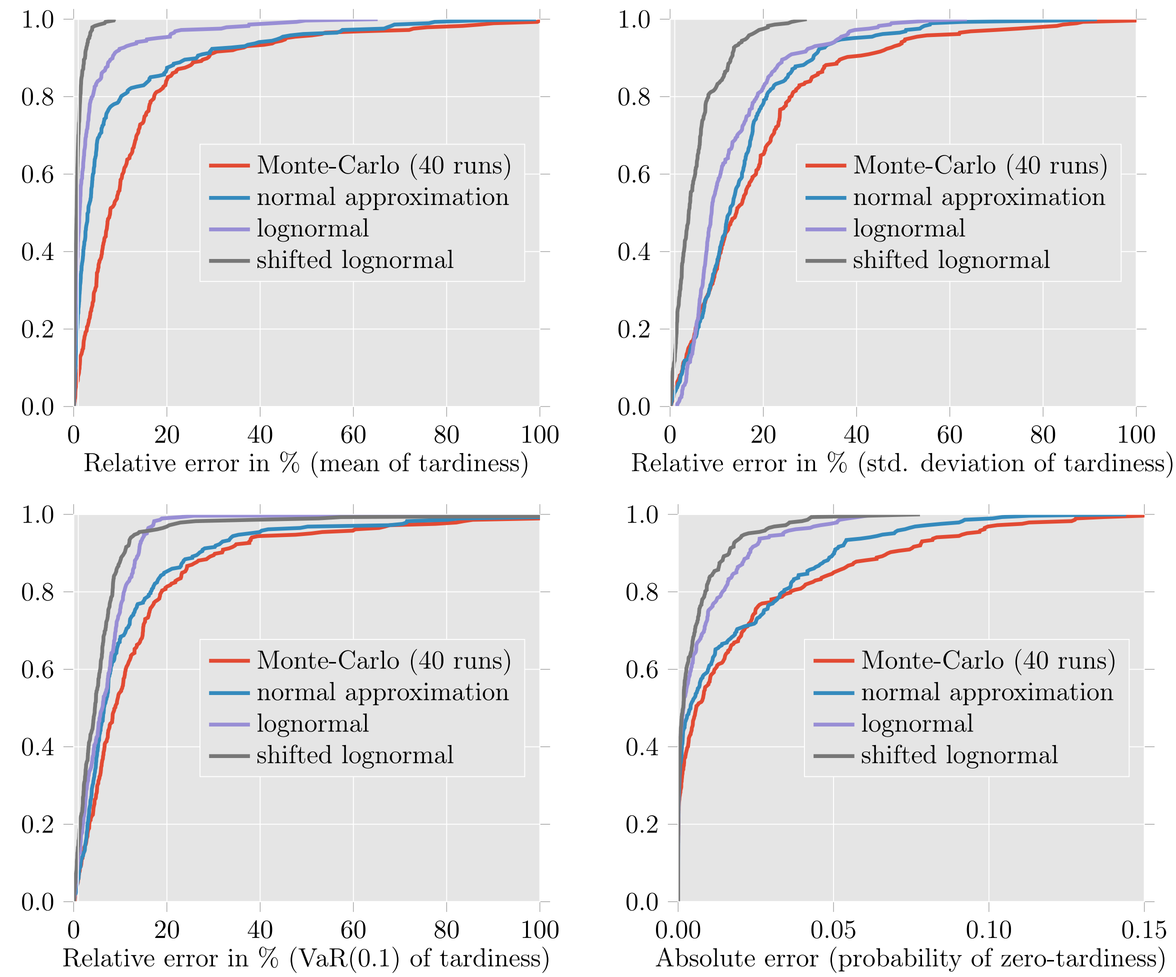

Figure 5: Distribution of the error for the estimations of the mean, standard deviation and VaR(0.1) of the overtime, and for the estimations of the probability of zero-overtime, over $N = 516$ instances.

they yield slightly better results than the approximation based on (non-shifted) lognormals. The fact that the estimates of the mean are better with the shifted lognormal comes from the fact that the standard deviations of the starting times Y_v are better approximated with this procedure. Indeed, to approximate $\mathbb{E}[D_v] = \mathbb{E}[(Y_v - d_v)^+]$ it is not enough to have a good estimate $\mathbb{E}[Y_v]$; we also need some information on how often $Y_v \geq d_v$, and this information is (partially) provided by the second moment of Y_v.

For example, our method provides an estimation of the mean overtime within 5% of the true value for 98% of the instances, while this number falls to 83%, 68%, and 34% of the instances for the lognormal, normal, and Monte-Carlo (with 40 runs) methods, respectively. The standard deviation of the estimation error of the mean overtime is 24.5 times larger with the Monte-Carlo approach than with the shifted log-normal approach. This suggests that we would require approximately $24.5^2 \simeq 600$ times more simulation runs to obtain estimates of a similar quality by using the Monte-Carlo method.

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A Computation of normal probabilities over $\mathcal{D}_\delta$

The next proposition expresses $P_\delta(\boldsymbol{\mu}, \Sigma)$ as a one-dimensional integral, thanks to a change of variable. Then, fast approximations of $P_\delta(\boldsymbol{\mu}, \Sigma)$ can be obtained from the Gauss-Legendre quadrature; see e.g. [GW84].

Proposition A.1. *Let $\delta \in \mathbb{R}, \boldsymbol{\mu} \in \mathbb{R}^2$ and consider a variance-covariance matrix*

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \in \mathbb{S}_+^2$$

for some $\sigma_1, \sigma_2 > 0$, $\rho \in (-1, 1)$. Then,

$$P_\delta(\boldsymbol{\mu}, \Sigma) = \int_{t=\frac{\delta-\mu_2}{\sigma_2}}^{\infty} \varphi(t)\, \Phi(f(t))\, dt, \tag{16}$$

where the function f is defined for all $t > \frac{\delta-\mu_2}{\sigma_2}$ by

$$f(t) := \frac{1}{\sigma_1\sqrt{1-\rho^2}} \left(\log\left(e^{\sigma_2 t+\mu_2} - e^\delta\right) - \mu_1 - \rho\sigma_1 t \right).$$

Proof. Consider the change of variables

$$X_1 = \frac{1}{\sqrt{1-\rho^2}} \left(\frac{W_1-\mu_1}{\sigma_1} - \rho\frac{W_2-\mu_2}{\sigma_2} \right), \quad X_2 = \frac{W_2-\mu_2}{\sigma_2},$$

which is well known to transform $\boldsymbol{W} \sim \mathcal{N}_2(\boldsymbol{\mu}, \Sigma)$ into a pair of independent and standardized normal variables, $\boldsymbol{X} \sim \mathcal{N}_2(0, I)$. Then, we have

$$\begin{aligned} \boldsymbol{W} \in \mathcal{D}_\delta \iff e^{W_2} \geq e^\delta + e^{W_1} &\iff e^{\sigma_2 X_2+\mu_2} \geq e^\delta + e^{\sigma_1\sqrt{1-\rho^2}X_1+\rho\sigma_1 X_2+\mu_1} \\ &\iff e^{\sigma_1\sqrt{1-\rho^2}X_1} \leq \left(e^{\sigma_2 X_2+\mu_2} - e^\delta\right) e^{-(\rho\sigma_1 X_2+\mu_1)} \\ &\iff X_1 \leq f(X_2). \end{aligned}$$

Note that $f(t)$ is defined for all $t > \frac{\delta-\mu_2}{\sigma_2}$. Hence,

$$\begin{aligned} P_\delta(\boldsymbol{\mu}, \Sigma) = \int_{\boldsymbol{y}\in\mathcal{D}_\delta} \varphi_2(\boldsymbol{y}; \boldsymbol{\mu}, \Sigma)\, d^2\boldsymbol{y} &= \int_{x_2=\frac{\delta-\mu_2}{\sigma_2}}^{\infty} \int_{x_1=-\infty}^{f(x_2)} \varphi_2(\boldsymbol{x}; \boldsymbol{0}, I)\, d^2\boldsymbol{x} \\ &= \int_{t=\frac{\delta-\mu_2}{\sigma_2}}^{\infty} \varphi(t)\, \Phi(f(t))\, dt, \end{aligned}$$

where we have used the fact that for all $\boldsymbol{x} \in \mathbb{R}^2$, $\varphi_2(\boldsymbol{x}; \boldsymbol{0}, I) = \varphi(x_1)\varphi(x_2)$.

□

B Covariance between dependent shifted lognormals

In Section 3.2 we need to compute cross-covariances between dependent shifted lognormal. Equation (15) expresses these covariances using the function $G(a, b, \boldsymbol{\mu}, \Sigma) := \mathbb{E}[\max(a, e^X)\max(b, e^Y)]$, where the expectation is taken with respect to $(X, Y) \sim \mathcal{N}_2(\boldsymbol{\mu}, \Sigma)$. We will give below a formula for $G(a, b, \boldsymbol{\mu}, \Sigma)$ relying on the bivariate normal CDF Φ_2. We need to distinguish 4 cases in function of the sign of a and b.

- If $a \leq 0, b \leq 0$, note that

$$e^X e^Y = e^{X+Y} \sim LN(\mu_1 + \mu_2, \Sigma_{1,1} + \Sigma_{2,2} + 2\Sigma_{1,2}),$$

Then, a formula for $G(a, b, \boldsymbol{\mu}, \Sigma)$ is easily obtained:

$$G(a, b, \boldsymbol{\mu}, \Sigma) = \mathbb{E}[e^{X+Y}] = e^{\mu_1+\mu_2+\frac{1}{2}(\Sigma_{1,1}+\Sigma_{2,2}+2\Sigma_{1,2})}.$$

- If $a \leq 0, b > 0$, we need to use the identity

$$\forall \boldsymbol{v} \in \mathbb{R}^n, \quad e^{\boldsymbol{v}^T \boldsymbol{x}} \varphi_n(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) = e^{\boldsymbol{\mu}^T \boldsymbol{v} + \frac{1}{2}\boldsymbol{v}^T \Sigma \boldsymbol{v}} \varphi_n(\boldsymbol{x}; \boldsymbol{\mu} + \Sigma \boldsymbol{v}, \Sigma). \tag{17}$$

Recall that $\boldsymbol{e}_1 = [1, 0]^T$ and $\boldsymbol{e}_2 = [0, 1]^T$. In addition, we write $\mathbf{1} = [1, 1]^T$.

$$\begin{aligned}
G(a, b, \boldsymbol{\mu}, \Sigma) &= \mathbb{E}[e^X \max(b, e^Y)] \\
&= \int_{\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 \leq \log b\}} b e^{x_1} \varphi_2(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) + \int_{\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 > \log b\}} e^{x_1+x_2} \varphi_2(\boldsymbol{x}; \boldsymbol{\mu}, \Sigma) \\
&= b e^{\mu_1 + \frac{1}{2}\Sigma_{1,1}} \int_{\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 \leq \log b\}} \varphi_2\left(\boldsymbol{x}; \boldsymbol{\mu} + \Sigma \boldsymbol{e}_1, \Sigma\right) \\
&\quad + e^{\boldsymbol{\mu}^T \mathbf{1} + \frac{1}{2}\mathbf{1}^T \Sigma \mathbf{1}} \int_{\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 > \log b\}} \varphi_2\left(\boldsymbol{x}; \boldsymbol{\mu} + \Sigma \mathbf{1}, \Sigma\right)
\end{aligned}$$

In the last expression, the integrals represent normal probabilities over half-spaces of $\mathbb{R}^2$. These probabilities can be expressed by using the standard normal CDF:

$$\begin{aligned}
G(a, b, \boldsymbol{\mu}, \Sigma) &= b e^{\mu_1 + \frac{1}{2}\Sigma_{1,1}} \Phi\left(\frac{\log b - \mu_2 - \Sigma_{1,2}}{\Sigma_{2,2}^{1/2}}\right) \\
&\quad + e^{\mu_1+\mu_2+\frac{1}{2}(\Sigma_{1,1}+\Sigma_{2,2}+2\Sigma_{1,2})} \left[1 - \Phi\left(\frac{\log b - \mu_2 - \Sigma_{1,2} - \Sigma_{2,2}}{\Sigma_{2,2}^{1/2}}\right)\right].
\end{aligned}$$

- Similarly, if $a > 0, b \leq 0$,

$$\begin{aligned}
G(a, b, \boldsymbol{\mu}, \Sigma) &= \mathbb{E}[e^Y \max(a, e^X)] \\
&= a e^{\mu_2 + \frac{1}{2}\Sigma_{2,2}} \Phi\left(\frac{\log a - \mu_1 - \Sigma_{1,2}}{\Sigma_{1,1}^{1/2}}\right) \\
&\quad + e^{\mu_1+\mu_2+\frac{1}{2}(\Sigma_{1,1}+\Sigma_{2,2}+2\Sigma_{1,2})} \left[1 - \Phi\left(\frac{\log a - \mu_1 - \Sigma_{1,2} - \Sigma_{1,1}}{\Sigma_{1,1}^{1/2}}\right)\right].
\end{aligned}$$

• Finally, for $a > 0, b > 0$ we need to integrate over the 4 quadrants $Q_1 = (-\infty, \log a] \times (-\infty, \log b]$, $Q_2 = (\log a, +\infty) \times (-\infty, \log b]$, $Q_3 = (-\infty, \log a] \times (\log b, +\infty)$ and $Q_4 = (\log a, +\infty) \times (\log b, +\infty)$:

$$\begin{aligned} G(a,b,\boldsymbol{\mu},\Sigma) = & \int_{Q_1} ab\, \varphi_2(\boldsymbol{x};\boldsymbol{\mu},\Sigma) + \int_{Q_2} be^{x_1}\, \varphi_2(\boldsymbol{x};\boldsymbol{\mu},\Sigma) \\ & + \int_{Q_3} ae^{x_2}\, \varphi_2(\boldsymbol{x};\boldsymbol{\mu},\Sigma) + \int_{Q_4} e^{x_1+x_2}\, \varphi_2(\boldsymbol{x};\boldsymbol{\mu},\Sigma) \end{aligned}$$

After using formula (17) and expressing probabilities over quadrant Q_i with the bivariate CDF, we obtain

$$\begin{aligned} G(a,b,\boldsymbol{\mu},\Sigma) = & \, ab\, \Phi_2(\boldsymbol{\ell};\boldsymbol{\mu},\Sigma) \\ & + be^{\mu_1+\frac{1}{2}\Sigma_{1,1}} \left[\Phi\left(\frac{\boldsymbol{e}_2^T(\boldsymbol{\ell}-\boldsymbol{\mu}-\Sigma\boldsymbol{e}_1)}{\Sigma_{2,2}^{1/2}} \right) - \Phi_2\left(\boldsymbol{\ell};\boldsymbol{\mu}+\Sigma\boldsymbol{e}_1,\Sigma\right) \right] \\ & + ae^{\mu_2+\frac{1}{2}\Sigma_{2,2}} \left[\Phi\left(\frac{\boldsymbol{e}_1^T(\boldsymbol{\ell}-\boldsymbol{\mu}-\Sigma\boldsymbol{e}_2)}{\Sigma_{1,1}^{1/2}} \right) - \Phi_2\left(\boldsymbol{\ell};\boldsymbol{\mu}+\Sigma\boldsymbol{e}_2,\Sigma\right) \right] \\ & + e^{\boldsymbol{\mu}^T\mathbf{1}+\frac{1}{2}\mathbf{1}^T\Sigma\mathbf{1}} \left[1 + \Phi_2\left(\boldsymbol{\ell};\boldsymbol{\mu}+\Sigma\mathbf{1},\Sigma\right) - \Phi\left(\frac{\boldsymbol{e}_1^T(\boldsymbol{\ell}-\boldsymbol{\mu}-\Sigma\mathbf{1})}{(\mathbf{1}^T\Sigma\mathbf{1})^{1/2}} \right) - \Phi\left(\frac{\boldsymbol{e}_2^T(\boldsymbol{\ell}-\boldsymbol{\mu}-\Sigma\mathbf{1})}{(\mathbf{1}^T\Sigma\mathbf{1})^{1/2}} \right) \right], \end{aligned}$$

where we have set $\boldsymbol{\ell} := [\log a, \log b]^T$.