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Cutting Planes for Families Implying Frankl's Conjecture

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Abstract. We find previously unknown families which imply Frankl's conjecture using an algorithmic framework. The conjecture states that for any non-empty union-closed (or Frankl) family there exists an element in at least half of the sets. Poonen's Theorem characterizes the existence of weights which determine whether a given Frankl family implies the conjecture for all Frankl families which contain it. A Frankl family is *Non–Frankl-Complete* (Non–FC), if it does not imply the conjecture in its elements for some Frankl family that contains it. We design a cutting-plane method that computes the explicit weights which imply the existence conditions of Poonen's Theorem. This method allows us to find a counterexample to a ten-year-old conjecture by R. Morris about the structure of generators for Non–FC-families.

1 Introduction

Frankl's (Union-Closed Sets) conjecture is a celebrated unsolved problem in combinatorics that was recently brought to the attention of a wider audience as a polymath project led by Timothy Gowers [1]. A non-empty finite family of finite sets $\mathcal{F}$ is Frankl if for every $A, B \in \mathcal{F}$ we have $A \cup B \in \mathcal{F}$. Frankl's conjecture states that for any Frankl family $\mathcal{F}$ there exists an element in at least $|\mathcal{F}|/2$ sets. The problem appears to have little structure – perphaps the very reason why a proof or disproof remains elusive. In this paper, we focus on what are referred to as *local configurations* in [3], namely Frankl families that imply the conjecture for all Frankl families which contain them. In this regard, given a Frankl family $\mathcal{A}$, Poonen's Theorem [10] characterizes the necessity of the implication by the existence of weights on the elements of $\mathcal{A}$ that obey certain inequalities. Following Vaughan [12], we say that a Frankl family of sets $\mathcal{A}$ is *Frankl-Complete* (FC), if for every Frankl family $\mathcal{F} \supseteq \mathcal{A}$ there exists an element in the sets of $\mathcal{A}$ which satisfies the Frankl conjecture. A Frankl family $\mathcal{A}$ is *Non–Frankl-Complete* (Non–FC), if there exists a Frankl family $\mathcal{F} \supseteq \mathcal{A}$ such that any element contained in the sets of $\mathcal{A}$ is in less than half of the sets of $\mathcal{F}$.

Using special structures and averaging arguments, previous researchers have determined a number of FC-families. In particular, Poonen proved that any Frankl family which contains three distinct 3-subsets of a 4-set safistifies the conjecture and Vaughan [12], [13], [14] proved that the conjecture holds for any Frankl family which contains all five of the 4-subsets of a 5-set, or ten of the

4-subsets of a 6-set, or three 3-subsets of a 7-set with a common element. With the help of a computer program, Morris [9] completely characterized FC-families on at most 5 elements. Finally, Marić, Živković, and Vučković [8] formalized a combinatorial search in the interactive theorem prover Isabelle/HOL and showed that all families containing four 3-subsets of a 7-set are FC-families. In this paper, we design a general computational framework which is able to precisely characterize FC or Non–FC-families by using exact integer programming, thus providing an algorithmic roadmap for improving many of the known results and settling previously unknown questions.

The connection between Frankl's conjecture and combinatorial optimization is well-established in [11], where the authors derive the equivalence of the problem with an integer program and investigate related conjectures. Furthermore, given a Frankl family $\mathcal{A}$, Poonen's Theorem yields a constructive proof to determine if $\mathcal{A}$ is FC or Non–FC in the form of a fractional polytope with a potentially exponential number of constraints. In general, this makes it difficult to explicitly exhibit the conditions which determine whether a given Frankl family implies the conjecture for all families that contain it. To overcome this, we design a cutting-plane method that computes the explicit weights which imply Poonen's existence conditions. In particular, this paves the way toward automated discovery of FC-families by computational integer programming, especially when coupled with an exact rational solver [4] and other verification routines. As a result, we are able to find previously unknown FC-families. Although our current implementation in SCIP 3.2.1 [6] (output rechecked with CPLEX 12.6.3 [5], Gurobi 6.5.2 [7], and finally exact SCIP [4]) allows us to characterize *any FC-family up to 9 elements* tested so far[1], within the confines of this paper we only feature a few new FC-families as an example of our method. The main contribution of our algorithm in this work is the construction of an explicit counterexample to a ten-year-old conjecture of Morris [9] about the structure of generators for Non–FC-families. We believe this application best illustrates the reach and potential of our method.

2 Cutting Plane for FC-families

As mentioned in section 1, Poonen's seminal article [10] precisely characterized conditions for FC-families. Poonen's theorem is at the basis of all subsequent approaches for classifying FC-families, which in turn form an integral part of checking whether the conjecture holds for $n \leq 11$ [2]. Let S_n denote the power set on n elements, and define $[n] := \{1, 2, \ldots, n\}$. We say that a family of sets $\mathcal{A}$ *covers* n elements if the union of all sets in $\mathcal{A}$ is $[n]$, and for families of sets $\mathcal{A}$ and $\mathcal{B}$ define $\mathcal{B} \uplus \mathcal{A} := \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$. Finally, for a family of sets $\mathcal{F}$ define $\mathcal{F}_i := \{F \in \mathcal{F} \mid i \in F\}$, and denote by $\mathcal{X}^{\mathcal{F}}$ the incidence vector of $\mathcal{F}$.

[1] Up to date, we have verified hundreds of *minimal non-isomorphic* (under some permutation of $[n]$) FC-families. A classification of such families will be featured in an upcoming paper by the author.

Theorem 1 (Poonen 1992). *Let $\mathcal{A}$ be a Frankl family that covers n elements. The following statements are equivalent:*

1. *For every Frankl family $\mathcal{F} \supseteq \mathcal{A}$, there exists $i \in [n]$ such that $|\mathcal{A}_i| \geq |\mathcal{A}|/2$.*

2. *There exist non-negative real numbers $c_1, \ldots, c_n$ with $\sum_{i\in[n]} c_i = 1$ such that for every Frankl family $\mathcal{B} \subseteq \mathcal{S}_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds*

$$\sum_{i\in[n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2. \tag{1}$$

The proof of the theorem includes a beautiful application of the separating hyperplane theorem and points, at least algorithmically, in the right way. Indeed, for a fixed Frankl family $\mathcal{A}$ that covers n elements, it is easy to see that the second statement above describes a convex polyhedron $P_c \subset \mathbb{R}^n$ defined by:

$$\left\{ \begin{array}{ll} \sum_{i\in[n]} c_i = 1; & \\ \sum_{i\in[n]} c_i |\mathcal{B}_i| \geq |\mathcal{B}|/2 & \forall \mathcal{B} \subseteq \mathcal{S}_n : \mathcal{B} \uplus \mathcal{A} = \mathcal{B}; \\ c_i \geq 0 & \forall i \in [n]; \end{array} \right\}$$

Furthermore since the coefficients are all integral, if there exists a feasible point of P_c, we can safely assume (via Fourier-Motzkin elimination) that the c_i are rational. Thus, we can use the simplex or interior point methods to find a feasible point of P_c, or show that one does not exist via the Farkas' Lemma. Furthermore let $P_{\bar{c}}$ denote the following integer program:

$$\begin{aligned} \min & \sum_{i\in[n]} \bar{c}_i & \\ \text{s.t.} & \sum_{i\in[n]} \bar{c}_i |\mathcal{B}_i| \geq (|\mathcal{B}|/2) \sum_{i\in[n]} \bar{c}_i & \forall \mathcal{B} \subseteq \mathcal{S}_n : \mathcal{B} \uplus \mathcal{A} = \mathcal{B} \\ & \sum_{i\in[n]} \bar{c}_i \geq 1 & \\ & \bar{c}_i \in \mathbb{Z}_{\geq 0} & \forall i \in [n] \end{aligned}$$

Observation 2 *Let $\mathcal{A}$ be a Frankl family that covers n elements. Then there exists a feasible point of P_c if and only if there exists a feasible solution of $P_{\bar{c}}$.*

Proof. We simply scale. More precisely, let p_c be a feasible point of P_c. We can safely assume that p is a rational vector, i.e. $p_c = \{c_1 = \frac{a_1}{b_1}, c_2 = \frac{a_1}{b_1}, \ldots, c_n = \frac{a_n}{b_n}\} \in \mathbb{Q}^n_{\geq 0}$. Define $g := lcm(b_1, b_2, \ldots, b_n)$, and $\bar{c}_i := gc_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$. It is easy to see that the defined $\bar{c}_i$ yield a feasible solution of $P_{\bar{c}}$. Let $p_{\bar{c}}$ be a feasible solution of $P_{\bar{c}}$, i.e. $p_{\bar{c}} = \{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n\} \in \mathbb{Z}^n_{\geq 0}$. Define $c_i := \bar{c}_i / (\sum_{i\in[n]} \bar{c}_i)$. It is easy to see that the defined c_i yield a feasible point of P_c. $\square$

Depending on $\mathcal{A}$ and n, there may be a large number of inequalites based on $\mathcal{B} \subseteq \mathcal{S}_n$ such that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$. In general, this makes the explicit enumeration of such inequalities impractical. Fortunately we have at our disposal a well-established technique that handles precisely this type of problem, namely the cutting-plane method. A *separation oracle* for a polyhedron $P \subset \mathbb{R}^n$ is an algorithm that, queried on $x \in \mathbb{R}^n$, either asserts that $x \in P$ or returns $h \in \mathbb{R}^n$ such that $hy < hx$ for all $y \in P$. The main idea behind the cutting-plane method is simple, yet powerful: If a polyhedron is described by a class of constraints too large to generate explicitely, start with a small subset of the constraints and generate the rest dynamically, as needed. As a result, the following question is of central interest: Given a feasible point $p^* \in \tilde{P}_c \supseteq P_c$, where $\tilde{P}_c$ is the polyhedron defined by a small subset of all possible inequalites of the type $\mathcal{B} \subseteq \mathcal{S}_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ and non-negative c_i that sum to one, how do we exhibit a Frankl family (coefficients for an inequality of type 1) which separates p^* from P_c or show that no such family exists? It is clear that the same applies to $P_{\bar{c}}$. In order to answer this question and design a separation oracle for P_c (or $P_{\bar{c}}$), we first need the following corollary of Poonen's Theorem, a version of which is already noted in [9]. We formalize it here for clarity and reference.

Corollary 1. *Let $\mathcal{A}$ be a Frankl family that covers n elements. The following are equivalent:*

1. *The Frankl conjecture holds for every Frankl family $\mathcal{F} \supseteq \mathcal{A}$. In particular, there exists $i \in [n]$ such that $|\mathcal{A}_i| \geq |\mathcal{A}|/2$.*
2. *For every $\mathcal{B} \subseteq S_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, we have $\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) \geq 0$ with $\sum_{i\in[n]} c_i = 1$, and $c_i \in \mathbb{Q}_{\geq 0}$, for all $i \in [n]$.*

Proof. Fix a Frankl family $\mathcal{B} \subseteq S_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$. Then the following holds,

$$\begin{aligned}
\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} c_i - \sum_{i\notin S} c_i\right) &= 2\sum_{S\in\mathcal{B}}\sum_{i\in S} c_i - \sum_{S\in\mathcal{B}} \left(\sum_{i\notin S} c_i + \sum_{i\in S} c_i\right) \\
&= 2\sum_{S\in\mathcal{B}}\sum_{i\in S} c_i - \sum_{S\in\mathcal{B}}\sum_{i\in[n]} c_i \\
&= 2\sum_{i\in[n]} c_i|\mathcal{B}_i| - |\mathcal{B}|\sum_{i\in[n]} c_i \geq 0 \\
&\iff \sum_{i\in[n]} c_i|\mathcal{B}_i| \geq |\mathcal{B}|/2.
\end{aligned}$$

Since the above holds for every Frankl family $\mathcal{B} \subseteq S_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the desired result follows from Poonen's Theorem. □

Corollary 1 combined with an integer programming approach to Frankl families inspired by [11], provides the basis of our separation oracle. Fix a Frankl family

$\mathcal{A} \subseteq \mathcal{S}_n$, and fix $\mathcal{C} := \{c_1 = \frac{a_1}{b_1}, c_2 = \frac{a_1}{b_1}, \ldots, c_n = \frac{a_n}{b_n}\} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i\in[n]} c_i = 1$. Furthermore, define $g := lcm(b_1, b_2, \ldots, b_n)$, and $\bar{c}_i := gc_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$ as previously. Let $FC(\mathcal{A}, \mathcal{C})_n$ denote the following polyhedron:

$$\left\{\begin{array}{ll} x_T + x_U \leq 1 + x_S & \forall T \cup U = S \in \mathcal{S}_n; \\ \sum_{S\in\mathcal{S}_n} \left(\sum_{i\in x_S} \bar{c}_i - \sum_{i\notin x_S} \bar{c}_i\right) x_S + 1 \leq 0; & \\ x_T \leq x_U & \forall S \cup T = U \in \mathcal{S}_n, S \in \mathcal{A}; \\ x_S \in \{0,1\} & \forall S \in \mathcal{S}_n; \end{array}\right\}$$

Suppose $FC(\mathcal{A}, \mathcal{C})_n$ is non-empty, and let $p^* \in FC(\mathcal{A}, \mathcal{C})_n$. Then $p^* = \mathcal{X}^{\mathcal{B}}$ where $\mathcal{B} \subseteq \mathcal{S}_n$. The first inequalities ensure that the chosen family $\mathcal{B}$ is Frankl, and we denote them as Union-Closed (UC) inequalities. The third class of inequalites ensures that $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, and we denote them as Fixed-Set (FS) inequalites. We explain the $\bar{c}_i$ inequality in the next theorem.

Proposition 1. *Let $\mathcal{A}$ be a Frankl family that covers n elements, and let $\mathcal{C} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i\in[n]} c_i = 1$. If $FC(\mathcal{A}, \mathcal{C})_n$ is unfeasible, then all Frankl families $\mathcal{F} \supseteq \mathcal{A}$ satisfy Frankl's conjecture.*

Proof. Suppose that $FC(\mathcal{A}, \mathcal{C})_n$ is unfeasible. Let $\tilde{P}$ be defined as $FC(\mathcal{A}, \mathcal{C})_n$ without the $\bar{c}_i$ inequality. It is easy to see that $\tilde{P}$ is non-empty since the all zero vector is feasible. Indeed, any $p^* = \mathcal{X}^{\mathcal{B}}$ where $\mathcal{B} \subseteq S_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ is feasible. Therefore if $FC(\mathcal{A}, \mathcal{C})_n$ is unfeasible this implies there exists no Frankl family $\mathcal{B} \subseteq S_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$ such that:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} \bar{c}_i - \sum_{i\notin S} \bar{c}_i\right) \leq -1.$$

Since $x_S = \{0,1\}$ for all $S \in \mathcal{S}_n$, and $\bar{c}_i \in \mathbb{Z}_{\geq 0}$ for all $i \in [n]$, this implies that for all Frankl families $\mathcal{B} \subseteq S_n$ with $\mathcal{B} \uplus \mathcal{A} = \mathcal{B}$, the following inequality holds:

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} \bar{c}_i - \sum_{i\notin S} \bar{c}_i\right) \geq 0.$$

It is easy to see that corollary 1 still holds if we replace c_i with $\bar{c}_i$. Therefore Poonen's Theorem implies that all Frankl families $\mathcal{F} \supseteq \mathcal{A}$ satisfy the Frankl conjecture. □

A natural candidate for checking the feasibility of $FC(\mathcal{A}, \mathcal{C})_n$, for some $\mathcal{A}$ and $\mathcal{C}$, is a standard branch and bound algorithm. Since we are mainly interested in proving that certain (previously unknown) Frankl families $\mathcal{A}$ are FC or Non–FC, we do not address questions of complexity, but simply optimize some linear objective function over $FC(\mathcal{A}, \mathcal{C})_n$ in a general purpose integer programming solver as specified in section 1. We do the same for optimizing over $P_{\bar{c}}$, and checking the feasibility of P_c.

Corollary 2. *Let $\mathcal{A}$ be a Frankl family that covers n elements. Given $\mathcal{C} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i\in[n]} c_i = 1$ as input, determining whether there exists a feasible point p^* of $FC(\mathcal{A}, \mathcal{C})_n$ is equivalent to a separation oracle for P_c.*

Proof. Keeping all the above in mind, it is clear that if $p^* = \mathcal{X}^{\mathcal{B}}$ is a feasible point of $FC(\mathcal{A}, \mathcal{C})_n$, we immediately obtain the following valid inequality for P_c (where we distingush the variable $\tilde{c}_i$ from the fixed c_i)

$$\sum_{i\in[n]} \tilde{c}_i |\mathcal{B}_i| \geq |\mathcal{B}|/2,$$

which separates p^* from $\tilde{P}_c \supseteq P_c$ since the following implications hold

$$\sum_{S\in\mathcal{B}} \left(\sum_{i\in S} \bar{c}_i - \sum_{i\notin S} \bar{c}_i \right) \leq -1 \iff |\mathcal{B}|/2 - \sum_{i\in[n]} c_i |\mathcal{B}_i| > 0.$$

Otherwise, if $FC(\mathcal{A}, \mathcal{C})_n$ is unfeasible, no such valid inequality exists. In this case, proposition 1 implies that $\mathcal{C} \in P_c$. $\square$

Furthermore, define as $FC(\mathcal{A}, \mathcal{C})^{Max}_n$ the binary program which maximizes the following linear objective function

$$\sum_{i\in[n]} c_i \left(\sum_{S\in\mathcal{S}_n} x_S - 2 \sum_{S\in\mathcal{S}_n : i \in S} x_S \right),$$

over $FC(\mathcal{A}, \mathcal{C})_n$.

Observation 3 *Let $\mathcal{A}$ be a Frankl family that covers n elements, and let $\mathcal{C} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i\in[n]} c_i = 1$. An optimal solution of $FC(\mathcal{A}, \mathcal{C})^{Max}_n$ returns a maximally violated inequality for P_c.*

It is easy to see that in the following algorithm the tuples $\left(P^I_c, FC(\mathcal{A}, \mathcal{C})^{Max}_n\right)$ and $(P_c, FC(\mathcal{A}, \mathcal{C})_n)$ [2] may be used interchangeably with appropriate adjustments.

[2] The reason for using the different formulations is entirely computational. For example, when using P_c, if the least common multiple of a given $\mathcal{C}$ is very large, it can spell numerical trouble for optimizing over $FC(\mathcal{A}, \mathcal{C})_n$.

Algorithm 1: Cutting planes for FC-families

Data: A Frankl family $\mathcal{A}$ that covers n elements.
Result: c_i for $\mathcal{A}$, or infeasible $P_k \supseteq P_c$.

$P_k \leftarrow \left(\sum_{i\in[n]} c_i = 1, c_i \geq 0, \forall i \in [n]\right)$, $\mathcal{C} \leftarrow \emptyset$, $g \leftarrow \emptyset$;
while $\exists\ C \in P_k$ **do**
  $g \leftarrow lcm(b_1, b_2, \ldots, b_n)$, where $C = \{c_1 = \frac{a_1}{b_1}, c_2 = \frac{a_1}{b_1}, \ldots, c_n = \frac{a_n}{b_n}\} \in \mathbb{Q}^n_{\geq 0}$;
  $\mathcal{C} \leftarrow \{gc_1, gc_2, \ldots, gc_n\}$;
  if $\exists\ p^* \in FC(\mathcal{A}, \mathcal{C})_n$ **then**
    $P_k \leftarrow P_k \cap \left(\sum_{i\in[n]} c_i|\mathcal{B}_i| \geq |\mathcal{B}|/2\right)$, where $p^* = \mathcal{X}^{\mathcal{B}}$;
  else
    return C
return $P_k \supseteq P_c$

Theorem 4. *Let $\mathcal{A}$ be a Frankl family that covers n elements. Then Algorithm 1 either outputs weights which prove that $\mathcal{A}$ is FC, or an infeasible system of constraints which proves that $\mathcal{A}$ is Non-FC.*

Proof. It is clear algorithm 1 finitely terminates. Let $\mathcal{A}$ be a Frankl family that covers n elements. Suppose $\mathcal{A}$ is a FC-family. By Poonen's Theorem there exist $c_i \geq 0$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$ that satisfy all inequalites of type 1. Given $\mathcal{A}$, at some iteration of algorithm 1, for some $\mathcal{C}$, by corollary 2 we arrive at $C \in P_c$, otherwise if the algorithm terminates without outputing some $C \in P_c$, it outputs an infeasible $P_k \supseteq P_c$ which implies that $\mathcal{A}$ is not an FC-family and we arrive at a contradiction. Therefore since $C \in P_c$, this implies that $FC(\mathcal{A}, \mathcal{C})_n$ is unfeasible. Suppose $\mathcal{A}$ is a Non–FC-family. This implies there exist no $c_i \geq 0$ for all $i \in [n]$ with $\sum_{i\in[n]} c_i = 1$ that satisfy all inequalites of type 1. By corollary 2 during all the iterations of algorithm 1 we have that $C \notin P_c$, otherwise we arrive at a contradiction. Therefore algorithm 1 terminates when $P_k = \emptyset$, which implies that $P_k \supseteq P_c$ is infeasible. □

3 Valid Inequalities

From the prospective of computational integer programming, inequalities which are added to a polyhedron are considered effective if they lead to a smaller branch and bound tree. In this sense, if we want to prove that the integer hull of a particular polyhedron is infeasible, the best we can do is show that the corresponding linear relaxation is infeasible. For all the results that we feature in this paper, the following inequalites [3] significantly reduce the size of the

[3] Only a small subset of the considered inequalities is added at the root node. Here we do not discuss *how* to decide *which* particular subset to add.

branch and bound tree. Given a family of sets $\mathcal{S}$, we say that $\mathcal{S}$ *generates* (or is a *generator* for) $\mathcal{F}$, denoted by $\langle \mathcal{S} \rangle = \mathcal{F}$, if $\mathcal{F}$ is the smallest Frankl family that contains $\mathcal{S}$. A family of sets $\mathcal{S}$ *generates* $\mathcal{F}$ *with* $\mathcal{A}$, denoted by $\langle \mathcal{S} \rangle_{\mathcal{A}} = \mathcal{F}$, if $\mathcal{F}$ is the smallest Frankl family that contains $\mathcal{S}$ such that $\mathcal{F} \uplus \mathcal{A} = \mathcal{F}$.

Proposition 2 (FC inequalities). *Let $\mathcal{A}$ be a Frankl family that covers n elements, and let $\mathcal{C} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i\in[n]} c_i = 1$. Suppose we have $S \in \mathcal{A}$, $S \cup U = F$, and $S \cup T = F$. Then the following inequality*

$$x_T + x_U - x_{T\cup U} - x_F \leq 0,$$

is valid for $FC(\mathcal{A}, \mathcal{C})_n$.

Proof. Suppose there exists a feasible solution of $FC(\mathcal{A}, \mathcal{C})_n$ which yields a Frankl family $\mathcal{F}$ such that the following inequality holds

$$x_T + x_U - x_{T\cup U} - x_F \geq 1.$$

This implies that the number of variables which equal one with positive coefficients is greater than the number of variables with negative coefficients which equal one. But if either x_T or x_U are one then x_F is one (if both are one then $x_{T\cup U}$ is one) and we arrive at a contradiction. $\square$

Definition 1 (FC-chain). *Let $\mathcal{A}$ be a Frankl family that covers n elements, and let $\mathcal{C} \in \mathbb{Q}^n_{\geq 0}$ with $\sum_{i\in[n]} c_i = 1$. Let $\mathcal{S}, \mathcal{S}' \subset \mathcal{S}_n$, $\mathcal{S} \cap \mathcal{S}' = \emptyset$. Given $B_i \in \mathcal{S}, B_j \in \mathcal{S}'$, we say B_i, B_j form a FC-chain which we denote by $B_i \longrightarrow B_j$, if there exist tuples $(B_i, B_1), (B_1, B_2), \ldots, (B_m, B_j)$, $B_k \in \mathcal{S}_n$ for all $1 \leq k \leq m$, such that for any tuple (B_i, B_{i+1}), at least one of the following conditions holds:*

1. *There exists $A \in \mathcal{A}$ such that $x_{B_i} \leq x_{B_{i+1}}$ is a valid inequality for $FC(\mathcal{A}, \mathcal{C})_n$.*
2. *There exists $B_k \in \langle \mathcal{S} \rangle_{\mathcal{A}}$ such that $x_{B_i} + x_{B_k} \leq 1 + x_{B_{i+1}}$ is a valid inequality for $FC(\mathcal{A}, \mathcal{C})_n$.*

We denote an explicit FC-chain by $B_i \xrightarrow{S} B_1 \xrightarrow{U} \ldots B_m \xrightarrow{T} B_j$, where S, U, T satify either condition listed above. When needed we specify which type of inequalies form an FC-chain by S^{UC}, U^{UC}, T^{UC} for UC inequalites, and S^{FS}, U^{FS}, T^{FS} for FS inequalites.

Proposition 3 (FC-chain inequalities). *Let $\mathcal{S}, \mathcal{S}' \subset \mathcal{S}_n$, $\mathcal{S} \cap \mathcal{S}' = \emptyset$, such that for any $S \in \mathcal{S}$, there exists $S' \in \mathcal{S}'$ where $S \longrightarrow S'$. Let $\mathcal{T} \subseteq \mathcal{S}$, and define $\mathcal{U}_T := \{S' \in \mathcal{S}' \mid \exists\, S \in \mathcal{T} : S \longrightarrow S'\}$. Suppose that $|\mathcal{T}| \leq |\mathcal{U}_T|$ for all $\mathcal{T} \subseteq \mathcal{S}$. Then the following inequality*

$$\sum_{S\in\mathcal{S}} x_S - \sum_{S\in\mathcal{S}'} x_S \leq 0,$$

is valid for $FC(\mathcal{A}, \mathcal{C})_n$.

Proof. Suppose there exists a feasible solution of $FC(\mathcal{A},\mathcal{C})_n$ which yields a Frankl family $\mathcal{F}$ such that the following inequality holds

$$\sum_{S\in\mathcal{S}\cap\mathcal{F}} x_S - \sum_{S\in\mathcal{S}'\cap\mathcal{F}} x_S \geq 1.$$

It is clear that $\mathcal{S}\cap\mathcal{F}\neq\emptyset$, otherwise we arrive at a contradiction. Therefore the inequality implies that the number of variables x_S which equal one, for all $S\in\mathcal{S}\cap\mathcal{F}$ is greater than the number of variables x_S which equal one, for all $S\in\mathcal{S}'\cap\mathcal{F}$. Let $\mathcal{T}\subseteq\mathcal{S}\cap\mathcal{F}$, and for all $S\in\mathcal{T}$, let $x_S=1$. Then $|\mathcal{T}|\leq|\mathcal{U}_T|$ holds by hypothesis. Furthermore by the definition of a FC-chain, for all $S'\in\mathcal{U}_T$ we conclude that $x_{S'}=1$. Thus we arrive at a contradiction. □

Remark 1. FC-chain inequalities are generalizations of FC inequalites and FS inequalities.

4 Generators for Non–FC-families

In this section we exhibit a counterexample to a conjecture of Morris [9] about generators for Non-FC-families.

Definition 2 (regular). *Let $\mathcal{S}$ be a family of sets that covers n elements. Suppose $\mathcal{S}$ is a minimal generator for a Frankl family $\mathcal{F}$, such that $\mathcal{F}$ is a Non–FC-family. Then $\mathcal{S}$ is* regular *if for any $\mathcal{A}\in\mathcal{S}$, $\mathcal{A}\neq\emptyset$, we have that $\langle(\mathcal{S}\setminus\{A\})\cup\{A\cup\{i\}\}\rangle$ is a Non–FC-family.*

Conjecture 1 (Morris 2006). $\mathcal{S}$ is regular for all $n\in\mathbb{Z}_{\geq1}$.

Morris [9] thought the conjecture plausible and checked that it held for all known families at the time. Our counterexample on six elements is minimal, in the sense that Morris [9] completely characterized FC-families on 5 elements. Let $\mathcal{S}=\{\emptyset,\{4,5,6\},\{1,3,4\},\{1,2,5,6\},\{1,2,3,4\}\}$. Furthermore, let $\mathcal{T}=\{\{1,2,4,5,6\},\{1,3,4,5,6\},\{1,2,3,4,5,6\}\}$. Then, we see that $\langle\mathcal{S}\rangle=\mathcal{S}\cup\mathcal{T}$. It is easy to see that $\mathcal{S}$ is a minimal generator for $\mathcal{S}\uplus\mathcal{T}$.

Proposition 4. *$\mathcal{S}\uplus\mathcal{T}$ is a Non–FC-family.*

Proof. The proof is the output of algorithm 1 with $\mathcal{S}\uplus\mathcal{T}$ as input, which is an infeasible system of constraints. We display an irreducible infeasible subset of the given system. We identify columns with zero one entries for each $S\in\mathcal{S}_6$. The six matrices featured below represent Frankl families. The top row keeps track of the number of sets in each family. In addition to rechecking with an exact rational solver [4] and other solvers, we check that each matrix is Frankl via simple external subroutines and finally by hand [4]. Furthermore, let $\mathcal{F}$ be a

[4] Using a subroutine to indicate the position of a set that is the union of two sets in the exhibited family, a determined reader can speed up the process of checking each matrix by hand.

family represented by one of the matrices below. It easy to see that $\mathcal{F} \uplus \langle \mathcal{S} \rangle = \mathcal{F}$ by inspection. In each matrix, we color columns which correspond to sets in $\mathcal{S}$, $\mathcal{T}$, red and blue, respectively. Each matrix yields an inequality of type 1 (multiplied by two) featured below it. The following system of constraints is infeasible in non-negative c_i for all $1 \leq i \leq 6$. For each row we display the Farkas dual values in square brackets. This yields a certificate of infeasibility.

$$[-684] : \mathtt{c}_1 + \mathtt{c}_2 + \mathtt{c}_3 + \mathtt{c}_4 + \mathtt{c}_5 + \mathtt{c}_6 = 1.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	1	0	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	0	1	0	1

$$[3] : 22\mathtt{c}_1 + 46\mathtt{c}_2 + 50\mathtt{c}_3 + 50\mathtt{c}_4 + 46\mathtt{c}_5 + 46\mathtt{c}_6 \geq 43.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	1	0	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	1	0	1	0	1

$$[1] : 46\mathtt{c}_1 + 14\mathtt{c}_2 + 42\mathtt{c}_3 + 42\mathtt{c}_4 + 42\mathtt{c}_5 + 42\mathtt{c}_6 \geq 39.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1
c_4	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1	1
c_5	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	1	0	0	1	1	0	0	1	1
c_6	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1

$$[4] : 52\mathtt{c}_1 + 46\mathtt{c}_2 + 16\mathtt{c}_3 + 28\mathtt{c}_4 + 52\mathtt{c}_5 + 52\mathtt{c}_6 \geq 46.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
c_4	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	1
c_5	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
c_6	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

$$[2] : 48\mathtt{c}_1 + 40\mathtt{c}_2 + 16\mathtt{c}_3 + 48\mathtt{c}_4 + 40\mathtt{c}_5 + 40\mathtt{c}_6 \geq 40.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1
c_4	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
c_5	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
c_6	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1

$$[3] : 44\mathtt{c}_1 + 44\mathtt{c}_2 + 42\mathtt{c}_3 + 48\mathtt{c}_4 + 20\mathtt{c}_5 + 52\mathtt{c}_6 \geq 42.$$

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
c_2	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
c_3	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1
c_4	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
c_5	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	0	1	1	0	1	1	0	1	1	0	1	1	0	1	1
c_6	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1

$$[3] : 44\mathtt{c}_1 + 44\mathtt{c}_2 + 42\mathtt{c}_3 + 48\mathtt{c}_4 + 52\mathtt{c}_5 + 20\mathtt{c}_6 \geq 42.$$

□

Proposition 5. *Let* $\mathcal{S}' = \{\emptyset, \{4,5,6\}, \{1,3,4\}, \{1,2,5,6\}, \{1,2,3,4,5\}\}$. *Then* $\langle \mathcal{S}' \rangle$ *is an FC-family.*

Proof. Let $\mathcal{C} = \{16, 8, 12, 20, 17, 15\}$. Then $FC(\langle \mathcal{S}' \rangle, \mathcal{C})_n$ is infeasible [5]. □

Corollary 3. $\mathcal{S}$ *is a counterexample to conjecture 1.*

Proof. $\mathcal{S}' = (\mathcal{S} \setminus \{1,2,3,4\}) \cup \{\{1,2,3,4\} \cup \{5\}\}$. □

We exhibit a few more new FC-families verified by our algorithm. As mentioned earlier, the computations are rechecked with a number of solvers, followed by a final check with an exact rational solver. Furthemore, for all the families shown here, by adding FC-chain inequalities at the root node, it is possible to reduce the resulting branch and bound tree to a few nodes, and check the irreducible infeasible subset of each leaf node (linear program) manually, or with an external subroutine.

Proposition 6. *Let* $\mathcal{S} = \{\{4,5,6,7\}, \{1,3,4,5\}, \{3,4,6,7\}, \{5,6,7\}, \{2,3,4\}\}$. *Then* $\langle \mathcal{S} \rangle$ *is an FC-family.*

Proof. Let $\mathcal{C} = \{4, 8, 11, 13, 10, 9, 9\}$. Then $FC(\langle \mathcal{S} \rangle, \mathcal{C})_n$ is infeasible. □

Proposition 7. *Let* $\mathcal{S} = \{\{4,5,6,7\}, \{1,2,5,7\}, \{3,4,5,7\}, \{5,6,7\}, \{4,6,7\}\}$. *Then* $\langle \mathcal{S} \rangle$ *is an FC-family.*

Proof. Let $\mathcal{C} = \{1, 1, 1, 3, 3, 3, 4\}$. Then $FC(\langle \mathcal{S} \rangle, \mathcal{C})_n$ is infeasible. □

Proposition 8. *Let* $\mathcal{S} = \{\{2,5,6,7\}, \{2,4,6\}, \{1,4,7,8\}, \{1,2,3,7\}, \{1,2,7,8\}\}$. *Then* $\langle \mathcal{S} \rangle$ *is an FC-family.*

Proof. Let $\mathcal{C} = \{13, 22, 7, 18, 17, 17, 10\}$. Then $FC(\langle \mathcal{S} \rangle, \mathcal{C})_n$ is infeasible. □

[5] In the appendix we explicitely show the infeasibility of $FC(\mathcal{S}', \mathcal{C})_n$ by making use of FC-chain inequalities and displaying irreducible infeasible subsets of constraints for the two leaf nodes of the resulting branch and bound tree.

Conclusion

In this work we design a cutting-plane algorithm that determines if a given Frankl family necessarily implies Frankl's conjecture. As a result, we exhibit a counterexample to a ten-year-old conjecture of Morris[9] about generators for Frankl families, and display a number of previously unknown families that imply Frankl's conjecture.

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5 Appendix

We identify sets in S_6 with the columns in the matrix below. For each column, the number on the top row represents its corresponding variable index in $FC(\mathcal{A}, \langle\mathcal{S}'\rangle)_n$. Column W corresponds to a weight vector for the elements in $[n]$. The columns representing families of sets $\mathcal{S}'$ and $\mathcal{T}$ are colored red and blue, respectively. As previously, $\langle\mathcal{S}'\rangle = \mathcal{S}' \cup \mathcal{T}$.

W	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
20	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
17	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
15	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

We show that $FC(\mathcal{A}, \langle\mathcal{S}'\rangle)_n$ is unfeasible by branching on x_0 and showing that the linear relaxations of the two subproblems are infeasible. We do this by explicitly exhibiting Farkas dual values (shown in square brackets) for each row of some irreducible infeasible subset of constraints. Define $FC(\mathcal{A}, \langle\mathcal{S}'\rangle)_n^1 := FC(\mathcal{A}, \langle\mathcal{S}'\rangle)_n \cap (x_0 = 1), FC(\mathcal{A}, \langle\mathcal{S}'\rangle)_n^0 := FC(\mathcal{A}, \langle\mathcal{S}'\rangle)_n \cap (x_0 = 0)$. Then it is clear that if $FC\left(\mathcal{A}, \langle\mathcal{S}'\rangle\right)_n^1$ and $FC\left(\mathcal{A}, \langle\mathcal{S}'\rangle\right)_n^0$ are infeasible, this implies $FC\left(\mathcal{A}, \langle\mathcal{S}'\rangle\right)_n$ is unfeasible. Let $\overline{FC\left(\mathcal{A}, \langle\mathcal{S}'\rangle\right)_n^1} \supseteq FC\left(\mathcal{A}, \langle\mathcal{S}'\rangle\right)_n^1$, and let the following be the linear relaxation of $\overline{FC\left(\mathcal{A}, \langle\mathcal{S}'\rangle\right)_n^1}$, defined by the following constraints (trivial ones not shown):

1. $[44] : \mathtt{x}_0 = 1$.
2. UC inequalites:
   $[-2] : \mathtt{x}_{11} + \mathtt{x}_{45} - \mathtt{x}_9 \leq 1$, $[-3] : \mathtt{x}_{13} + \mathtt{x}_{59} - \mathtt{x}_9 \leq 1$,
   $[-2] : \mathtt{x}_{14} + \mathtt{x}_{43} - \mathtt{x}_{10} \leq 1$, $[-1] : \mathtt{x}_{22} + \mathtt{x}_{61} - \mathtt{x}_{20} \leq 1$,
   $[-3] : \mathtt{x}_{23} + \mathtt{x}_{60} - \mathtt{x}_{20} \leq 1$, $[-1] : \mathtt{x}_{35} + \mathtt{x}_{45} - \mathtt{x}_{33} \leq 1$,
   $[-3] : \mathtt{x}_{35} + \mathtt{x}_{62} - \mathtt{x}_{34} \leq 1$, $[-6] : \mathtt{x}_{37} + \mathtt{x}_{59} - \mathtt{x}_{33} \leq 1$,
   $[-1] : \mathtt{x}_{38} + \mathtt{x}_{43} - \mathtt{x}_{34} \leq 1$, $[-3] : \mathtt{x}_{38} + \mathtt{x}_{45} - \mathtt{x}_{36} \leq 1$,
   $[-1] : \mathtt{x}_{38} + \mathtt{x}_{61} - \mathtt{x}_{36} \leq 1$, $[-1] : \mathtt{x}_{39} + \mathtt{x}_{44} - \mathtt{x}_{36} \leq 1$,
   $[-2] : \mathtt{x}_{42} + \mathtt{x}_{55} - \mathtt{x}_{34} \leq 1$, $[-2] : \mathtt{x}_{53} + \mathtt{x}_{43} - \mathtt{x}_{33} \leq 1$,
   $[-5] : \mathtt{x}_{54} + \mathtt{x}_{43} - \mathtt{x}_{34} \leq 1$, $[-3] : \mathtt{x}_{44} + \mathtt{x}_{55} - \mathtt{x}_{36} \leq 1$,
   $[-4] : \mathtt{x}_{47} + \mathtt{x}_{49} - \mathtt{x}_{33} \leq 1$.
3. FS inequalities:
   $[0] : \mathtt{x}_{47} - \mathtt{x}_1 \leq 0$, $[-6] : \mathtt{x}_{63} - \mathtt{x}_1 \leq 0$, $[-14] : \mathtt{x}_7 - \mathtt{x}_4 \leq 0$,
   $[-1] : \mathtt{x}_{55} - \mathtt{x}_4 \leq 0$, $[-16] : \mathtt{x}_{63} - \mathtt{x}_{12} \leq 0$, $[-12] : \mathtt{x}_{14} - \mathtt{x}_2 \leq 0$,
   $[-3] : \mathtt{x}_{46} - \mathtt{x}_2 \leq 0$, $[-24] : \mathtt{x}_{47} - \mathtt{x}_3 \leq 0$, $[-12] : \mathtt{x}_{61} - \mathtt{x}_{17} \leq 0$,

$[-21] : \mathtt{x}_{62} - \mathtt{x}_{18} \leq 0$, $[-19] : \mathtt{x}_{62} - \mathtt{x}_{18} \leq 0$, $[-4] : \mathtt{x}_{63} - \mathtt{x}_{19} \leq 0$,
$[-24] : \mathtt{x}_{31} - \mathtt{x}_{24} \leq 0$, $[-1] : \mathtt{x}_{37} - \mathtt{x}_{32} \leq 0$, $[-4] : \mathtt{x}_{38} - \mathtt{x}_{32} \leq 0$,
$[-23] : \mathtt{x}_{39} - \mathtt{x}_{32} \leq 0$, $[-16] : \mathtt{x}_{47} - \mathtt{x}_{40} \leq 0$, $[-11] : \mathtt{x}_{55} - \mathtt{x}_{48} \leq 0$,
$[-8] : \mathtt{x}_{63} - \mathtt{x}_{56} \leq 0$.

4. FC inequalities:
$[-2] : \mathtt{x}_{15} + \mathtt{x}_{53} - \mathtt{x}_{1} - \mathtt{x}_{5} \leq 0$, $[-7] : \mathtt{x}_{15} + \mathtt{x}_{57} - \mathtt{x}_{1} - \mathtt{x}_{9} \leq 0$,
$[-9] : \mathtt{x}_{58} + \mathtt{x}_{15} - \mathtt{x}_{1} - \mathtt{x}_{9} \leq 0$, $[-2] : \mathtt{x}_{15} + \mathtt{x}_{59} - \mathtt{x}_{1} - \mathtt{x}_{11} \leq 0$,
$[-7] : \mathtt{x}_{45} + \mathtt{x}_{23} - \mathtt{x}_{1} - \mathtt{x}_{5} \leq 0$, $[-5] : \mathtt{x}_{60} + \mathtt{x}_{23} - \mathtt{x}_{20} - \mathtt{x}_{16} \leq 0$,
$[-1] : \mathtt{x}_{61} + \mathtt{x}_{23} - \mathtt{x}_{21} - \mathtt{x}_{16} \leq 0$, $[-5] : \mathtt{x}_{15} + \mathtt{x}_{53} - \mathtt{x}_{1} - \mathtt{x}_{5} \leq 0$,
$[[-3] : \mathtt{x}_{27} + \mathtt{x}_{61} - \mathtt{x}_{25} - \mathtt{x}_{16} \leq 0$, $[0] : \mathtt{x}_{46} + \mathtt{x}_{29} - \mathtt{x}_{1} - \mathtt{x}_{9} \leq 0$,
$[-5] : \mathtt{x}_{54} + \mathtt{x}_{29} - \mathtt{x}_{20} - \mathtt{x}_{16} \leq 0$, $[-6] : \mathtt{x}_{29} + \mathtt{x}_{59} - \mathtt{x}_{16} - \mathtt{x}_{25} \leq 0$,
$[-4] : \mathtt{x}_{43} + \mathtt{x}_{30} - \mathtt{x}_{10} - \mathtt{x}_{8} \leq 0$, $[-2] : \mathtt{x}_{53} + \mathtt{x}_{30} - \mathtt{x}_{16} - \mathtt{x}_{20} \leq 0$,
$[-7] : \mathtt{x}_{59} + \mathtt{x}_{30} - \mathtt{x}_{16} - \mathtt{x}_{26} \leq 0$, $[-4] : \mathtt{x}_{31} + \mathtt{x}_{60} - \mathtt{x}_{16} - \mathtt{x}_{28} \leq 0$,
$[-1] : \mathtt{x}_{43} + \mathtt{x}_{45} - \mathtt{x}_{8} - \mathtt{x}_{41} \leq 0$, $[-1] : \mathtt{x}_{43} + \mathtt{x}_{62} - \mathtt{x}_{8} - \mathtt{x}_{42} \leq 0$,
$[-3] : \mathtt{x}_{47} + \mathtt{x}_{62} - \mathtt{x}_{8} - \mathtt{x}_{46} \leq 0$, $[-3] : \mathtt{x}_{51} + \mathtt{x}_{62} - \mathtt{x}_{16} - \mathtt{x}_{50} \leq 0$,
$[-7] : \mathtt{x}_{7} + \mathtt{x}_{54} - \mathtt{x}_{4} - \mathtt{x}_{6} \leq 0$, $[-9] : \mathtt{x}_{51} + \mathtt{x}_{53} - \mathtt{x}_{48} - \mathtt{x}_{49} \leq 0$.

5. Feasibility inequality:
$[-0.5] : 58\mathtt{x}_{1} + 54\mathtt{x}_{2} + 24\mathtt{x}_{3} + 48\mathtt{x}_{4} + 18\mathtt{x}_{5} + 14\mathtt{x}_{6} - 16\mathtt{x}_{7} + 64\mathtt{x}_{8}$
$+34\mathtt{x}_{9} + 30\mathtt{x}_{10} + 24\mathtt{x}_{12} - 6\mathtt{x}_{13} - 10\mathtt{x}_{14} - 40\mathtt{x}_{15} + 72\mathtt{x}_{16} + 42\mathtt{x}_{17}$
$+38\mathtt{x}_{18} + 8\mathtt{x}_{19} + 32\mathtt{x}_{20} + 2\mathtt{x}_{21} - 2\mathtt{x}_{22} - 32\mathtt{x}_{23} + 48\mathtt{x}_{24} + 18\mathtt{x}_{25}$
$+14\mathtt{x}_{26} - 16\mathtt{x}_{27} + 8\mathtt{x}_{28} - 22\mathtt{x}_{29} - 26\mathtt{x}_{30} - 56\mathtt{x}_{31} + 56\mathtt{x}_{32} + 26\mathtt{x}_{33}$
$+22\mathtt{x}_{34} - 8\mathtt{x}_{35} + 16\mathtt{x}_{36} - 14\mathtt{x}_{37} - 18\mathtt{x}_{38} - 48\mathtt{x}_{39} + 32\mathtt{x}_{40} + 2\mathtt{x}_{41}$
$-2\mathtt{x}_{42} - 32\mathtt{x}_{43} - 32\mathtt{x}_{43} - 8\mathtt{x}_{44} - 38\mathtt{x}_{45} - 42\mathtt{x}_{46} - 72\mathtt{x}_{47} + 40\mathtt{x}_{48}$
$+10\mathtt{x}_{49} + 6\mathtt{x}_{50} - 24\mathtt{x}_{51} - 30\mathtt{x}_{53} - 34\mathtt{x}_{54} - 64\mathtt{x}_{55} + 16\mathtt{x}_{56} - 14\mathtt{x}_{57}$
$-18\mathtt{x}_{58} - 48\mathtt{x}_{59} - 24\mathtt{x}_{60} - 54\mathtt{x}_{61} - 58\mathtt{x}_{62} - 88\mathtt{x}_{63} \leq -89$.

Let $\overline{FC\left(\mathcal{A}, \mathcal{S}'\right)_n^0} \supseteq FC\left(\mathcal{A}, \mathcal{S}'\right)_n^0$, and let the following be the linear relaxation of $\overline{FC\left(\mathcal{A}, \mathcal{S}'\right)_n^0}$, defined by the following constraints (trivial ones not shown):

1. $[-186.5] : \mathtt{x}_{0} = 0$
2. FS inequalities:
$[-7.5] : \mathtt{x}_{1} - \mathtt{x}_{0} \leq 0$, $[-10] : \mathtt{x}_{6} - \mathtt{x}_{0} \leq 0$, $[-8.5] : \mathtt{x}_{11} - \mathtt{x}_{0} \leq 0$,
$[0] : \mathtt{x}_{19} - \mathtt{x}_{0} \leq 0$, $[-8.5] : \mathtt{x}_{23} - \mathtt{x}_{0} \leq 0$, $[-4] : \mathtt{x}_{35} - \mathtt{x}_{0} \leq 0$,
$[-7] : \mathtt{x}_{37} - \mathtt{x}_{0} \leq 0$, $[-9] : \mathtt{x}_{38} - \mathtt{x}_{0} \leq 0$, $[-24] : \mathtt{x}_{39} - \mathtt{x}_{0} \leq 0$,
$[-2.5] : \mathtt{x}_{41} - \mathtt{x}_{0} \leq 0$, $[-16] : \mathtt{x}_{44} - \mathtt{x}_{0} \leq 0$, $[-21] : \mathtt{x}_{46} - \mathtt{x}_{0} \leq 0$,
$[-15] : \mathtt{x}_{47} - \mathtt{x}_{0} \leq 0$, $[-6] : \mathtt{x}_{50} - \mathtt{x}_{0} \leq 0$, $[-19] : \mathtt{x}_{55} - \mathtt{x}_{0} \leq 0$,
$[-5.5] : \mathtt{x}_{56} - \mathtt{x}_{0} \leq 0$, $[-17] : \mathtt{x}_{59} - \mathtt{x}_{0} \leq 0$, $[-16] : \mathtt{x}_{61} - \mathtt{x}_{0} \leq 0$,
$[-11] : \mathtt{x}_{62} - \mathtt{x}_{0} \leq 0$, $[-23] : \mathtt{x}_{63} - \mathtt{x}_{0} \leq 0$, $[-12] : \mathtt{x}_{13} - \mathtt{x}_{12} \leq 0$,
$[-12.5] : \mathtt{x}_{14} - \mathtt{x}_{2} \leq 0$, $[-12.5] : \mathtt{x}_{22} - \mathtt{x}_{18} \leq 0$, $[-6.5] : \mathtt{x}_{62} - \mathtt{x}_{18} \leq 0$,
$[-1] : \mathtt{x}_{42} - \mathtt{x}_{40} \leq 0$, $[-7] : \mathtt{x}_{51} - \mathtt{x}_{48} \leq 0$, $[-7.5] : \mathtt{x}_{29} - \mathtt{x}_{17} \leq 0$,
$[-5.5] : \mathtt{x}_{43} - \mathtt{x}_{8} \leq 0$, $[-5.5] : \mathtt{x}_{61} - \mathtt{x}_{9} \leq 0$, $[0] : \mathtt{x}_{63} - \mathtt{x}_{56} \leq 0$.
3. FC inequalities:
$[-7.5] : \mathtt{x}_{15} + \mathtt{x}_{45} - \mathtt{x}_{1} - \mathtt{x}_{13} \leq 0$, $[-9] : \mathtt{x}_{15} + \mathtt{x}_{53} - \mathtt{x}_{1} - \mathtt{x}_{5} \leq 0$,
$[-3.5] : \mathtt{x}_{15} + \mathtt{x}_{57} - \mathtt{x}_{1} - \mathtt{x}_{9} \leq 0$, $[-7.5] : \mathtt{x}_{23} + \mathtt{x}_{62} - \mathtt{x}_{16} - \mathtt{x}_{22} \leq 0$,
$[-8] : \mathtt{x}_{27} + \mathtt{x}_{45} - \mathtt{x}_{1} - \mathtt{x}_{9} \leq 0$, $[-8.5] : \mathtt{x}_{31} + \mathtt{x}_{43} - \mathtt{x}_{1} - \mathtt{x}_{11} \leq 0$,
$[-1] : \mathtt{x}_{31} + \mathtt{x}_{53} - \mathtt{x}_{16} - \mathtt{x}_{21} \leq 0$, $[-3.5] : \mathtt{x}_{45} + \mathtt{x}_{57} - \mathtt{x}_{8} - \mathtt{x}_{41} \leq 0$,
$[-9] : \mathtt{x}_{55} + \mathtt{x}_{58} - \mathtt{x}_{16} - \mathtt{x}_{50} \leq 0$, $[-17] : \mathtt{x}_{7} + \mathtt{x}_{54} - \mathtt{x}_{4} - \mathtt{x}_{6} \leq 0$,
$[-5] : \mathtt{x}_{51} + \mathtt{x}_{53} - \mathtt{x}_{48} - \mathtt{x}_{49} \leq 0$.

4. FC-chain inequalities (it is easy to check that the explicit chains work for all subsets):
$[-1.5] : \mathsf{x}_{29} + \mathsf{x}_{47} + \mathsf{x}_{61} + \mathsf{x}_{63} - \mathsf{x}_{8} - \mathsf{x}_{13} - \mathsf{x}_{17} - \mathsf{x}_{56} \leq 0$,
$(29 \xrightarrow{19^{FS}} 17)$, $(63 \xrightarrow{8^{FS}} 8)$, $(47 \xrightarrow{8^{FS}} 8)$, $(61 \xrightarrow{56^{FS}} 56)$, $(63 \xrightarrow{56^{FS}} 56)$,
$(47 \xrightarrow{29^{UC}} 13)$, $(61 \xrightarrow{19^{FS}} 17)$.
$[-4] : \mathsf{x}_{29} + \mathsf{x}_{61} + \mathsf{x}_{62} - \mathsf{x}_{16} - \mathsf{x}_{17} - \mathsf{x}_{28} \leq 0$,
$(29 \xrightarrow{16^{FS}} 16)$, $(61 \xrightarrow{19^{FS}} 17)$, $(62 \xrightarrow{19^{FS}} 50 \xrightarrow{16^{FS}} 16)$,
$(29 \xrightarrow{62^{UC}} 28)$, $(29 \xrightarrow{16^{FS}} 17)$.
$[-7.5] : \mathsf{x}_{30} + \mathsf{x}_{31} + \mathsf{x}_{47} + \mathsf{x}_{63} - \mathsf{x}_{8} - \mathsf{x}_{14} - \mathsf{x}_{16} - \mathsf{x}_{24} \leq 0$,
$(63 \xrightarrow{8^{FS}} 8)$, $(47 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{16^{FS}} 16)$, $(47 \xrightarrow{30^{UC}} 14)$, $(30 \xrightarrow{16^{FS}} 16)$,
$(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{16^{FS}} 16)$, $(31 \xrightarrow{56^{FS}} 24)$.
$[-4] : \mathsf{x}_{30} + \mathsf{x}_{31} + \mathsf{x}_{55} - \mathsf{x}_{19} - \mathsf{x}_{22} - \mathsf{x}_{24} \leq 0$,
$(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{19^{FS}} 19)$, $(55 \xrightarrow{30^{UC}} 22)$, $(55 \xrightarrow{19^{FS}} 19)$.
$[-7] : \mathsf{x}_{30} + \mathsf{x}_{31} + \mathsf{x}_{59} - \mathsf{x}_{16} - \mathsf{x}_{24} - \mathsf{x}_{26} \leq 0$,
$(30 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{56^{FS}} 24)$, $(31 \xrightarrow{16^{FS}} 16)$, $(30 \xrightarrow{16^{FS}} 16)$,
$(59 \xrightarrow{16^{FS}} 16)$, $(59 \xrightarrow{30^{UC}} 26)$.
$[0] : \mathsf{x}_{47} + \mathsf{x}_{54} + \mathsf{x}_{63} - \mathsf{x}_{8} - \mathsf{x}_{16} - \mathsf{x}_{38} \leq 0$,
$(63 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{16^{FS}} 16)$, $(47 \xrightarrow{8^{FS}} 8)$, $(54 \xrightarrow{16^{FS}} 16)$, $(54 \xrightarrow{47^{UC}} 38)$.
$[-12] : \mathsf{x}_{47} + \mathsf{x}_{60} + \mathsf{x}_{63} - \mathsf{x}_{8} - \mathsf{x}_{44} - \mathsf{x}_{56} \leq 0$.
$(47 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{8^{FS}} 8)$, $(63 \xrightarrow{56^{FS}} 56)$, $(60 \xrightarrow{56^{FS}} 56)$, $(60 \xrightarrow{47^{UC}} 44)$.
5. Feasibility inequality:
$[-0.5] : 58\mathsf{x}_{1} + 54\mathsf{x}_{2} + 24\mathsf{x}_{3} + 48\mathsf{x}_{4} + 18\mathsf{x}_{5} + 14\mathsf{x}_{6} - 16\mathsf{x}_{7} + 64\mathsf{x}_{8}$
$+34\mathsf{x}_{9} + 30\mathsf{x}_{10} + 24\mathsf{x}_{12} - 6\mathsf{x}_{13} - 10\mathsf{x}_{14} - 40\mathsf{x}_{15} + 72\mathsf{x}_{16} + 42\mathsf{x}_{17}$
$+38\mathsf{x}_{18} + 8\mathsf{x}_{19} + 32\mathsf{x}_{20} + 2\mathsf{x}_{21} - 2\mathsf{x}_{22} - 32\mathsf{x}_{23} + 48\mathsf{x}_{24} + 18\mathsf{x}_{25}$
$+14\mathsf{x}_{26} - 16\mathsf{x}_{27} + 8\mathsf{x}_{28} - 22\mathsf{x}_{29} - 26\mathsf{x}_{30} - 56\mathsf{x}_{31} + 56\mathsf{x}_{32} + 26\mathsf{x}_{33}$
$+22\mathsf{x}_{34} - 8\mathsf{x}_{35} + 16\mathsf{x}_{36} - 14\mathsf{x}_{37} - 18\mathsf{x}_{38} - 48\mathsf{x}_{39} + 32\mathsf{x}_{40} + 2\mathsf{x}_{41}$
$-2\mathsf{x}_{42} - 32\mathsf{x}_{43} - 32\mathsf{x}_{43} - 8\mathsf{x}_{44} - 38\mathsf{x}_{45} - 42\mathsf{x}_{46} - 72\mathsf{x}_{47} + 40\mathsf{x}_{48}$
$+10\mathsf{x}_{49} + 6\mathsf{x}_{50} - 24\mathsf{x}_{51} - 30\mathsf{x}_{53} - 34\mathsf{x}_{54} - 64\mathsf{x}_{55} + 16\mathsf{x}_{56} - 14\mathsf{x}_{57}$
$-18\mathsf{x}_{58} - 48\mathsf{x}_{59} - 24\mathsf{x}_{60} - 54\mathsf{x}_{61} - 58\mathsf{x}_{62} - 88\mathsf{x}_{63} \leq -1$.