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A Combinatorial Proof of a Kőnig-type Theorem for Unimodular Hypergraphs

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Perfect f-Matchings and f-Factors in Hypergraphs - A Combinatorial Approach

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We state purely combinatorial proofs for Kőnig- and Hall-type theorems for a wide class of combinatorial optimization problems. Our methods rely on relaxations of the matching and vertex cover problem and, moreover, on the strong coloring properties admitted by bipartite graphs and their generalizations.

1 Introduction

One of the most significant results in graph theory and combinatorial optimization is Kőnig's Theorem on matchings in bipartite graphs. It says that the cardinality of a maximum matching equals the cardinality of a minimum vertex cover.

There are various ways of proving this result, for example using linear programming duality. Another - less common - way of proving this result is to use the edge-coloring properties of bipartite graphs.

A key step in this proof is the following trick. Suppose there is some edge $e^* = uv$ in a bipartite graph $G = (V, E)$ and two matchings M_u and M_v of the same size, say $|M_u| = k = |M_v|$. Assume that M_u does not cover u and M_v does not cover v and consider the graph $H := (V, \{e^*\} \cup M_u \cup M_v)$. It is bipartite and has maximum degree two, so there is a proper 2-edge-coloring of this graph. That is, the edges of H can be partitioned into two matchings, one of which is larger than k. An inductive application of this augmentation step can be used to prove Kőnig's Theorem.

The aim of this paper is to generalize this technique from bipartite graphs to a richer class of combinatorial structures, namely unimodular hypergraphs. These are hypergraphs whose vertex-hyperedge-incidence matrix is totally unimodular. We obtain

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Kőnig- and Hall-type theorems including weights and capacities in full generality on these hypergraphs. While such results are well known and easy to prove using linear programming methods, our proofs are solely based on coloring properties and thus reveal the combinatorial explanation of these Kőnig- and Hall-type theorems.

1.1 Overview/contributions.

In Section 2, we introduce f-matchings, f-factors, and d-covers, as well as unimodular and balanced hypergraphs. Furthermore, we survey their coloring notions, in particular, de Werra's coloring property of unimodular hypergraphs. Subsequently, in Section 3, we show how to improve matchings and vertex covers via de Werra's coloring property. In Section 3.2, we state our main result - the combinatorial proof of a general theorem of the Kőnig-type, Theorem 3.1. The last part of Section 3 is devoted to an application of our methods, i.e., we state a completely new combinatorial proof for a min-max result for balanced hypergraphs. All proofs in Section 3 rely on the fact that unimodular hypergraphs admit so-called equitable colorings. A combinatorial explanation of this fact, based on Seymour's decomposition of regular matroids, is sketched in Section 4. Finally, in Section 5, we characterize the existence of matchings and factors in unimodular hypergraphs.

Again, we emphasize that most of our results are well known and/or easy to prove with linear programming methods. Our main contribution lies in the combinatorial ideas stated and applied in this work.

2 Preliminaries

Given an undirected graph $G = (V, E)$, a *matching* is a subset of edges $M \subseteq E$ such that each vertex $v \in V$ is incident to at most one edge in M. Dually, a *vertex cover* is a subset of vertices $C \subseteq V$ such that each edge $e \in E$ is covered by C, meaning that e has at least one end-vertex in C. The famous theorem of König states that any bipartite graph admits a matching and a vertex cover of equal size. This min-max result can be extended to a weighted and capacitated version which reads as follows.

Min-max theorem for bipartite graphs. *Let $A \in \{0,1\}^{|V|\times|E|}$ be the vertex-edge incidence matrix of a bipartite graph $G = (V, E)$, i.e., $A_{v,e} = 1$ if and only if $v \in e$. Given edge-weights $f : E \to \mathbb{Z}_+$ and vertex-weights $d : V \to \mathbb{Z}_+$, there exist integral optimal solutions $y^* \in \mathbb{Z}^{|E|}$ and $x^* \in \mathbb{Z}^{|V|}$ to the following primal-dual pair of linear programs*

$$\max\{d^T y \mid Ay \leq f,\ y \geq 0\} = \min\{f^T x \mid A^T x \geq d,\ x \geq 0\}.$$

In Section 2.3 we state a more general version of the theorem together with further reference.

2.1 Hypergraphs and TUM matrices.

In general, an integral matrix $A \in \mathbb{Z}^{m\times n}$ is *totally unimodular (TUM)* if and only if the determinant of each square submatrix is in $\{-1, 0, 1\}$. This means in particular, that all entries of A are in $\{-1, 0, 1\}$ as well. For example, each vertex-edge incidence matrix of a bipartite graph is known to be TUM. As it is well-known, and usually proved by using linear-programming techniques, the above min-max theorem can be extended from vertex-edge incidence matrices of bipartite graphs to TUM matrices in general.

In this paper, we restrict our considerations to *binary* matrices $A \in \{0, 1\}^{|V|\times|E|}$ as such matrices can be interpreted as vertex-edge incidence matrices of *hypergraphs* $H = (V, E)$ with $E \subseteq 2^V$ via

$$A_{v,e} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{else.} \end{cases}$$

Another reason for this restriction is that our current methods only work in the case of binary TUM matrices / unimodular hypergraphs.

Hypergraphs generalize undirected graphs in the sense that (hyper-)edges may correspond to arbitrary subsets of vertex set V, instead of only subsets of size two. We call a hypergraph $H = (V, E)$ *unimodular* if and only if its incidence matrix is TUM. Given any hypergraph $H = (V, E)$ with incidence-matrix $A \in \{0, 1\}^{|V|\times|E|}$, as well as weight functions $f : E \to \mathbb{Z}_+$ and $d : V \to \mathbb{Z}_+$, we call any feasible integral solution $y \in \mathbb{Z}_+^{|E|}$ of $Ay \le f$ an *f-matching*, and any feasible integral solution $x \in \mathbb{Z}_+^{|V|}$ of $A^T x \ge d$ a *d-cover*. Given an additional function $g : E \to \mathbb{Z}_+$, we call any feasible integral solution $y \in \mathbb{Z}_+^{|E|}$ of $g \le Ay \le f$ a *(g, f)-matching*. Any (f, f) matching is called *perfect f-matching*. Furthermore, if the (g, f)-matching needs to obey certain capacity constraints of the type $y(e) \le c(e)$ for all $e \in E$, we call any feasible integral solution $y \in \mathbb{Z}_+^{|E|}$ of $\{g \le Ay \le f,\ y \le c\}$ a *c-capacitated (g, f)-matching*. In the special case where $c \equiv 1$, we call a c-capacitated (g, f)-matching just a *(g, f)-factor*, or simply *f-factor* if $c \equiv 1$ and $g \equiv f$.

2.2 Coloring properties of unimodular hypergraphs.

Given a hypergraph $H = (V, E)$, an *equitable 2-coloring* is a partition of the vertex set $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ such that

$$\left\lfloor \frac{|e|}{2} \right\rfloor \le |e \cap V_i| \le \left\lceil \frac{|e|}{2} \right\rceil,\ i = 1, 2$$

for every hyperedge e of E. Note that a graph G admits an equitable 2-coloring if and only if G is bipartite. Recall the following characterization of TUM matrices by Ghouila-Houri (since we could not obtain the original source [GH62] we refer to [Ber89]).

Theorem 2.1. *[GH62] A matrix A is totally unimodular if and only if for every submatrix A', say of dimension $m \times n$, there is a vector $v \in \{-1, 1\}^n$ such that $A'v \in \{-1, 0, 1\}^m$.*

As a consequence, the following characterization of unimodular hypergraphs by equitable 2-colorings is true.

Corollary 2.2. *A hypergraph H is unimodular if and only if for every vertex subset $S \subseteq V$ the induced subhypergraph $H[S] = (S, E[S])$ admits an equitable 2-coloring.*

De Werra strengthened this coloring property. He proved the following.

Theorem 2.3. *[dW71] An unimodular hypergraph $H = (V, E)$ has an equitable vertex k-coloring for $k \geq 2$; i.e., a partition $(C_1, C_2, \cdots, C_k)$ of V such that*

$$\left\lfloor \frac{|e|}{k} \right\rfloor \leq |e \cap C_i| \leq \left\lceil \frac{|e|}{k} \right\rceil, \ i = 1, 2, \ldots, k$$

for every hyperedge e of H.

2.3 Min-max result for unimodular hypergraphs

The main goal of this paper is to provide a combinatorial proof of the theorem stated below. Further reference can be found in Chapter 83 of the book by Schrijver [Sch03]. Our proof relies solely on de Werra's coloring property of unimodular hypergraphs.

Min-max theorem for binary TUM matrices. *Let $A \in \{0, 1\}^{|V| \times |E|}$ be TUM. Given edge-weights $f : E \to \mathbb{Z}_+$ and vertex-weights $d : V \to \mathbb{Z}_+$, there exist integral optimal solutions $y^* \in \mathbb{Z}^{|E|}$ and $x^* \in \mathbb{Z}^{|V|}$ to the following primal-dual pair of linear programs*

$$\max\{d^T y \mid Ay \leq f, \ y \geq 0\} = \min\{f^T x \mid A^T x \geq d, \ x \geq 0\}.$$

Loosely speaking, de Werra's coloring property (i.e., Theorem 2.3) is the only way that we use the total unimodularity of the hypergraph. This is why, in Section 4, we give a purely combinatorial proof of Theorem 2.1. This proof is a consequence of Seymour's decomposition of totally unimodular matrices which is also purely combinatorial.

Remark. Since submatrices of totally unimodular matrices are totally unimodular, we obtain that partial subhypergraphs are again equitable colorable. Multiplying vertices and/or edges, i.e., replacing vertices and edges by copies, does not affect the unimodularity and therefore the colorability of a hypergraph, too. Applying these equitable coloring results to the dual hypergraph, yields results for edge colorings, namely:

A unimodular hypergraph $H = (V, E)$ has an equitable edge k-coloring for $k \geq 2$; that is, a partition $(C_1, C_2, \cdots, C_k)$ of E such that

$$\left\lfloor \frac{\deg_H(v)}{k} \right\rfloor \leq |\{C_i | v \in e \in C_i\}| \leq \left\lceil \frac{\deg_H(v)}{k} \right\rceil, \ i = 1, 2, \ldots, k$$

for every vertex v of H.

2.4 Balanced hypergraphs.

Balanced hypergraphs form a superclass of unimodular hypergraphs whose definition goes back to Berge [Ber89]. Given a hypergraph $H = (V, E)$, a *cycle of length* k in H is an alternating sequence $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ such that $v_{k+1} = v_1$, the vertices $v_1, v_2, \dots, v_k$ are distinct, the hyperedges $e_1, e_2, \dots, e_k$ are distinct, and $v_i, v_{i+1} \in e_i$ for $i = 1, \dots, k$. A cycle $v_1, e_1, v_2, e_2, \dots, v_k, e_k, v_{k+1}$ is called *strong* if $e_i \cap \{v_1, \dots, v_k\} = \{v_i, v_{i+1}\}$ for all $i = 1, \dots, k$.

Definition 2.1. A hypergraph $H = (V, E)$ is *balanced* if H does not contain any strong cycle of odd length.

Balanced hypergraphs admit strong coloring properties. Most of them were found by Berge [Ber89]. Here, we need the notion of *good colorings*:

Definition 2.2. A *good edge k-coloring* for $k \geq 2$ is a partition $(C_1, C_2, \cdots, C_k)$ of E in color classes such that every vertex $v \in V$ is incident to $\min\{\deg_H(v), k\}$ color classes. Similar, a *good vertex k-coloring* for $k \geq 2$ is a partition $(C_1, C_2, \cdots, C_k)$ of V in color classes such that every edge intersects $\min\{|e|, k\}$ color classes.

Theorem 2.4. *[Ber89] A balanced hypergraph $H = (V, E)$ admits good vertex and edge k-colorings for $k \geq 2$.*

3 A combinatorial proof of a "general theorem of the Kőnig-type"

We consider a more general version of matching and covering problems by introducing additional penalty costs $p : E \to \mathbb{Z}_+$ and $q : V \to \mathbb{Z}_+$ to pay for violations of the packing and covering constraints. In this section, we provide a combinatorial proof of the following theorem, relying solely on de Werra's coloring property of TUM matrices.

Theorem 3.1 (Min-max theorem for binary TUM matrices with penalty costs)**.** *Let $A \in \{0,1\}^{|V| \times |E|}$ be TUM. Given weight functions $f : E \to \mathbb{Z}_+$, $d : V \to \mathbb{Z}_+$, and penalty cost functions $p : E \to \mathbb{Z}_+$ and $q : V \to \mathbb{Z}_+$, there exist integral optimal solutions (y^*, z^*) and (x^*, w^*) to the following primal-dual pair of linear programs*

$$\max\{d^T y - q^T z \mid Ay - Iz \leq f,\ y, z \geq 0,\ y \leq p\} \tag{1}$$

$$= \min\{f^T x + p^T w \mid A^T x + Iw \geq d,\ x, w \geq 0,\ x \leq q\}. \tag{2}$$

Note that Theorem 3.1 can easily be derived by linear-programming techniques since the relaxations of the underlying constraint matrices remain TUM. Theorem 3.1 generalizes the usual min-max theorem for binary TUM matrices: for large enough penalty costs, any pair of an optimal integral primal and dual solution correspond to an optimal f-matching and an d-cover, respectively. Throughout, we call the subvector y of any feasible integral solution (y, z) of the primal problem (1) a *relaxed f-matching*, since exceeding the right-hand-side f can be compensated by z, but results in penalty costs $q^T z$.

Similarly, the subvector x of any feasible integral solution (x, w) of the dual problem (2) is called *relaxed d-cover*. Interpreting the strong duality combinatorially, the above fact can equivalently be formalized as the following Kőnig-type result for unimodular hypergraphs:

An optimum d-weighted, relaxed f-matching with penalty costs q (for violating the f-matching condition) which is p-capacitated has the same value as an optimum f-weighted, relaxed d-cover with penalty costs p (for violating the d-cover condition) which is q-capacitated.

Given d, f, p, q and an unimodular hypergraph H with incidence matrix A as above, we define the violation of a relaxed f-matching $y \in \mathbb{Z}_+^{|E|}$ at a vertex v and the violation of a relaxed d-cover $x \in \mathbb{Z}_+^{|V|}$ at an edge e to be

$$\mathrm{viol}_v(y) := \left[\sum_{e:v\in e} y(e) - f(v)\right]_+ \quad \text{and} \quad \mathrm{viol}_e(x) := \left[d(e) - \sum_{v\in e} x(v)\right]_+ ,$$

respectively. Accordingly, we define the (Lagrangian) weight functions

$$L(y) := d^T y - q^T \mathrm{viol}(y) \quad \text{and} \quad \tilde{L}(x) := f^T x + p^T \mathrm{viol}(x).$$

3.1 The Coloring Trick

Let us first assume that the penalty costs satisfy the assumption

$$(A) \quad q^T A > d \quad \text{and} \quad Ap > f.$$

We will see later that this assumption is not restrictive. In the following two lemmata and their proofs, we show how equitable colorings can be used to improve a set of relaxed f-matchings or d-covers.

Lemma 3.2. *Suppose there exists a vertex $v_0 \in V$ contained in exactly k hyperedges $\{e_1, \ldots, e_k\} \subseteq E$ and k relaxed d-covers $x^{(1)}, \ldots, x^{(k)}$ satisfying*

- *$v_0 \in e_i$ and $\sum_{v\in e_i} x_v^{(i)} < d(e_i)$ for each $i \in [k]$, and*
- *$x^{(j)}(v_0) < q(v_0)$ for at least one $j \in [k]$.*

Then there is a q-capacitated relaxed d-cover x^* with $\tilde{L}(x^*) < \frac{1}{k} \sum_{i=1}^{k} \tilde{L}(x^{(i)})$.

Proof. Let us define a function $\mathrm{mult} : V \to \mathbb{N}_0$ with

$$\mathrm{mult}(v) := \begin{cases} \sum_{i=1}^{k} x^{(i)}(v) & \text{for } v \neq v_0, \\ \sum_{i=1}^{k} x^{(i)}(v_0) + 1 & \text{for } v = v_0. \end{cases}$$

Consider the hypergraph H^{mult} which arises from H by replacing each vertex v by $\mathrm{mult}(v)$ copies $v_1, \ldots, v_{\mathrm{mult}(v)}$ (i.e., vertices v with $\mathrm{mult}(v) = 0$ are deleted,) and adding for each $v \in V$ the additional edge $\bar{e}_v := \{v_1, \dots, v_{\text{mult}(v)}\}$. Note that H^{mult} remains unimodular since those two transformations do not affect the unimodularity. By de Werra's coloring property there exists an equitable k-coloring $(C_1, \dots, C_k)$ of the vertices in H^{mult}. Consider the associated color-vectors $c^{(1)}, \dots, c^{(k)} : V \to \mathbb{Z}_+$ with entries $c^{(i)}(v) = |C_i \cap \bar{e}_v|$ for each $v \in V$ and each $i \in [k]$. That is, $c^{(i)}(v)$ is the number of copies of v of color C_i. Each of these color vectors is a q-capacitated d-cover, since H^{mult} contains at most $q(v) \cdot k$ copies of each vertex $v \in V$ so that de Werra's coloring property ensures that the colors are distributed equitable among the k vectors, implying $c^{(i)} \leq q$ for all $i \in [k]$. In the remainder of the proof, we show that

$$\sum_{i=1}^{k} \tilde{L}(c^{(i)}) < \sum_{i=1}^{k} \tilde{L}(x^{(i)}), \tag{3}$$

implying that at least one of the q-capacitated relaxed f-matchings $c^{(i)}$ must have smaller $\tilde{L}$-weight than the average $\frac{1}{k}\sum_{i=1}^{k} \tilde{L}(x^{(i)})$. By construction, we know that

$$\sum_{i=1}^{k} f^T x^{(i)} + f(v_0) = \sum_{i=1}^{k} f^T c^{(i)}.$$

For each individual edge $g \in E$, let us consider the sum of violations $\sum\limits_{i=1}^{k} \text{viol}_g(x^{(i)})$ and $\sum\limits_{i=1}^{k} \text{viol}_g(c^{(i)})$ caused by g. By de Werra's coloring property, none of the edges $g \in E$ with $\sum\limits_{i=1}^{k} \sum\limits_{v \in g} x^{(i)}(v) \geq k \cdot d(g)$ causes a penalty cost for any of the color-vectors $c^{(1)}, \dots, c^{(k)}$, i.e., for each such edge g and each $i \in [k]$ we have $\text{viol}_g(c^{(i)}) \leq 0$. So let us consider an edge $g \in E$ with $\sum\limits_{i=1}^{k} \sum\limits_{v \in g} x^{(i)}(v) < k \cdot d(g)$. For each such edge $g \in E$ and each $i \in [k]$, de Werra's equitable coloring condition ensures $d(g) \geq \sum\limits_{v \in g} c^{(i)}(v)$. Observe that

$$\sum_{i=1}^{k} \text{viol}_g(x^{(i)}) = \sum_{i=1}^{k} \left[d(g) - \sum_{v \in g} x^{(i)}(v) \right]_+ \geq \sum_{i=1}^{k} \left(d(g) - \sum_{v \in g} x^{(i)}(v) \right).$$

By construction, we know that

$$\sum_{i=1}^{k} \left(d(g) - \sum_{v \in g} x^{(i)}(v) \right) = \sum_{i=1}^{k} \left(d(g) - \sum_{v \in g} c^{(i)}(v) \right) (+1 \text{ if } v_0 \in g).$$

Since $d(g) \geq \sum\limits_{v \in g} c^{(i)}(v)$, we derive

$$\sum_{i=1}^{k} \left(d(g) - \sum_{v \in g} c^{(i)}(v) \right) = \sum_{i=1}^{k} \left[d(g) - \sum_{v \in g} c^{(i)}(v) \right]_+.$$

It follows that

$$\sum_{i=1}^{k} \text{viol}_g(x^{(i)}) \geq \sum_{i=1}^{k} \text{viol}_g(c^{(i)})(+1 \text{ if } v_0 \in g).$$

Summarizing we obtain

$$\sum_{i=1}^{k} \tilde{L}(x^{(i)}) + f(v_0) - \sum_{e:v_0 \in e} p(e) \geq \sum_{i=1}^{k} \tilde{L}(c_i).$$

Combining this with our assumption (A) leads to $\sum\limits_{i=1}^{k} \tilde{L}(x^{(i)}) > \sum\limits_{i=1}^{k} \tilde{L}(c^{(i)})$, as desired. □

Lemma 3.3. *Suppose there exist a hyperedge $e_0 \in E$ with $e_0 = \{v_1, \dots, v_k\}$ and k relaxed f-matchings $y^{(1)}, \dots, y^{(k)}$ satisfying*

- $\sum_{e \in E:\, v_i \in e} y^{(i)}(e) < f(v_i)$ *for all $i \in [k]$, and*
- $y^{(j)}(e_0) < p(e_0)$ *for at least one $j \in [k]$.*

Then there is a p-capacitated and relaxed f-matching y^* with $L(y^*) > \frac{1}{k} \sum\limits_{i=1}^{k} L(y^{(i)})$.

Proof. The main ideas are contained in the proof of the previous lemma and have to be applied to the dual setting. Nevertheless, we will explain the necessary steps: Again, we define a function mult : $E \to \mathbb{N}_0^m$ with

$$\text{mult}(e) := \begin{cases} \sum\limits_{i=1}^{k} y^{(i)}(e) & \text{for } e \neq e_0, \\ \sum\limits_{i=1}^{k} y^{(i)}(e_0) + 1 & \text{for } e = e_0, \end{cases}$$

and consider the hypergraph which arises from H in the following way: Introduce a new vertex v_e for each edge $e \in E$ and replace e by $e \cup \{v_e\}$. After that multiply the edges with mult$(e) > 0$. (Edges e with mult$(e) = 0$ are deleted.) Then, an equitable edge coloring in k colors yields edge-color classes which can be interpreted as new f-matchings $c^{(1)}, \dots, c^{(k)}$. Because of the auxiliary vertices v_e for $e \in E$, these f-matchings are p-capacitated. As above, we compare the weights of the color classes with the weights of the $y^{(i)}$.

1. $\sum\limits_{i=1}^{k} d^T y^{(i)} + d(e_0) = \sum\limits_{i=1}^{k} d^T c^{(i)}$.
2. Let a vertex $u \in V$ be given with

$$\sum_{i=1}^{k} \sum_{e:u \in e} y^{(i)}(e) > k \cdot f(u). \tag{4}$$

If no vertex u satisfying (4) exists, there is no penalty for the color classes.

Otherwise,

$$\begin{aligned}
\sum_{i=1}^{k} \mathrm{viol}_u(y^{(i)}) = \sum_{i=1}^{k} \left[\sum_{e:u\in e} y^{(i)}(e) - f(u)\right]_+ \geq & \sum_{i=1}^{k} \left(\sum_{e:u\in e} y^{(i)}(e) - f(u)\right) \\
= & \sum_{i=1}^{k} \left(\sum_{e:u\in e} c^{(i)}(e) - f(u)\right) (-1,\ \text{if } u \in e_0) \\
\overset{(4)}{=} & \sum_{i=1}^{k} \left[\sum_{e:u\in e} c^{(i)}(e) - f(u)\right]_+ (-1,\ \text{if } u \in e_0) \\
= & \sum_{i=1}^{k} \mathrm{viol}_u(c^{(i)})(-1,\ \text{if } u \in e_0).
\end{aligned}$$

and we obtain (penalty costs are negative here)

$$\sum_{i=1}^{k} L(y^{(i)}) + d(e_0) - \sum_{v:v\in e_0} q(v) \leq \sum_{i=1}^{k} L(c^{(i)}).$$

Combining this with Condition (A) leads to

$$\sum_{i=1}^{k} L(c^{(i)}) > \sum_{i=1}^{k} L(y^{(i)})$$

which yields the desired result. □

3.2 The Main Proof

Now, we come to the announced general result of the Kőnig-type, i.e., the proof of Theorem 3.1, stating

$$\max\{L(y) \mid y : E \to \mathbb{N}_0, p - \text{capacitated}\} = \min\{\tilde{L}(x) \mid x : V \to \mathbb{N}_0, q - \text{capacitated}\}.$$

Proof. Let a p-capacitated relaxed f-matching y and a q-capacitated and relaxed d-cover

x be given. We obtain

$$\begin{aligned}
L(y) &= \sum_{e\in E} y(e)d(e) - \sum_{v\in V}\left[\sum_{e:v\in e} y(e) - f(v)\right]_+ \cdot q(v) \\
&\leq \sum_{e\in E} y(e)\left(\sum_{v:v\in e} x(v)\right) + \sum_{e\in E} y(e)\left[d(e) - \sum_{v:v\in e} x(v)\right]_+ \\
&\quad - \sum_{v\in V}\left[\sum_{e:v\in e} y(e) - f(v)\right]_+ \cdot x(v) \\
&\leq \sum_{e\in E} y(e)\left(\sum_{v:v\in e} x(v)\right) + \sum_{e\in E}\left[d(e) - \sum_{v:v\in e} x(v)\right]_+ \cdot p(e) \\
&\quad - \sum_{v\in V}\left(\sum_{e:v\in e} y(e) - f(v)\right) \cdot x(v) \\
&= \sum_{v\in V} x(v)f(v) + \sum_{e\in E}\left[d(e) - \sum_{v:v\in e} x(v)\right]_+ \cdot p(e) = \tilde{L}(x).
\end{aligned}$$

This proves

$$\max\{L(y) \mid y : E \to \mathbb{N}_0, p-\text{capacitated}\} \leq \min\{\tilde{L}(x) \mid x : E \to \mathbb{N}_0, q-\text{capacitated}\}.$$

We prove the reverse inequality by induction over $|V|+|E|+\sum_{e\in E} p(e)$. For the induction basis one has to consider an empty hypergraph for which the statement obviously holds. As a first step we convince ourselves that $q^T A > d$ and $Ap > f$, i.e., condition (A), can be assumed w.l.o.g. As long as $Ap > f$ is not true, i.e., whenever there exists some $v \in V$ with $\sum_{e:v\in e} p(e) \leq f(v)$, we can remove vertex v from H, apply the induction hypothesis and set $x(v) = 0$. Now, suppose that $q^T A > d$ is not true, i.e., there exists some $e \in E$ with $\sum_{v:v\in e} q(v) \leq d(e)$. In this case, consider the hypergraph $H' = (V, E \setminus \{e\})$ and the function

$$f'(v) := \begin{cases} f(v) & \text{for } v \notin e, \\ f(v) - p(e) & \text{for } v \in e \text{ and } p(e) \leq f(v), \\ 0 & \text{for } v \in e \text{ and } p(e) > f(v). \end{cases}$$

By induction hypothesis, the min-max result holds for $H', f', d\mid_{E\setminus\{e\}}, p\mid_{E\setminus\{e\}}, q$. Take two optimum solutions x and y of this reduced problem and set $y(e) := p(e)$ and leave x unchanged. This yields the desired inequality for H and the original functions because L increases by

$$d(e)p(e) + \sum_{\substack{v:v\in e\\ p(e)\geq f(v)}} (f(v) - p(e))q(v)$$

and $\tilde{L}$ increases by

$$\begin{aligned}
&\sum_{\substack{v:v\in e\\ p(e)\geq f(v)}} f(v)x(v) + \sum_{\substack{v:v\in e\\ p(e)<f(v)}} p(e)x(v) + \left[d(e) - \sum_{v:v\in e} x(v)\right]_+ p(e)\\
= \ & d(e)p(e) + \sum_{\substack{v:v\in e\\ p(e)\geq f(v)}} (f(v) - p(e))x(v).
\end{aligned}$$

It follows that condition (A) is fulfilled. Therefore, we can apply Lemma 3.2 which gives us an edge $e^* \in E$ covered by every optimum d-cover. (If there is no such edge, we obtain a vertex v_0 fulfilling the conditions of Lemma 3.2 and, therefore, a contradiction because these d-covers cannot be optimum.) We reduce the penalty costs of this edge by one. We denote this changed penalty function by p', i.e.,

$$p'(e) := \begin{cases} p(e) & e \neq e^* \\ p(e^*) - 1 & e = e^* \end{cases},$$

and the weight functions corresponding to f, d, p', and q by L' and $\tilde{L}'$. By induction hypothesis, we obtain an optimum relaxed f-matching y' and an optimum d-cover x' with

$$\tilde{L}'(x') = L'(y').$$

Let an optimum relaxed f-matching y concerning L be given. Then

$$\begin{aligned}
L(y) \ \geq \ & L'(y') = \tilde{L}'(x') = \sum_{v\in V} x'(v)f(v) + \sum_{e\in E}\left[d(e) - \sum_{v:v\in e} x'(v)\right]_+ \cdot p'(e)\\
= \ & \sum_{v\in V} x'(v)f(v) + \sum_{e\in E}\left[d(e) - \sum_{v:v\in e} x'(v)\right]_+ \cdot p(e) - \left[d(e^*) - \sum_{v:v\in e^*} x'(v)\right]_+ .
\end{aligned}$$

If e^* is covered by x', we directly obtain $L(y) \geq \tilde{L}(x')$. If e^* is not covered by x', we conclude that x' is not an optimum cover. Then

$$L(y) \geq \tilde{L}(x') - d(e^*) + \sum_{v:v\in e^*} x'(v) \geq \tilde{L}(x) + \underbrace{1 - d(e^*) + \sum_{v:v\in e^*} x'(v)}_{\leq 0}$$

for an optimum d-cover x. Suppose that $\sum_{v:v\in e^*} x'(v) \leq d(e^*) - k$ holds for all optimum solutions x' for $\tilde{L}'$ and $k \geq 2$. (If there is a solution x' with $\sum_{v:v\in e^*} x'(v) = d(e^*) - 1$ we are done.) We apply a coloring trick similar as in Lemma 3.2 and consider H^{mult} arising from $k-1$ copies of a relaxed d-cover x' and one copy of the d-cover x. Here, x' is an optimum solution for $\tilde{L}'$ with $\sum_{v:v\in e^*} x'(v) = d(e^*) - k$. Analogously as in Lemma 3.2, we choose an equitable vertex k coloring in H^{mult} such that the color classes are q

capacitated covers (auxiliary edges are needed). We denote our new covers arising from the color classes by $c^{(1)}, \dots, c^{(k)}$. Summing the weight functions of the original k cover, we obtain

$$(k-1)\tilde{L}'(x') + \tilde{L}'(x) \leq (k-1)\tilde{L}'(x') + \tilde{L}'(x') + k - 1 = k\tilde{L}'(x') + k - 1.$$

(If $\tilde{L}'(x) \geq \tilde{L}'(x') + k = \tilde{L}(x')$, then x' would be optimum concerning $\tilde{L}$ and does not cover e^*. This is impossible by assumption.) Therefore, we obtain at least one relaxed d-cover c^* having the same weight as x' concerning $\tilde{L}'$. Furthermore, since

$$\frac{1}{k}\left(\sum_{i=1}^{k}\sum_{v:v\in e^*} c^{(i)}(v)\right) \geq \frac{1}{k}((k-1)(d(e^*)-k) + d(e^*)) = d(e^*) - k + 1$$

the equitability yields $\sum_{v:v\in e^*} c^*(v) > d(e^*) - k$. This contradicts our assumption and the proof is completed.

□

3.3 A Similar Approach for Balanced Hypergraphs

A similar approach can be used for the class of balanced hypergraphs. Consider the following pair of linear programs with balanced constraint matrix A :

$$\begin{array}{llcll} \max & (d,-q)^T(y,z) & \qquad & \min & 1^T x \\ \text{s.t.} & Ay - Iz \leq 1 & & \text{s.t.} & A^T x \geq d, \\ & 0 \leq y, z & & & 0 \leq x \leq q, \end{array}$$

In [ST15] a coloring trick with good colorings was used to prove a duality theorem for this program pair. Here, we explain in somewhat more detail an approach for the following two programs:

$$\begin{array}{llcll} \max & 1^T y & \qquad & \min & (f,p)^T(x,w) \\ \text{s.t.} & Ay \leq f & & \text{s.t.} & A^T x + Iw \geq 1, \\ & 0 \leq y \leq p & & & 0 \leq x, w \end{array}$$

Interpreted combinatorially and assuming integrality conditions, we have p capacitated f-matchings on the left and relaxed 1-covers on the right. The corresponding weight functions are

$$L(y) := \sum_{e\in E} y(e)$$

and the second one on the set of relaxed vertex covers

$$\tilde{L}(x) := \sum_{v\in V} x(v)f(v) + \sum_{e\in E}\left[1 - \sum_{v:v\in e} x(v)\right]_+ \cdot p(e).$$

Similar to Lemma 3.2, the following assertion holds for balanced hypergraphs:

Assume that there is a vertex $v_0 \in V$ of degree k with the following property: There exist 1-covers $x^{(i)}$ for all edges $e_i \in E$ with $v_0 \in e_i$ such that $\sum_{v\in e_i} x^{(i)}(v) = 0$. Then, there is a q capacitated and relaxed 1-cover x^* of weight smaller than the average $\frac{1}{k}\sum_{i=1}^{k}\tilde{L}(x_i)$. In order to prove this assertion, we define again a multiplication function $\text{mult}: V \to \mathbb{N}_0^n$ which is defined as follows

$$\text{mult}(v) := \begin{cases} \sum_{i=1}^{k} x^{(i)}(v) & \text{for } v \neq v_0, \\ 1 & \text{for } v = v_0. \end{cases}$$

As above, we consider the hypergraph $H^{\text{mult}} := (V^{\text{mult}}, E^{\text{mult}})$ which arises from H by replacing each vertex v with $\text{mult}(v) > 0$ by $\text{mult}(v)$ copies $v_1, \ldots, v_{\text{mult}(v)}$. (We do not need any auxiliary edges here.) Then, we consider a good k-coloring of H^{mult} and denote by $C_1, \ldots, C_k$, its color classes. Under Condition (A), we can prove that, compared to the original $x^{(i)}$, their aggregated penalty costs are reduced by more than $f(v_0)$ while the f-weights are only increased by $f(v_0)$. Now, we state a calculation for the aggregated penalty costs:
The inequality

$$\sum_{\substack{e\in E^{\text{mult}}\\ v_0\notin e}} \min\{|e|,k\}p(e) + \sum_{\substack{e\in E^{\text{mult}}\\ v_0\in e}} (\min\{|e|,k\}-1)p(e) \geq \sum_{e\in E^{\text{mult}}} |\{x^{(i)} \mid x^{(i)} \text{ covers } e\}|p(e)$$

holds because of the defining properties of a good vertex coloring. It is equivalent to

$$\sum_{e\in E^{\text{mult}}} \min\{|e|,k\}p(e) - \sum_{\substack{e\in E^{\text{mult}}\\ v_0\in e}} p(e) \geq \sum_{e\in E^{\text{mult}}} |\{x^{(i)} \mid x^{(i)} \text{ covers } e\}|p(e).$$

From this follows $\left(\text{subtract } \left(\sum_{e\in E^{mult}} k|e|\right) \text{ and use Condition (A)}\right)$

$$\sum_{e\in E^{\text{mult}}} |\{C_i \mid e\cap C_i = \emptyset\}|p(e) + f(v_0) < \sum_{e\in E^{\text{mult}}} |\{x^{(i)} \mid x^{(i)} \text{ does not cover } e\}|p(e).$$

As above, we obtain a color class which has - interpreted as cover - a better weight than the average of the $x^{(i)}$.
Now, we will state a combinatorial proof for an integral min-max result for the following program pair:

$$\begin{array}{lll} \max & 1^T y \\ \text{s.t.} & Ay \leq f \\ & 0 \leq y \leq p \end{array} \qquad \begin{array}{lll} \min & (f,p)^T(x,w) \\ \text{s.t.} & A^T x + Iw \geq 1, \\ & 0 \leq x, w \end{array}$$

with balanced constraint matrix A. The result is due to Fulkerson, Hoffman, and Oppenheim, who showed it by using linear programming arguments (see [FHO74]). Here, we proceed along the lines of Theorem 3.1: That the maximum is at most as large as the minimum can be shown as in Theorem 3.1. The reverse inequality can be proven by induction over $|V| + \sum_{e\in E} p(e)$.

As above, we can assume that Condition (A) holds, otherwise we reduce the problem and apply induction. Then, the coloring trick presented above gives us an edge e^* which is covered by every optimum d-cover. We reduce the penalty costs of this edge by one and denote this new function by p', the weight functions corresponding to f, p' and 1 by L' and $\tilde{L}'$. By the induction hypothesis, we obtain an optimum relaxed f-matching y' and an optimum d-cover x' with

$$\tilde{L}'(x') = L'(y').$$

Let, moreover, optimum solutions x, y concerning $L, \tilde{L}$ be given.

$$\begin{aligned}
L(y) \quad &\geq \quad L'(y') = \tilde{L}'(x') = \sum_{v \in V} x'(v)f(v) + \sum_{e \in E} \left[1 - \sum_{v: v \in e} x'(v)\right]_+ \cdot p'(e) \\
&= \quad \sum_{v \in V} x'(v)f(v) + \sum_{e \in E} \left[1 - \sum_{v: v \in e} x'(v)\right]_+ \cdot p(e) - \left[1 - \sum_{v: v \in e^*} x'(v)\right]_+ \\
&= \quad \tilde{L}(x') - \left[1 - \sum_{v: v \in e^*} x'(v)\right]_+ \\
&\geq \quad \begin{cases} \tilde{L}(x) & \text{if } e^* \text{ is covered by } x' \\ \tilde{L}(x) + 1 - 1 & \text{if } e^* \text{ is not covered by } x'. \end{cases}
\end{aligned}$$

4 A combinatorial proof of the Ghouila-Houri Theorem

In this section, in order to give a full combinatorial proof of Theorem 3.1, we show how the result of Ghouila-Houri, Theorem 2.1, can be shown in a purely combinatorial way. For this, however, we rely on Seymour's decomposition of totally unimodular matrices [Sey80]. We remark that, while being highly non-trivial, Seymour's proof is purely combinatorial.

4.1 Seymour's decomposition

A corollary of Seymour's decomposition theorem for regular matroids is the following decomposition of totally unimodular matrices [Sey80]. Basically, the result says that a totally unimodular matrix is either one of two small special matrices, M_1 or M_2, a so-called network matrix, or can be decomposed by the operations defined below. Here we stick to the presentation given by Schrijver [Sch86].

The following two matrices are the basic matrices M_1 and M_2.

$$M_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Next, we need to define network matrices. For this, let $D = (V, A)$ be a directed graph and let $T = (V, A')$ be a directed tree on the same vertex set. We now define a matrix $M \in \{-1, 0, 1\}^{A' \times A}$ as follows. Let $a = (u, v) \in A$ and $a' \in A'$.

$$M_{a',a} = \begin{cases} 1 & \text{if the unique } u - v \text{ path in } T \text{ passes through } a' \text{ forwardly,} \\ -1 & \text{if the unique } u - v \text{ path in } T \text{ passes through } a' \text{ backwardly,} \\ 0 & \text{if the unique } u - v \text{ path in } T \text{ does not pass through } a'. \end{cases}$$

A matrix is called a *network matrix* if and only if it can be constructed in the above way.

Finally, the operations to generate a totally unimodular matrix from smaller ones are as follows:

(a) permuting rows and/or columns,

(b) taking the transpose,

(c) multiplying a row or column by -1,

(d) pivoting: replacing the matrix $\begin{pmatrix} \lambda & b \\ a & D \end{pmatrix}$ by the matrix $\begin{pmatrix} -\lambda & \lambda b \\ \lambda a & D - \lambda ab \end{pmatrix}$,

(e) adding an all-zero row or column, or a row or column with exactly one non-zero, either 1 or -1,

(f) repeating a row or column

(g) 1-sum: replacing the matrices A and B by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$,

(h) 2-sum: replacing the matrices $\begin{pmatrix} A & a \end{pmatrix}$ and $\begin{pmatrix} B \\ b \end{pmatrix}$ by $\begin{pmatrix} A & ab \\ 0 & B \end{pmatrix}$,

(i) 3-sum: replacing the matrices $\begin{pmatrix} A & a & a \\ c & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & b \\ d & d & B \end{pmatrix}$ by $\begin{pmatrix} A & ab \\ dc & B \end{pmatrix}$.

Here, A, B, and C are matrices, a and d are column vectors, b and c are row vectors, and $\lambda \in \{-1, 1\}$ is a scalar.

Theorem 4.1 (Seymour [Sey80])**.** *A matrix M is totally unimodular if and only if it arises from the matrices M_1, M_2, and network matrices by the operations (a)–(i) defined above.*

4.2 The proof of Theorem 2.1

In view of the decomposition theorem it suffices to show that there is a vector $v \in \{-1,1\}^n$ such that $Mv \in \{-1,0,1\}$, when M is the totally unimodular matrix at hand. In the following, we say that M satisfies the Ghouila-Houri property if there exists such a vector v.

The proof that M_1, M_2, and their respective transpose matrices admit the Ghouila-Houri property is simple and skipped here. The same goes for the invariance of the Ghouila-Houri property under the operations (a), (c), as well as (e)–(g). Regarding operation (b), taking the transpose, we just show in the coming proofs that indeed every totally unimodular matrix and its transpose enjoy the Ghouila-Houri property.

We now establish the Ghouila-Houri property for network matrices, and their transpose matrices.

Lemma 4.2. *Network matrices as well as their transpose matrices enjoy the Ghouila-Houri property.*

Proof. Let D, T, and M be defined like above.

We first establish the claim for network matrices. For this, we go on by induction on $|V|+|A|$. Let u be a leaf vertex of T, and let e be the arc incident to u in T. Without loss of generality, $e=(u,v)$ for some $v \in V$. Let $a_1,\dots,a_k$ be the arcs incident to u in D. After flipping the sign of the corresponding columns of M, we may assume that all of these arcs are outgoing arcs, so they are of the form $a_i=(u,v_i)$, $i=1,\dots,k$.

If $k=0$, we are done by applying induction on $D-u$ and $T-u$. So, assume for now that $k=1$. This means that the row corresponding to e has only one non-zero entry. Consequently, we may delete this row and proceed. In the graph, this means we apply induction on the graphs obtained from D and T by contracting the edge e to a single vertex. This may create a loop, in case $v_1=v$, but in this case we can just delete the loop without causing trouble.

So, assume $k \geq 2$. Let $\hat{A}=(A \setminus \{a_1,a_2\}) \cup \{a=(v_1,v_2)\}$ and let $\hat{D}=(V,\hat{A})$. Let $\hat{M}$ be the network matrix obtained from $\hat{D}$ and T. By induction, $\hat{M}$ has the Ghouila-Houri property, and we may pick a vector $\hat{v} \in \{-1,0,1\}^{|\hat{A}|}$ with $\hat{M}\hat{v} \in \{-1,0,1\}^{|A'|}$. Without loss of generality, we may assume that $\hat{v}_a=1$.

We define a vector $v \in \{-1,0,1\}^{|A|}$ by putting $v_b=\hat{v}_b$ for all $b \in A\setminus\{a_1,a_2\}$, $v_{a_1}=-1$, and $v_{a_2}=1$. Next we show that the vector v satisfies $Mv \in \{-1,0,1\}^{|A'|}$, as desired. To this end, let $f \in A'$ be arbitrary.

If $M_{f,a_1}=M_{f,a_2}=0$, then also $\hat{M}_{f,a}=0$, and so $(Mv)_f=(\hat{M}\hat{v})_f$ and we are done. Otherwise, if $M_{f,a_1}=1$ but $M_{f,a_2}=0$, we have $\hat{M}_{f,a}=-1$. Thus,

$$(Mv)_f=(Mv)_f+M_{f,a_1}+\hat{M}_{f,a}=(Mv)_f-v_{a_1}\cdot M_{f,a_1}+\hat{v}_a\cdot\hat{M}_{f,a}=(\hat{M}\hat{v})_f,$$

as desired. The cases when $M_{f,a_1}=-1$ but $M_{f,a_2}=0$, or $M_{f,a_1}=0$ but $M_{f,a_2}\neq 0$ are handled analogously. Note, that the cases $M_{f,a_1}=-M_{f,a_2}=\pm 1$ cannot occur. Next we discuss the transpose matrices of network matrices. Pick a bipartition (U,W) of the vertices of T, *i.e.*, a partition $V=U\cup W$ such that every arc of T joins a vertex of U

and a vertex of W. After flipping the sign of some rows of M, we may assume that all arcs of T are directed from U to W.

It remains to define a vector $v \in \{-1,1\}^{A'}$ such that $M^T v \in \{-1,0,1\}^A$. We may just pick the all-ones vector for v, as every path in T is alternating between forward and backward edges. This completes the proof. $\square$

It remains to prove that the Ghouila-Houri property is preserved under the operations (d), (h), and (i). We discuss the pivoting operation, (d), as the other operations are dealt with in a similar fashion.

Recall that we replace the $m \times n$-matrix $M = \begin{pmatrix} \lambda & b \\ a & D \end{pmatrix}$ by the matrix $M' = \begin{pmatrix} -\lambda & \lambda b \\ \lambda a & D - \lambda ab \end{pmatrix}$. After possibly flipping the sign of the first column, we may assume that $\lambda = 1$. Let $v \in \{-1,1\}^n$ be such that $Mv \in \{-1,0,1\}^m$, say $v = (v_1,\dots,v_n)^T$, and let $v' = (v_2,\dots,v_n)$.

If $bv' = 0$, then obviously v does the job, since $(ab)v' = 0$. If $bv' \in \{-2,2\}$, say $bv' = 2$, then $v_1 = -1$. Thus, the vector u with $u_1 = 1$ and $u_i = v_i$, $i = 2,\dots,n$, satisfies $M'u \in \{-1,0,1\}^m$. The case of $bv' = -2$, is dealt with in a similar fashion.

The last case is that of $bv' \in \{-1,1\}$. We may assume that $bv' = 1$, as the case $bv' = -1$ is similar. Consequently, $v_1 = -1$, and so we have $M'v = Mv$, which implies the desired property.

Together with Lemma 4.2, we derive that totally unimodular matrices have the Ghoulia-Houri property.

5 Existence of (g,f)-factors in unimodular hypergraphs

As an application of Theorem 3.1 we provide conditions characterizing the feasibility of

$$g \le Ay \le f, 0 \le y \le c \tag{5}$$

with a totally unimodular binary matrix A. Similar characterizations have been proven by Hoffman [Hof60]. He called his result (proven in [Hof76]) *"the most general theorem of the Hall type"*. His proof highly depends on linear programming methods. In the following, we develop a combinatorial approach. In terms of hypergraphs, we are characterizing the existence of (capacitated) f-,(g,f)-matchings and f-factors.

For unimodular hypergraphs the existence of a c-capacitated (g,f)-factor can all be boiled down to the existence of a perfect f-matching in an auxiliary unimodular hypergraph. Hence, we start by characterizing the existence of perfect f-matchings. Therefore we need the notion of parallelization of a hypergraph:

Definition 5.1. Let $H = (V,E)$ be a hypergraph, $\lambda \ge 0$ an integer, and $v \in V$. *Expanding* a vertex v by λ means replacing v by λ new vertices $v^1,\dots,v^\lambda$, and each hyperedge e which contains v by λ new hyperedges $e^1 = e \setminus \{v\} \cup \{v^1\}$,..., $e^\lambda = e \setminus \{v\} \cup \{v^\lambda\}$. If $\lambda = 0$, we delete v and all hyperedges e containing v. Given $f : V \to \mathbb{Z}_+$

the *parallelization* of H by f is the hypergraph H^f obtained from H by expanding each vertex v by $f(v)$.

It is easy to see that a hypergraph H has a perfect f-matching if and only if its parallelization H^f has a perfect matching. However, if H is unimodular it is possible that H^f is not unimodular, even not balanced. Thus, we cannot apply Hall's condition for balanced hypergraphs (see [CCKV96]). However, we can use Theorem 3.2 to obtain a "defect" version of Kőnig's Theorem for H^f, similar to the one for ordinary 1-matchings in balanced hypergraphs [ST15]. With this "defect" version we prove a direct generalization of the condition in [CCKV96] to perfect f-matchings in unimodular hypergraphs.

In the following, let $\nu_V(H)$ denote the maximum number of vertices covered by a matching of H. Please note that $\nu_V(H^f)$ is equal to the optimum value of

$$\begin{array}{ll} \max & \epsilon^T y \\ \text{s.t.} & Ay \leq f, \\ & 0 \leq z \end{array}$$

with ϵ being the vector of edge cardinalities of the hypergraph H. In other words, $\nu_V(H^f)$ is the maximum edge cardinality sum of an f-matching, i.e., a value of $f(V)$ would correspond to a perfect f-matching. As usual, we define $h(U) := \sum_{v \in U} h(v)$ for functions $h : V \to \mathbb{Z}_+$ and $U \subseteq V$.

Corollary 5.1. *Let $H = (V, E)$ be an unimodular hypergraph, and $f : V \to \mathbb{Z}_+$ a given function. $\nu_V(H^f) \leq f(V) - k$ if and only if there exists $x : V \to \mathbb{Z}_+$ with $\sum_{v \in e} x(v) \geq |e|$ for all $e \in E$, $0 \leq x(v) \leq k + 1$ for all $v \in V$, and $\sum_{v \in V} f(v)x(v) \leq f(V) - k$.*

Proof. "$\Leftarrow$": Let x be as stated in the corollary and $y : E \to \mathbb{Z}$ be an f-matching with $\sum_{e \in E} |e|y(e) = \nu_V(H^f)$. Then

$$\sum_{e \in E} |e|y(e) \leq \sum_{e \in E} \left(\sum_{v \in e} x(v) \right) y(e) = \sum_{v \in V} x(v) \sum_{e : v \in e} y(e) \leq \sum_{v \in V} f(v)x(v) \leq f(V) - k.$$

"$\Rightarrow$": Apply Theorem 3.1 with $d(e) = |e|$ for all $e \in E$, $q \equiv k + 1$, f, and $p \equiv \infty$. Let $y \in \mathbb{Z}^E$ be an optimum relaxed f-matching and $z \in \mathbb{Z}^V$ the corresponding vector of penalties. If $z \equiv 0$, then y is an f-matching and by assumption $\sum_{e \in E} |e|f(e) \leq f(V) - k$.

Otherwise, we have

$$\begin{aligned}
&\sum_{e\in E}|e|y(e)-\sum_{v\in V}z(v)(k+1)\\
&=\sum_{e\in E}|e|y(e)-\sum_{v\in V}\left[\sum_{e\ni v}y(e)-f(v)\right]_+\cdot(k+1)\\
&=\sum_{v\in V}\sum_{e\ni v}y(e)-\sum_{v\in V}\left[\sum_{e\ni v}y(e)-f(v)\right]_+\cdot(k+1)\\
&=\sum_{v:\sum\limits_{e\ni v}y(e)\le f(v)}\sum_{e\ni v}y(e)+\sum_{v:\sum\limits_{e\ni v}y(e)>f(v)}f(v)-\sum_{v:\sum\limits_{e\ni v}y(e)\ge f(v)}\left(\sum_{e\ni v}y(e)-f(v)\right)\cdot k\\
&\le\sum_{v\in V}f(v)-\sum_{v:\sum\limits_{e\ni v}y(e)\ge f(v)}\left(\sum_{e\ni v}y(e)-f(v)\right)\cdot k\\
&\le f(V)-k.
\end{aligned}$$

The last inequality holds because of $z(v)>0$ for some $v\in V$. In both cases, Theorem 3.2 guarantees the existence of a d-cover $x:E\to\mathbb{Z}$ as desired. □

Now, we can proceed as in [ST15].

Theorem 5.2. *Let $H=(V,E)$ be an unimodular hypergraph, and $f:V\to\mathbb{Z}_+$ be a given function. H has a perfect f-matching if and only if for all disjoint subsets $X,Y\subseteq V$ with $f(X)>f(Y)$ there exists a hyperedge $e\in E$ with $|e\cap X|>|e\cap Y|$.*

Proof. First, suppose H has a perfect f-matching y and let $X,Y\subseteq V$ with $f(X)>f(Y)$, then

$$\sum_{e\in E}|e\cap X|y(e)=\sum_{v\in X}\sum_{e:v\in e}y(e)=\sum_{v\in X}f(v)>\sum_{v\in Y}f(v)=\sum_{e\in E}|e\cap Y|y(e).$$

Thus, there exists a hyperedge $e\in E$ with $|e\cap X|>|e\cap Y|$.

Now, suppose H has no perfect f-matching. Then $\nu_V(H^f)\le f(V)-1$, and by Corollary 5.1 there exists $x:V\to\mathbb{Z}$ with $\sum_{v\in e}x(v)\ge|e|$ for all $e\in E$, $0\le x(v)\le 2$ for all $v\in V$, and $\sum_{v\in V}x(v)\le f(V)-1$. Set $X:=\{v\in V\mid x(v)=0\}$ and $Y:=\{v\in V\mid x(v)=2\}$, then

$$2f(Y)+f(V\setminus(X\cup Y))<f(V),$$

which implies $f(Y)<f(X)$. Furthermore, for every $e\in E$ we have

$$2|e\cap Y|+|e\setminus(X\cup Y)|\ge|e|,$$

which shows that $|e\cap Y|\ge|e\cap X|$ for every $e\in E$. □

In the remainder, we reduce the capacitated (g, f)-matching problem to the perfect f-matching problem. This gives the following corollary which, in matrix language, characterizes when system (5) has a solution.

Corollary 5.3. *Let $H = (V, E)$ be an unimodular hypergraph, $g, f : V \to \mathbb{Z}_+$ functions with $g(v) \leq f(v)$ for all $v \in V$, and $c : E \to \mathbb{Z}_+$.*
H has a c-capacitated (g, f)-matching if and only if

$$g(X) - f(Y) \leq \sum_{e \in E : |e \cap X| \geq |e \cap X|} c(e)(|e \cap X| - |e \cap Y|) \tag{6}$$

holds for all disjoint $X, Y \subseteq V$.

Proof. If H has a c-capacitated (g, f)-matching an easy calculation shows that inequality (6) holds for all disjoint $X, Y \subseteq V$.

For the other direction, we reduce the existence of a c-capacitated (g, f)-matching to the existence of a perfect f-matching in an auxiliary hypergraph. For every vertex $v \in V$ let v' be a copy of v, and for every hyperedge $e \in E$ let v_e be a new vertex. We set $V' := \{v' : v \in V\}$ and $V_E := \{v_e : e \in E\}$. Now, we define an auxiliary hypergraph $\tilde{H} := (\tilde{V}, \tilde{E})$ with vertex function $\tilde{f} : \tilde{V} \to \mathbb{Z}_+$ by $\tilde{V} := V \cup V' \cup V_E$, $\tilde{E} := \{e \cup \{v_e\}, \{v_e\} : e \in E\} \cup \{\{v, v'\}, \{v'\} : v \in V\}$, and

$$\tilde{f}(\tilde{v}) := \begin{cases} f(v) & \text{if } \tilde{v} = v \in V, \\ f(v) - g(v) & \text{if } \tilde{v} = v' \in V', \\ c(e) & \text{if } \tilde{v} = v_e \in V_E \end{cases}.$$

We show that $\tilde{H}$ has a perfect $\tilde{f}$-matching if and only if H has a c-capacitated (g, f)-matching. First, let $y : E \to \mathbb{Z}_+$ be a c-capacitated (g, f)-matching of H. We define a function $\tilde{y}$ on $\tilde{E}$ by

$$\tilde{y}(\tilde{e}) := \begin{cases} y(e) & \text{if } \tilde{e} = e \cup \{v_e\}, \\ c(e) - y(e) & \text{if } \tilde{e} = \{v_e\}, \\ f(v) - y(\delta_H(v)) & \text{if } \tilde{e} = \{v, v'\}, \\ y(\delta_H(v)) - g(v) & \text{if } \tilde{e} = \{v'\} \end{cases},$$

where $\delta_H(v) := \{e \in E(H) : v \in e\}$ denotes the set of all hyperedges in H containing vertex v. With this definition $\tilde{y} \geq 0$ holds as y is a c-capacitated (g, f)-matching. It remains to show that $\tilde{y}(\delta_{\tilde{H}}(\tilde{v})) = \tilde{f}(\tilde{v})$ for all $\tilde{v} \in \tilde{V}$. If $v \in V$, then

$$\tilde{y}(\delta_{\tilde{H}}(v)) = y(\delta_H(v)) + \tilde{y}(\{v, v'\}) = y(\delta_H(v)) + f(v) - y(\delta_H(v)) = \tilde{f}(v).$$

For $v' \in V'$, we have

$$\tilde{y}(\delta_{\tilde{H}}(v')) = \tilde{y}(\{v, v'\}) + \tilde{y}(\{v'\}) = f(v) - y(\delta_H(v)) + y(\delta_H(v)) - g(v) = f(v) - g(v) = \tilde{f}(v'),$$

and for $v_e \in E$ we get

$$\tilde{y}(\delta_{\tilde{H}}(v_e)) = \tilde{y}(\{v_e\}) + \tilde{y}(e \cup \{v_e\}) = c(e) - y(e) + y(e) = c(e) = \tilde{f}(v_e).$$

Thus, $\tilde{y}$ is a perfect $\tilde{f}$-matching of $\tilde{H}$.

On the other hand, let a perfect $\tilde{f}$-matching $\tilde{y}$ be given. Set $y(e) := \tilde{y}\left(e \cup \{v_e\}\right)$ for all $e \in E$. Then y is c-capacitated because $y(e) \leq \tilde{y}\left(\{v_e\}\right) + \tilde{y}\left(e \cup \{v_e\}\right) = c(e)$. Furthermore, for every $v \in V$ it holds that

$$y\left(\delta_H(v)\right) = \tilde{y}\left(\delta_{\tilde{H}}(v)\right) - \tilde{y}\left(\{v, v'\}\right) \begin{cases} \leq \tilde{f}(v) = f(v) \\ \geq f(v) - (f(v) - g(v)) = g(v) \end{cases},$$

where we use that $\tilde{y}\left(\{v, v'\}\right) \leq \tilde{y}\left(\delta_{\tilde{H}}(v')\right) = f(v) - g(v)$. Thus, y is a c-capacitated (g, f)-matching of H.

It is easy to show that $\tilde{H}$ is unimodular if H is unimodular. In particular, if $\tilde{H}$ has no perfect $\tilde{f}$-matching there exists disjoint sets $\tilde{X}, \tilde{Y} \subseteq \tilde{V}$ such that

$$\tilde{f}(\tilde{X}) > \tilde{f}(\tilde{Y}) \text{ and} \tag{7}$$

$$|\tilde{e} \cap \tilde{X}| \leq |\tilde{e} \cap \tilde{Y}| \quad \text{for all } \tilde{e} \in \tilde{E}. \tag{8}$$

By inequality (8) applied to the hyperedges of size one, $\tilde{X}$ cannot contain any vertices from V' and V_E, i.e., $\tilde{X} \subseteq V$. Set $X := \tilde{X}$ and $Y := \tilde{Y} \cap V$. Then

$$|e \cap Y| = |\left(e \cup \{v_e\}\right) \cap \tilde{Y}| - |\{v_e\} \cap \tilde{Y}| \geq |e \cap X| - |\{v_e\} \cap \tilde{Y}|$$

for all $e \in E$. This implies that $|e \cap X| - |e \cap Y| \leq 1$ with equality if and only if $v_e \in \tilde{Y}$. Furthermore, inequality (8) for $\tilde{e} = \{v, v'\}$ implies that $v' \in \tilde{Y}$ for all $v \in X$. These observations together with (7) lead to

$$\begin{aligned} f(X) - f(Y) &= \tilde{f}(\tilde{X}) - \tilde{f}(\tilde{Y}) + \sum_{v' \in \tilde{Y} \cap V'} (f(v) - g(v)) + \sum_{v_e \in \tilde{Y} \cap V_E} c(e) \\ &> \sum_{v \in X} (f(v) - g(v)) + \sum_{e \in E : |e \cap X| \geq |e \cap Y|} c(e) \left(|e \cap X| - |e \cap Y|\right). \end{aligned}$$

This shows that X, Y are disjoint subsets of V violating condition (6). □

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