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Abstract

We investigate a special class of quadratic Hamiltonians on $so(4)$ and $so(3,1)$ and describe Hamiltonians that have additional polynomial integrals. One of the main results is a new integrable case with an integral of sixth degree.

Key words: integrable quadratic Hamiltonians, polynomial integrals, classification

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1 Introduction

In this paper we consider the following family of Poisson brackets

$$\{M_i, M_j\} = \varepsilon_{ijk}\, M_k, \qquad \{M_i, \gamma_j\} = \varepsilon_{ijk}\, \gamma_k, \qquad \{\gamma_i, \gamma_j\} = \kappa\, \varepsilon_{ijk}\, M_k. \tag{1.1}$$

Here M_i and γ_i are components of 3-dimensional vectors $\mathbf{M}$ and $\mathbf{\Gamma}$, ε_{ijk} is the totally skew-symmetric tensor, κ is a parameter. It is well-known that any linear Poisson bracket is defined by an appropriate Lie algebra. The cases $\kappa = 0$, $\kappa > 0$ and $\kappa < 0$ correspond to the Lie algebras $e(3)$, $so(4)$ and $so(3,1)$. In this paper we assume $\kappa \neq 0$.

Bracket (1.1) has the two Casimir functions

$$J_1 = (\mathbf{M}, \mathbf{\Gamma}), \qquad J_2 = \kappa|\mathbf{M}|^2 + |\mathbf{\Gamma}|^2,$$

where $(\cdot, \cdot)$ stands for the standard dot product in $\mathbb{R}^3$. Hence, for the Liouville integrability of the equations of motion only one additional integral that is functionally independent of the Hamiltonian and the Casimir functions is necessary.

The simplest nontrivial class of quadratic homogeneous Hamiltonians of the form

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{M},\, B\mathbf{\Gamma}) + (\mathbf{\Gamma},\, C\mathbf{\Gamma}), \tag{1.2}$$

where A, B and C are constant 3×3-matrices, has many important applications in rigid body dynamics. We call Hamiltonians (1.2) having an additional polynomial integral *integrable.* We say that (1.2) possesses an additional integral of degree k if there is no non-trivial integral of degree less than k. In this paper all coefficients of both the Hamiltonian and the additional integral are supposed to be real constants.

The case where the Hamiltonian (1.2) has a linear additional integral of motion has been investigated by Poincare [1].

There are two classical integrable cases, one found by Frahm-Schottky and one by Steklov, where the additional integral of motion is of second degree. It was proved in [2] that any Hamiltonian (1.2) possessing an additional second degree integral is equivalent to one of these two cases.

In 1986 Adler-van Moerbeke [3] and Reyman-Semenov-tian Shansky [4] independently found a Hamiltonian of the form (1.2) with a fourth degree additional integral.

For the Frahm-Schottky, Steklov, and Adler-van Moerbeke-Reyman-Semenov-tian Shansky cases all matrices A, B and C are diagonal. This special subclass of "diagonal" Hamiltonians (1.2) was investigated by many authors but no new integrable cases were found. Probably only these three integrable cases exist among "diagonal" Hamiltonians.

In the paper [5] the first integrable "non-diagonal" Hamiltonian (1.2) with $\kappa = 0$ (i.e. a Hamiltonian of the Kirchhoff type describing the motion of a rigid body in ideal fluid) was found. This Hamiltonian has a fourth degree additional integral. A generalization of this Hamiltonian to the case $\kappa \neq 0$ was reported in [6]. In the paper [8] the Hamiltonian has been rewritten in the form

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \tag{1.3}$$

where A is a constant symmetric matrix, $\mathbf{b} \neq 0$ is a constant vector and $\times$ stands for the cross product. It turns out that this class is very rich in integrable cases. In the paper [9] all Hamiltonians (1.3) with a quartic additional integral were described. Moreover, it was mentioned in [10, 9, 11] that the general Sklyanin brackets [12] for the XXX-magnetic model lead to some integrable Hamiltonians of the same kind.

The goal of our paper is a systematic investigation of integrable real Hamiltonians (1.3) and their inhomogeneous generalizations

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{b},\, \mathbf{M} \times \mathbf{\Gamma}) + (\mathbf{k},\, \mathbf{M}) + (\mathbf{n},\, \mathbf{\Gamma}), \tag{1.4}$$

where $\mathbf{k}$ and $\mathbf{n}$ are constant vectors.

It is remarkable that all known Hamiltonians (1.3) possessing additional polynomial integrals have also some **linear** partial integrals

$$P = (\mathbf{u},\, \mathbf{M}) + (\mathbf{v},\, \boldsymbol{\Gamma}), \tag{1.5}$$

where $\mathbf{u}$ and $\mathbf{v}$ are constant vectors. Namely, for some constant vectors $\mathbf{p}$ and $\mathbf{q}$ the following relation

$$\{H,\, P\} = \Big[(\mathbf{p},\, \mathbf{M}) + (\mathbf{q},\, \boldsymbol{\Gamma})\Big] \cdot P \tag{1.6}$$

holds. This implies that the corresponding equations of motion preserve the constraint $P = 0$. These linear partial integrals turn out to be factors of the polynomial integrals for all homogeneous Hamiltonians considered in this paper.

Because of this reason we start our study in subsection 2.1 devoted to linear partial integrals. An interesting subclass of one-parametric families of Hamiltonians (1.3) arises there. In the next subsection 2.2 we apply the Kowalewski-Lyapunov test, which is well-known in the Painleve analysis, to find all possibly integrable families from this subclass. Because of a continuous parameter in the Hamiltonian this test becomes extremely efficient. The main result of this consideration is a new integrable Hamiltonian on $so(3,1)$ with an additional sixth degree integral.

In section 3 we are dealing with inhomogeneous Hamiltonians (1.4). As we claim in section 4, there are no Hamiltonians of the form (1.3) having additional integrals of degrees from 1 to 8 other than examples described in section 2. We have also verified that we found in section 3 *all* Hamiltonians (1.4) having additional integrals of degrees from 1 to 6.

All computations for the paper have been done by the specialized computer algebra package CRACK. It is designed to solve overdetermined polynomial differential and algebraic systems with an emphasis on extremely large problems. A major concern for any operations performed by the program is the complexity of resulting expressions. In [14] an overview of the package and examples for its use in the classification of integrable systems are given.

In this paper we present integrable Hamiltonians and the corresponding additional integrals in a general vector form, which is invariant with respect to the orthogonal transformations. For computations with vectorial expressions extra code was written.

2 Homogeneous integrable cases

2.1 Linear partial integrals

In this section we describe all Hamiltonians of the form (1.3) having linear partial integrals (1.5). Relation (1.6) is equivalent to a system of bi-linear algebraic equations for coefficients of the Hamiltonian and components of vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{p}$ and $\mathbf{q}$. Below we present the result of our investigation of this system.

One can check that there are two different possibilities, either case 1: $\mathbf{v} = \mathbf{q} = 0$, or case 2: $\mathbf{q} = \mathbf{b}$ from formula (1.3).

Case 1: Calculations show that any Hamiltonian having a linear partial integral with $\mathbf{v} = \mathbf{q} = 0$ belongs to a class of "vectorial" Hamiltonians of the form

$$H = c_1\Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 + c_2\Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big), \tag{2.7}$$

where $\mathbf{b}$ and $\mathbf{a}$ are constant vectors, c_i are constant scalars. In this case $\mathbf{u} = \mathbf{b}$ and $\mathbf{p} = c_2\,\mathbf{a}\times\mathbf{b}$. In other words, for Hamiltonian (2.7) we have

$$\{H,\, (\mathbf{b},\, \mathbf{M})\} = c_2(\mathbf{a}\times\mathbf{b},\, \mathbf{M})\cdot(\mathbf{b},\, \mathbf{M}). \tag{2.8}$$

In the next subsection we will present four different integrable Hamiltonians of the form (2.7) that possess additional integrals of degrees 1, 3, 4 and 6.

Case 2: In the case $\mathbf{q} = \mathbf{b}$ the following conditions

$$(\mathbf{b},\, \mathbf{v}) = (\mathbf{b},\, \mathbf{u}) = 0, \qquad \mathbf{p} = \xi\mathbf{b},$$

where ξ is a scalar, have to be fulfilled.

Case 2a: If the vectors $\mathbf{v}$ and $\mathbf{u}$ are not parallel, then without loss of generality we may assume that $\mathbf{b} = \mathbf{u}\times\mathbf{v}$. It turns out that in this case $\xi = 0$ and $\mathbf{u}$ and $\mathbf{v}$ are arbitrary vectors such that $(\mathbf{u}, \mathbf{v}) = 0$. The Hamiltonian is given by

$$H = \frac{1}{2}|\mathbf{u}|^2|\mathbf{M}|^2 + \frac{1}{2}\Big(\mathbf{u},\, \mathbf{M}\Big)^2 - \frac{\kappa}{2}\Big(\mathbf{v},\, \mathbf{M}\Big)^2 + \Big(\mathbf{u}\times\mathbf{v},\, \mathbf{M}\times\mathbf{\Gamma}\Big). \tag{2.9}$$

The partial integral $P = (\mathbf{u},\, \mathbf{M}) + (\mathbf{v},\, \mathbf{\Gamma})$ satisfies the relation

$$\{H,\, P\} = (\mathbf{u}\times\mathbf{v},\, \mathbf{\Gamma})\cdot P.$$

This Hamiltonian has the following additional integral of fourth degree:

$$I = \Big(P|\mathbf{M}|^2 - 2(\mathbf{v},\, \mathbf{M})(\mathbf{M},\, \mathbf{\Gamma})\Big)\cdot P. \tag{2.10}$$

This integrable case was found in a different non-vector form in [5, 6]. A Lax operator is presented in [7].

Case 2b: The other possibility is that the vectors $\mathbf{v}$ and $\mathbf{u}$ are parallel:

$$(\mathbf{b},\, \mathbf{v}) = 0, \qquad \mathbf{p} = \xi\mathbf{b}, \qquad \mathbf{u} = \eta\mathbf{v}.$$

It turns out that in this case $\xi = \eta$ and $\kappa = \eta^2$. We see that a real linear integral exists only for the $so(4)$-version $\kappa > 0$ of bracket (1.1). The Hamiltonian is given by the formula

$$H = -2\eta\Big(\mathbf{v},\, \mathbf{M}\Big)\Big(\mathbf{z},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big), \qquad \mathbf{b} = \mathbf{v}\times\mathbf{z} \tag{2.11}$$

where $\mathbf{v}$ and $\mathbf{z}$ are arbitrary constant vectors. The partial integral $P = (\mathbf{v},\, \eta\mathbf{M} + \mathbf{\Gamma})$ satisfies

$$\{H,\, P\} = (\mathbf{b},\, \eta\mathbf{M} + \mathbf{\Gamma}) \cdot P.$$

This Hamiltonian has the following additional integral of fourth degree (cf. [12, 10]):

$$I = \Big(\mathbf{z},\, (\eta\mathbf{M} - \mathbf{\Gamma})|\mathbf{M}|^2 + 2\mathbf{M}\,(\mathbf{M},\, \mathbf{\Gamma})\Big) \cdot P. \tag{2.12}$$

The eigenvalues α_i of the matrix A from (1.2) satisfy the following relations

$$\alpha_3 = 0, \qquad \alpha_1\alpha_2 = -\eta^2|\mathbf{b}|^2.$$

Diagonalizing the matrix A, we get the possible canonical form of the Hamiltonian (2.11)

$$H = \eta(cM_1^2 - \frac{1}{c}M_2^2) + M_1\gamma_2 - M_2\gamma_1.$$

Although it is real only for the $so(4)$-bracket, after the renormalization $c\eta \to c$ of the arbitrary parameter c we get the Hamiltonian (see [9])

$$H = cM_1^2 - \frac{\kappa}{c}M_2^2 + M_1\gamma_2 - M_2\gamma_1,$$

which is real for any bracket (1.1). In particular, if $\kappa = 0$, we have a new integrable case on $e(3)$ with a fourth degree integral.

2.2 Kowalewski-Lyapunov test and a new integrable case.

In this section we show that the class of Hamiltonians (2.7) contains a number of integrable cases. The first example of this kind was found in [9]:

Example 1. Consider the $so(3,1)$-version $\kappa < 0$ of bracket (1.1). The Hamiltonian

$$H = \Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 - \Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big), \tag{2.13}$$

where the vector $\mathbf{b}$ is arbitrary and the length of the vector $\mathbf{a} = (a_1, a_2, a_3)$ is related to the Poisson bracket parameter κ by

$$a_1^2 + a_2^2 + a_3^2 = -\kappa, \tag{2.14}$$

possesses the additional quartic integral

$$I = \Big(\mathbf{b},\, \mathbf{M}\Big)^2\Big[2\Big(\mathbf{a},\, \mathbf{M}\times\mathbf{\Gamma}\Big) - \Big(\mathbf{a},\, \mathbf{M}\Big)^2 - \kappa|\mathbf{M}|^2 + |\mathbf{\Gamma}|^2\Big].$$

Recently in the paper [11] the following integrable case has been found:

Example 2. The Hamiltonian

$$H = \Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 - 2\Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big), \tag{2.15}$$

has under condition (2.14) the additional cubic integral

$$I = \Big(\mathbf{b},\, \mathbf{M}\Big)\Big[2\,\Big(\mathbf{a},\, \mathbf{M}\times\mathbf{\Gamma}\Big) - \kappa|\mathbf{M}|^2 + |\mathbf{\Gamma}|^2\Big].$$

In the two examples mentioned above, constraint (2.14) is necessary for integrability. We are going to find all integrable Hamiltonians similar to (2.13) and (2.15). To do that we apply the Kowalewski-Lyapunov test to the class of Hamiltonians (2.7) assuming that the additional condition (2.14) is valid.

Suppose we have a dynamical system

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X}), \qquad \mathbf{X} = (x_1, \ldots, x_N), \qquad \mathbf{F} = (f_1, \ldots f_N) \tag{2.16}$$

where f_i are homogeneous quadratic polynomials of $\mathbf{X}$. Solutions of the form

$$\mathbf{X}_0 = \frac{1}{t}\mathbf{K} \tag{2.17}$$

for system (2.16) with $\mathbf{K}$ being a constant vector are called *Kowalewski solutions*. Substituting (2.17) into (2.16), one obtains a system of algebraic equations for possible vectors $\mathbf{K}$.

The linearization $\mathbf{X} = \mathbf{X_0} + \varepsilon\mathbf{\Psi}$ of system (2.16) on a Kowalewski solution $\mathbf{X_0}$ obeys

$$\frac{d\mathbf{\Psi}}{dt} = \frac{1}{t}S(\mathbf{\Psi}), \tag{2.18}$$

where S is a constant $N \times N$-matrix depending on the Kowalewski solution.

Solutions of (2.18) have the form $\mathbf{\Psi} = \mathbf{s}\,t^{-k}$, where k is an eigenvalue and $\mathbf{s}$ is an eigenvector of the matrix S. The number $1-k$ is called *Kowalewski exponent*.

According to the Kowalewski-Lyapunov test, system (2.16) is "integrable" if for any Kowalewski solution all corresponding Kowalewski exponents belong to an *a priori* fixed number set $\mathcal{A}$. The structure of $\mathcal{A}$ is closely related to analytic properties of general solution for (2.16). The usual choice $\mathcal{A} = \mathbb{Z}$ is associated with the requirement that the general solution should be single-valued. The latter is a standard assumption for the Painleve analysis. The most general version $\mathcal{A} = \mathbb{Q}$ is associated with general solutions having algebraic branch points. But the main property for us is that $\mathcal{A}$ cannot be too wide. In particular, it cannot contain any open subset of $\mathbb{C}$ or $\mathbb{R}$ Therefore for any one-parameter family of integrable (in the Kowalewski-Lyapunov sense) dynamical systems (2.16) the Kowalewski exponents must not depend on the parameter. This gives us strong necessary integrability conditions for one-parametric families of homogeneous quadratic dynamical systems.

Given c_1 and c_2, the Hamiltonian (2.7) with (2.14) depends on one essential continuous parameter. Indeed, the length of $\mathbf{a}$ is fixed by (2.14), the length of $\mathbf{b}$ can be normalized by the scaling of the Hamiltonian. The transformations

$$\mathbf{M} \to T\mathbf{M}, \qquad \mathbf{\Gamma} \to T\mathbf{\Gamma} \tag{2.19}$$

for any constant orthogonal matrix T preserve brackets (1.1) and the form of Hamiltonian (2.7). Two vectors of a fixed length have only one invariant (the angle between them) with respect to orthogonal transformation (2.19).

We want to find all pairs c_1 and c_2 in (2.7) such that the Hamiltonian is integrable for any value of the angle.

Using transformation (2.19), we may reduce $\mathbf{b}$ and $\mathbf{a}$ to

$$\mathbf{b} = (0, 0, 1), \qquad \mathbf{a} = (a_1, 0, a_3). \tag{2.20}$$

Taking into account the constraint (2.14), we see that now a_3 remains to be the only free parameter in (2.7). For a generic Hamiltonian of this kind the Kowalewski exponents for the equations of motion depend continuously on a_3. For an integrable Hamiltonian these exponents must not depend on the parameter at all.

Theorem 1. Suppose all Kowalewski exponents for Hamiltonian (2.7) with (2.14), (2.20) do not depend on a_3; then the pair of constants c_1, c_2 belongs (up to the transformation $c_1 \to -c_1, \quad c_2 \to -c_2$, which corresponds to $\mathbf{a} \to -\mathbf{a}$) to the following list

a) c_1 - arbitrary, $\quad c_2 = 0$;
b) $c_1 = 1$, $\quad c_2 = -2$;
c) $c_1 = 1$, $\quad c_2 = -1$;
d) $c_1 = 1$, $\quad c_2 = -\frac{1}{2}$;
e) $c_1 = 1$, $\quad c_2 = 1$.

To prove this statement, we have to investigate the Kowalewski exponents on all solutions of the form

$$M_i = \frac{m_i}{t}, \qquad \gamma_i = \frac{g_i}{t} \tag{2.21}$$

for the equations of motion defined by Hamiltonian (2.7), (2.14). To get the list of Theorem 1, it turns out to be enough to investigate the Kowalewski exponents on special solutions (2.21), which satisfy $g_3 = 1$. There exist two classes of such solutions. For the first class we have $c_2(a_1 m_2 - a_2 m_1) = 1$. The second class is defined by $m_3 = 0$. For solutions of the first class, besides the cases of Theorem 1, the only extra surviving possibility is $c_1 = 1/2$, $c_2 = -1$. But this case does not pass the Kowalewski-Lyapunov test on the solutions of the second class. For each pair (c_1, c_2) of the remaining list a)-e) of Theorem 1 we verify that for all Kowalewski solutions the Kowalewski exponents do not depend on a_3. All computations are straightforward. A technical problem is that the explicit form of solutions of the first class involves radicals. To avoid this difficulty, we used calculations based on the Groebner basis technique.

Comments. The Hamiltonian a) belongs to the family

$$H = c_1 |\mathbf{b}|^2 \, |\mathbf{M}|^2 + c_2 \Big(\mathbf{b},\, \mathbf{M}\Big)^2 + \Big(\mathbf{b},\, \mathbf{M} \times \mathbf{\Gamma}\Big), \tag{2.22}$$

which possesses (without restriction (2.14)) the linear integral of motion $I = (\mathbf{b},\, \mathbf{M})$.

The Hamiltonians b) and c) are described in Examples 2 and 1, correspondingly. The possibility d) leads to the following new integrable case with an additional integral of sixth degree:

Example 3. The Hamiltonian

$$H = \Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 - \frac{1}{2}\Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M}\times\mathbf{\Gamma}\Big), \tag{2.23}$$

under condition (2.14) has the additional sixth degree integral

$$I = \Big(\mathbf{b},\mathbf{M}\Big)^2\Big[\Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times\mathbf{a}\Big)^2\mathbf{M}^2 + 2\Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times\mathbf{a}\Big)\Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times(\mathbf{M}\times\mathbf{\Gamma})\Big)$$
$$-\mathbf{\Gamma}^2\Big(\mathbf{M},\mathbf{b}\times\mathbf{a}\Big)^2 - \Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times\mathbf{\Gamma}\Big)^2 - \mathbf{M}^2\mathbf{\Gamma}^2\Big(\mathbf{b},\mathbf{a}\Big)^2 - \kappa\mathbf{M}^2\mathbf{\Gamma}^2\mathbf{b}^2\Big].$$

Possibly this integrable case is related to the simple Lie algebra $g(2)$.

In [13] a generalization of the general scheme by Sklyanin [12] has been proposed. In [11] the authors found a separation of variables for Hamiltonian (2.15) from Example 2 in the framework of this approach. Probably a separation of variables for the Hamiltonians from Examples 1 and 3 could be found after some development of these ideas.

All Kowalewski exponents for cases a)-c) are integers. In case d) there are two solutions of the class 2. For these solutions we have

$$\det(S - k\,\mathrm{Id}) = (k-1)^3(k+2)(k-\frac{1}{2})(k+\frac{3}{2})$$

and

$$\det(S - k\,\mathrm{Id}) = (k-1)^3(k+2)(k+\frac{1}{2})^2.$$

In other words, some of the Kowalewski exponents are half-integers.

Case e) is a mysterious one. We have verified that the Hamiltonian has no polynomial additional integrals of degree not greater than 8. On the other hand, on all Kowalewski solutions all Kowalewski exponents are integers. It would be interesting to verify whether the equations of motion in the case e) satisfy the standard Painleve test.

3 Inhomogeneous integrable Hamiltonians.

3.1 Admissible linear terms for integrable homogeneous Hamiltonians

In this subsection, for integrable homogeneous Hamiltonians H of the form (2.22), (2.15), (2.13), (2.11), (2.9) or (2.23), we find possible linear terms

$$T = (\mathbf{k},\, \mathbf{M}) + (\mathbf{n},\, \mathbf{\Gamma}), \tag{3.24}$$

where $\mathbf{k}$ and $\mathbf{n}$ are constant vectors, such that the Hamiltonian

$$\tilde{H} = H + T$$

has an additional integral of the same degree as H.

Proposition 1. The following linear terms are admissible (in the above sense):

1) for Hamiltonian (2.22) $\quad T = p_1(\mathbf{b},\, \mathbf{M}) + p_2(\mathbf{b},\, \mathbf{\Gamma})$;

2) for Hamiltonian (2.15) with (2.14) $\quad T = (\mathbf{k},\, \mathbf{M}) + p_1(\mathbf{b},\, \mathbf{\Gamma})$;

3) for Hamiltonian (2.13) with (2.14) $\quad T = (p_1\mathbf{a} + p_2\mathbf{a} \times \mathbf{b},\, \mathbf{M}) + p_3(\mathbf{b},\, \mathbf{\Gamma})$;

4) for Hamiltonian (2.11) $\quad T = (\mathbf{k},\, \mathbf{M}) + p_1(\mathbf{v} \times \mathbf{z},\, \mathbf{\Gamma})$;

5) for Hamiltonian (2.9) with $(\mathbf{u}, \mathbf{v}) = 0$ $\quad T = p_1(\mathbf{u},\, \mathbf{M}) + p_2(\mathbf{u} \times \mathbf{v},\, \mathbf{\Gamma})$;

6) for Hamiltonian (2.23) with (2.14) $\quad T = p_1(\mathbf{a} \times \mathbf{b},\, \mathbf{M})$,

where $\mathbf{k}$ is an arbitrary vector and p_1, p_2, p_3 are arbitrary constants.

We present the explicit form of the additional integrals for the non-homogeneous Hamiltonians $\tilde{H}$ of Proposition 1 in appendix A.

3.2 A deformation of the Poincare model

The formula (2.22) describes Hamiltonians (1.3) that have linear additional integrals. We shall call (2.22) the Poincare model.

For the special pair $c_1 = 1, c_2 = -1/2$ the Poincare model is superintegrable. It means that besides the linear integral the Hamiltonian has a polynomial integral of degree higher than 1.

Proposition 2. The Hamiltonian

$$H_{\text{hom}} = |\mathbf{b}|^2\, |\mathbf{M}|^2 - \frac{1}{2}\Big(\mathbf{b},\, \mathbf{M}\Big)^2 + \Big(\mathbf{b},\, \mathbf{M} \times \mathbf{\Gamma}\Big) \tag{3.25}$$

under condition $|\mathbf{b}|^2 = -\kappa$ has the following additional integral of degree 4, functionally independent of H, the Casimirs and the linear integral $(\mathbf{b},\, \mathbf{M})$:

$$\begin{aligned} I \;=\; & (\mathbf{k},\, \mathbf{M})\,\big[(\mathbf{k},\, \mathbf{M})|\mathbf{b}|^2 - 2(\mathbf{k},\, \mathbf{b})(\mathbf{b},\, \mathbf{M})\big] \cdot \big[|\mathbf{\Gamma}|^2 + |\mathbf{b}|^2|\mathbf{M}|^2\big] \\ & + |\mathbf{M}|^2(\mathbf{b},\, \mathbf{M})^2\,\big[(\mathbf{k},\, \mathbf{b})^2 + |\mathbf{k}|^2|\mathbf{b}|^2\big] - \Big(\mathbf{k} \times \mathbf{b},\, \mathbf{M} \times \mathbf{\Gamma}\Big)^2 \\ & + 2\Big(\mathbf{k} \times \mathbf{b},\, \mathbf{b} \times \mathbf{M}\Big) \cdot \big[|\mathbf{M}|^2(\mathbf{k}, \mathbf{b} \times \mathbf{\Gamma}) - (\mathbf{M},\, \mathbf{\Gamma})(\mathbf{k}, \mathbf{b} \times \mathbf{M})\big]\,, \end{aligned}$$

where $\mathbf{k}$ is an arbitrary constant vector.

It follows from the Jacobi identity that $I_1 = \{(\mathbf{b}, \mathbf{M}),\, I\}$ is a first integral as well. It turns out that $I_1 \neq 0$, which means that the Poisson subalgebra of polynomial integrals for H_{hom} is non-commutative. For any X let us denote the Poisson bracket $\{(\mathbf{b}, \mathbf{M}),\, X\}$ by X'. One can check that the integral I satisfies the relation $I''' + 4|\mathbf{b}|^2 I' = 0$.

The derivation $X \to X'$ can be regarded as a linear operator on the finite dimensional vector space S_n of all homogeneous polynomials of degree n depending on components of $\mathbf{M}$ and $\mathbf{\Gamma}$. The operator spectrum is $\mu_k = ik|\mathbf{b}|$, $1 \leq k \leq n$. Probably this operator plays an important role in the theory of the vector Hamiltonians (2.7). Many terms in the integrable Hamiltonians from this class admit simple descriptions in terms of this operator. For example, the linear polynomial $(\mathbf{k}, \mathbf{b} \times \mathbf{M})$, where $\mathbf{k}$ is an arbitrary constant vector, is the general linear solution of the equation $X'' + |\mathbf{b}|^2 X = 0$. This fact and formula (2.8) imply that the arbitrary Hamiltonian (2.7) satisfies the equation $X''' + |\mathbf{b}|^2 X' = 0$. The additional integral from Example 1 satisfies the same equation and so on.

The Hamiltonian (3.25) admits the following inhomogeneous integrable extension:

$$\tilde{H} = H_{\text{hom}} + (\mathbf{k} \times \mathbf{b},\, \mathbf{M}) + p_1(\mathbf{b},\, \mathbf{\Gamma}), \tag{3.26}$$

where $\mathbf{k}$ is an arbitrary constant vector, p_1 is an arbitrary constant. Hamiltonian (3.26) under condition (2.14) has an additional integral of degree 4, given in the appendix A.

4 Classification results

It is very likely that all integrable Hamiltonians of the form (1.3) and (1.4) are exhausted by the examples presented in sections 2 and 3.

Theorem 2. Suppose a Hamiltonian of the form (1.3) with real coefficients has an additional polynomial integral of degree from 1 to 8; then the Hamiltonian belongs to the six families (2.22), (2.15), (2.13), (2.11), (2.9) or (2.23).

Scheme of the proof. Using transformations (2.19), one can reduce any (real) Hamiltonian (1.3) to

$$H = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + a_4 M_1 M_3 + a_5 M_2 M_3 + M_1 \gamma_2 - M_2 \gamma_1. \tag{4.27}$$

In this canonical form the vector $\mathbf{b}$ is normalized to $(0, 0, 1)$. The alternative idea of bringing the matrix A to the diagonal form lead to overwhelming computational complexity.

Without loss of generality we may assume that the additional polynomial integral is homogeneous. Given the degree m of the additional integral I, we form the general homogeneous m-th degree polynomial of six variables M_i, γ_i with undetermined coefficients. The condition $\{I, H\} = 0$ gives rise to a bi-linear system of algebraic equations for both coefficients of H and I. Of course, this system can be solved "by hand" only for small m. If $m = 6$ the algebraic system contains 791 bi-linear equations for 458 unknown coefficients. This system can not be currently solved by standard computer algebra systems. Also all attempts by the authors to use the two best known packages specialized in the solution of polynomial algebraic systems failed.

The computation was performed using the computer algebra package CRACK. For $m < 6$ the calculations were performed automatically and for $m \geq 6$ with manual interaction.

Following the same line we have obtained

Theorem 3. Suppose a Hamiltonian of the form (1.4) with real coefficients has an additional polynomial integral of degree from 1 to 6; then the Hamiltonian belongs to the seven families described in Propositions 1 and 2.

5 Conclusion: unsolved problems

This paper as well as [9] belongs mostly to so called "experimental" mathematical physics. The result of the experiment is a new interesting class (1.4) of quadratic Hamiltonians. This class contains several new integrable cases. Theorems 1-3 give reasons to believe that we found all real integrable Hamiltonians of the form (1.4). However, this should be proved. To do that, one can apply the Painleve approach or methods developed in [15].

The separation of variables for several models from our paper is also an open problem.

Besides Hamiltonians listed in Theorem 3, there exist integrable Hamiltonians of the form (1.4) with complex coefficients. For example, the Hamiltonian (2.7), (2.14) with $c_1 = 1, c_2 = -\frac{2}{3}$ has an additional integral of sixth degree under condition $|\mathbf{b} \times \mathbf{a}| = 0$. To find the complex Hamiltonians, one should consider two different normalizations of the vector $\mathbf{b}$. The first is $\mathbf{b} = (0, 0, b_3)$, which corresponds to the possibility $|\mathbf{b}| \neq 0$ and gives rise to the normal form (4.27). The second normalization $\mathbf{b} = (0, b_2, i\, b_2)$ corresponds to $|\mathbf{b}| = 0$. In the case of fourth degree additional integrals all complex integrable Hamiltonians have been found in [9].

Probably all Hamiltonians from our paper have their quantum counterparts (for the case of fourth degree integral see [9]). It would be interesting to find the corresponding quantum Hamiltonians and integrals as well as the quantum separation of variables.

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Appendix A

Here we present an explicit form of additional integrals for inhomogeneous Hamiltonians from Propositions 1 and 2. Each integral I of degree m is a sum $I = \sum_{i=1}^{m} I_i$, where I_i is a polynomial of degree i

homogeneous in $\mathbf{M}, \mathbf{\Gamma}$.

Case 1 of Proposition 1. $I = I_1 = (\mathbf{b}, \mathbf{M})$.

Case 2 of Proposition 1. I_3 is presented in Example 2,

$$
\begin{aligned}
I_2 &= 2p_1(\mathbf{b},\mathbf{M})(\mathbf{a},\mathbf{\Gamma}) - (\mathbf{k},\mathbf{a})|\mathbf{M}|^2 - (\mathbf{k},\mathbf{M}\times\mathbf{\Gamma}), \\
I_1 &= -\kappa p_1^2(\mathbf{b},\mathbf{M}) - p_1(\mathbf{k},\mathbf{\Gamma}).
\end{aligned}
$$

Case 3 of Proposition 1. I_4 is presented in Example 1,

$$
\begin{aligned}
I_3 &= 2(\mathbf{b},\mathbf{M})\ \Big\{p_1\Big[(\mathbf{a},\mathbf{M})^2 + \kappa|\mathbf{M}|^2 - |\mathbf{\Gamma}|^2 - 2(\mathbf{a},\mathbf{M}\times\mathbf{\Gamma})\Big] \\
&\quad + p_2\Big[(\mathbf{b}\times\mathbf{a},\ \mathbf{M}\times\mathbf{\Gamma}) + (\mathbf{a},\mathbf{M})(\mathbf{a},\mathbf{b}\times\mathbf{M})\Big] + p_3(\mathbf{b},\mathbf{M})(\mathbf{a},\mathbf{\Gamma})\Big\}, \\
I_2 &= p_1^2\Big[2|\mathbf{\Gamma}|^2 - (\mathbf{a},\mathbf{M})^2 + 2(\mathbf{a},\mathbf{M}\times\mathbf{\Gamma})\Big] - 2p_2p_3(\mathbf{b},\mathbf{M})(\mathbf{a},\mathbf{b}\times\mathbf{\Gamma}) \\
&\quad + p_2^2\Big[|\mathbf{b}|^2(\mathbf{a},\mathbf{M})^2 - \kappa(\mathbf{b},\mathbf{M})^2 - 2(\mathbf{a},\mathbf{b})(\mathbf{a},\mathbf{M})(\mathbf{b},\mathbf{M})\Big] - 4p_1p_3(\mathbf{b},\mathbf{M})(\mathbf{a},\mathbf{\Gamma}) \\
&\quad - p_3^2\kappa(\mathbf{b},\mathbf{M})^2 + 2p_1p_2\Big[(\mathbf{a}\times\mathbf{b},\ \mathbf{M}\times\mathbf{\Gamma}) - (\mathbf{a},\mathbf{M})(\mathbf{a},\mathbf{b}\times\mathbf{M})\Big] \\
I_1 &= 2p_1p_3\Big[p_1(\mathbf{a},\mathbf{\Gamma}) + p_2(\mathbf{a},\mathbf{b}\times\mathbf{\Gamma}) + p_3\kappa(\mathbf{b},\mathbf{M})\Big].
\end{aligned}
$$

Case 4 of Proposition 1. $I_4 = \eta|\mathbf{a}\times\mathbf{b}|^2 I$ where I is given by (2.12),

$$
\begin{aligned}
I_3 &= p_1\eta\Big((\mathbf{v},\ \mathbf{z})^2 - |\mathbf{v}|^2|\mathbf{z}|^2\Big)\cdot\Big[\eta|\mathbf{M}|^2(\mathbf{v},\mathbf{z}\times\mathbf{\Gamma}) - 2(\mathbf{v},\ \mathbf{\Gamma})(\mathbf{z},\mathbf{M}\times\mathbf{\Gamma}) + |\mathbf{\Gamma}|^2(\mathbf{v},\mathbf{z}\times\mathbf{M})\Big] \\
&\quad + \eta^2|\mathbf{M}|^2\Big[(\mathbf{z},\ \mathbf{M})(\mathbf{v}\times\mathbf{z},\ \mathbf{k}\times\mathbf{v}) - (\mathbf{v},\ \mathbf{M})(\mathbf{v}\times\mathbf{z},\ \mathbf{k}\times\mathbf{z})\Big] \\
&\quad + 2\eta\Big[(\mathbf{v},\ \mathbf{M})(\mathbf{z},\ \mathbf{k})(\mathbf{v}\times\mathbf{z},\ \mathbf{\Gamma}\times\mathbf{M}) + (\mathbf{v},\ \mathbf{M})(\mathbf{k},\ \mathbf{M})(\mathbf{v}\times\mathbf{z},\ \mathbf{z}\times\mathbf{\Gamma}) \\
&\quad - (\mathbf{v},\ \mathbf{M})(\mathbf{k},\ \mathbf{\Gamma})(\mathbf{v}\times\mathbf{z},\ \mathbf{z}\times\mathbf{M}) \\
&\quad - \frac{1}{2}(\mathbf{k},\ \mathbf{\Gamma})|\mathbf{M}|^2|\mathbf{v}\times\mathbf{z}|^2 - (\mathbf{z}\times\mathbf{M},\ \mathbf{M}\times\mathbf{\Gamma})(\mathbf{v}\times\mathbf{z},\ \mathbf{v}\times\mathbf{k})\Big] \\
&\quad - (\mathbf{v},\mathbf{z}\times\mathbf{k})(\mathbf{v},\mathbf{z}\times\mathbf{M})|\mathbf{\Gamma}|^2, \\
I_2 &= p_1^2\eta|\mathbf{v}\times\mathbf{z}|^2\Big[\eta^2(\mathbf{v}\times\mathbf{M},\ \mathbf{z}\times\mathbf{M}) + (\mathbf{v},\ \mathbf{\Gamma})(\mathbf{z},\ \mathbf{\Gamma})\Big] - p_1\eta\Big[\Big((\mathbf{v},\mathbf{M})(\mathbf{z},\mathbf{\Gamma}) \\
&\quad + (\mathbf{v},\mathbf{\Gamma})(\mathbf{z},\mathbf{M})\Big)(\mathbf{v},\mathbf{z}\times\mathbf{k}) + (\mathbf{v}\times\mathbf{z},\ \mathbf{z}\times\mathbf{k})(\mathbf{v},\mathbf{M}\times\mathbf{\Gamma}) \\
&\quad + (\mathbf{v}\times\mathbf{z},\ \mathbf{v}\times\mathbf{k})(\mathbf{z},\mathbf{M}\times\mathbf{\Gamma})\Big] - \eta|\mathbf{M}|^2(\mathbf{v}\times\mathbf{k},\ \mathbf{z}\times\mathbf{k}) + (\mathbf{v},\mathbf{z}\times\mathbf{k})(\mathbf{k},\mathbf{M}\times\mathbf{\Gamma}), \\
I_1 &= p_1(\mathbf{v},\mathbf{z}\times\mathbf{k})\cdot\Big(p_1\eta^2(\mathbf{v},\mathbf{z}\times\mathbf{M}) + (\mathbf{k},\ \mathbf{\Gamma})\Big).
\end{aligned}
$$

Case 5 of Proposition 1. I_4 is is given by (2.10), $I_1 = 0$,

$$
\begin{aligned}
I_3 &= 2p_1\Big(P|\mathbf{M}|^2 - (\mathbf{v},\mathbf{M})(\mathbf{M},\mathbf{\Gamma})\Big) + 2p_2\Big((\mathbf{M},\mathbf{\Gamma})(\mathbf{u},\mathbf{v}\times\mathbf{M}) - P(\mathbf{M},\mathbf{\Gamma}\times\mathbf{v})\Big),\\
I_2 &= p_1^2|\mathbf{M}|^2 + p_2^2\Big(|\mathbf{v}|^2|\mathbf{\Gamma}|^2 + \kappa(\mathbf{v},\mathbf{M})^2 - (\mathbf{v},\mathbf{\Gamma})^2\Big) - 2p_1p_2(\mathbf{M},\mathbf{\Gamma}\times\mathbf{v}),
\end{aligned}
$$

where $P = (\mathbf{u},\mathbf{M}) + (\mathbf{v},\mathbf{\Gamma})$.

Case 6 of Proposition 1. I_6 is presented in Example 3, $\;I_1 = 0$,

$$
\begin{aligned}
I_5 = {} & 4p_1(\mathbf{b},\mathbf{M})\Big\{\Big[(\mathbf{a},\mathbf{b})|\mathbf{M}|^2(\mathbf{a}\times\mathbf{b},\,\mathbf{a}\times\mathbf{M})\,(\mathbf{a},\mathbf{b}\times\mathbf{M}) + |\mathbf{\Gamma}|^2(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\,(\mathbf{a},\mathbf{b}\times\mathbf{M})\Big]\\
&+(\mathbf{a}\times\mathbf{b},\,\mathbf{M}\times\mathbf{\Gamma})\,\Big[(\mathbf{b},\mathbf{\Gamma})(\mathbf{a},\mathbf{b}\times\mathbf{M}) - (\mathbf{b},\mathbf{M})(\mathbf{a},\mathbf{b}\times\mathbf{\Gamma})\Big]\\
&+(\mathbf{b},\mathbf{M})(\mathbf{M},\mathbf{\Gamma})\Big[(\mathbf{a},\,\mathbf{b})(\mathbf{a}\times\mathbf{b},\,\mathbf{a}\times\mathbf{M}) - \kappa(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\Big]\\
&-(\mathbf{a},\,\mathbf{b})(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\Big[2(\mathbf{a},\,\mathbf{M})(\mathbf{M},\,\mathbf{\Gamma}) - (\mathbf{a},\,\mathbf{\Gamma})|\mathbf{M}|^2\Big]\\
&+|\mathbf{M}|^2(\mathbf{a}\times\mathbf{b},\,\mathbf{a}\times\mathbf{M})\Big[|\mathbf{b}|^2(\mathbf{a},\,\mathbf{\Gamma}) - 2(\mathbf{a},\,\mathbf{b})(\mathbf{b},\,\mathbf{\Gamma})\Big]\Big\},
\end{aligned}
$$

$$
\begin{aligned}
I_4 = {} & p_1^2\Big\{4\Big[(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\,\Big((\mathbf{a},\,\mathbf{\Gamma})(\mathbf{b},\,\mathbf{M}) + 2(\mathbf{a},\,\mathbf{b})(\mathbf{M},\,\mathbf{\Gamma})\Big) - (\mathbf{b},\,\mathbf{\Gamma})(\mathbf{b},\,\mathbf{M})\,(\mathbf{a}\times\mathbf{b},\,\mathbf{a}\times\mathbf{M})\\
&-2(\mathbf{a},\,\mathbf{b})|\mathbf{M}|^2(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{\Gamma})\Big]\cdot(\mathbf{a},\mathbf{b}\times\mathbf{M}) + 4(\mathbf{b},\,\mathbf{M})\Big[(\mathbf{b},\,\mathbf{M})(\mathbf{a}\times\mathbf{b},\,\mathbf{a}\times\mathbf{M})\\
&-(\mathbf{a},\,\mathbf{M})(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\Big]\cdot(\mathbf{a},\mathbf{b}\times\mathbf{\Gamma}) - 8(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{\Gamma})(\mathbf{M},\,\mathbf{\Gamma})\\
&+\Big[4(\mathbf{a},\,\mathbf{b})^2(\mathbf{a},\,\mathbf{M})(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M}) - 4(\mathbf{b},\,\mathbf{M})\Big(2(\mathbf{a},\,\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2\Big)(\mathbf{a}\times\mathbf{b},\,\mathbf{a}\times\mathbf{M})\\
&+4(\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{\Gamma})^2\Big]|\mathbf{M}|^2 - 4(\mathbf{a},\,\mathbf{b})^2\Big((\mathbf{a},\,\mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2\Big)|\mathbf{M}|^4\Big\},
\end{aligned}
$$

$$
I_3 = 8p_1^3|\mathbf{a}\times\mathbf{b}|^2\Big((\mathbf{a},\mathbf{b})|\mathbf{M}|^2(\mathbf{a},\mathbf{b}\times\mathbf{M}) + (\mathbf{b}\times\mathbf{a},\,\mathbf{b}\times\mathbf{\Gamma})|\mathbf{M}|^2 + (\mathbf{a}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})(\mathbf{M},\mathbf{\Gamma})\Big),
$$

$$
I_2 = -4p_1^4|\mathbf{a}\times\mathbf{b}|^2\Big((\mathbf{a},\mathbf{b})^2|\mathbf{M}|^2 - |\mathbf{b}|^2|\mathbf{\Gamma}|^2\Big).
$$

The Case of Proposition 2. $I_1 = 0$; I_4 is related to integral I from Proposition 2 by

$$
\begin{aligned}
I_4 = {} & |\mathbf{b}|^2 I + (\mathbf{b},\mathbf{M})^2\Big[(\mathbf{k}\times\mathbf{b},\,\mathbf{b}\times\mathbf{\Gamma})\,(\mathbf{k},\mathbf{b}\times\mathbf{M}) - (\mathbf{k}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\,(\mathbf{k},\mathbf{b}\times\mathbf{\Gamma})\\
&+\Big((\mathbf{k},\,\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2\Big)\cdot\Big(|\mathbf{\Gamma}|^2 - \frac{1}{4}(\mathbf{b},\mathbf{M})^2\Big) + |\mathbf{k}|^2|\mathbf{b}|^2\Big(|\mathbf{\Gamma}|^2 + \kappa|\mathbf{M}|^2\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
I_3 = {} & p_1\Big[\Big((\mathbf{k},\,\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2\Big)(\mathbf{b},\mathbf{M})^2(\mathbf{b},\mathbf{\Gamma}) - 2|\mathbf{b}|^2\,(\mathbf{k}\times\mathbf{b},\,\mathbf{M}\times\mathbf{\Gamma})\,\Big((\mathbf{k},\mathbf{b}\times\mathbf{\Gamma}) + (\mathbf{k}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\Big)\Big]\\
&+\Big((\mathbf{k},\,\mathbf{b})^2 - |\mathbf{k}|^2|\mathbf{b}|^2\Big)\,(\mathbf{b},\mathbf{M})\,\Big[(\mathbf{b},\mathbf{M})\,(\mathbf{k},\mathbf{b}\times\mathbf{M}) + 2(\mathbf{k}\times\mathbf{b},\,\mathbf{\Gamma}\times\mathbf{M})\Big],
\end{aligned}
$$

$$
\begin{aligned}
I_2 \;=\;& p_1^2\Big[|\mathbf{b}|^2(\mathbf{k}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})^2+(\mathbf{k}\times\mathbf{b},\,\mathbf{b}\times\boldsymbol{\Gamma})^2+((\mathbf{k},\,\mathbf{b})^2-|\mathbf{k}|^2|\mathbf{b}|^2)\,|\mathbf{b}\times\boldsymbol{\Gamma}|^2\\
&+|\mathbf{k}|^2|\mathbf{b}|^4(|\boldsymbol{\Gamma}|^2+\kappa|\mathbf{M}|^2)\Big]-2p_1\Big((\mathbf{k},\,\mathbf{b})^2-|\mathbf{k}|^2|\mathbf{b}|^2\Big)\,(\mathbf{b},\mathbf{M})\,(\mathbf{k},\mathbf{b}\times\boldsymbol{\Gamma})+\\
&\Big((\mathbf{k},\,\mathbf{b})^2-|\mathbf{k}|^2|\mathbf{b}|^2\Big)\Big[(\mathbf{b},\mathbf{M})\,(\mathbf{k}\times\mathbf{b},\,\mathbf{k}\times\mathbf{M})-(\mathbf{k},\mathbf{M})\,(\mathbf{k}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M})\Big].
\end{aligned}
$$

Appendix B

With the comments in this appendix we only want to give some impression on features of the package CRACK that we used for the solution of the overdetermined algebraic systems $\{I,\,H\}=0$. The description is neither complete not detailed.

1. Flexibility (different possibilities to change or adapt the strategy):

- **a)** `CRACK` has a priority list, which contains among other things, 10 different types of substitutions, two types of factorizations, three types of shortening of equations, various types of Groebner basis calculation steps, and the possibility to call external specialized packages.

  One can set the different priorities for these internal procedures before the session but also change them interactively during the computation. In addition, the program is able to make minor changes in the priority list in accordance to its analysis of the current situation and history of the session. An important point for the analysis is that each equation has a property list describing the previous use of the equation as well as its potential for further computation.

  The program has automatic and interactive modes and the possibility to switch between both during the computation. In the interactive mode it provides extended support for inspecting the system of equations, to specify the very next steps in detail or modify the solution strategy in general.

- **b)** After a decision is made what kind of step should come next (based on the priority list) more heuristic knowledge decides how to do it. For example, if a factorization is to come next then a refined weighting scheme determines which equation to factorize and in which order to set its factors to zero.

- **c)** During computation, inequalities are actively gathered and simplified in order to reduce the number of branches.

2. The program is very cautious about expression swell and for its prevention it is able to

- **a)** compute a tight upper bound for the system's length after an intended substitution and hence choose substitutions giving minimal growth,

- **b)** find linear combinations of pairs of equations in an attempt to shorten and simplify the system recursively during computation, and

c) it takes advantage of the initial bi-linear form of the system and keeps it linear in the unknown coefficients of the first integral at all times.

3. Several facilities are provided, e.g.

a) to find re-parametrizations of solutions in order to merge them,

b) safety precautions: to interrupt intermediate steps automatically if they (unexpectedly) take too much time, to catch all interactive input, to conveniently store and load backups,

c) an automatic preparation of web pages with results of the computation.

4. Different branches of the general solution can be investigated at the same time in parallel on a cluster of computers.

To obtain a free copy of this REDUCE package contact T. Wolf under `twolf@brocku.ca` .

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