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Abstract

The numerical treatment of equivariant parameter-dependent nonlinear equation systems, and even more its automation requires the intensive use of group theory. This paper illustrates the group theoretic computations which are done in the preparation of the numerical computations. The bifurcation graph which gives the bifurcation subgroups is determined from the interrelationship of the irreducible representations of a group and its subgroups. The Jacobian is transformed to block diagonal structure using a modification of the transformation which transforms to block diagonal structure with respect to a supergroup. The principle of conjugacy is used everywhere to make symbolic and numerical computations even more efficient. Finally, when the symmetry reduced problems and blocks of Jacobian matrices are evaluated numerically, the fact that the given representation is a quasi-permutation representation is exploited automatically.

Contents

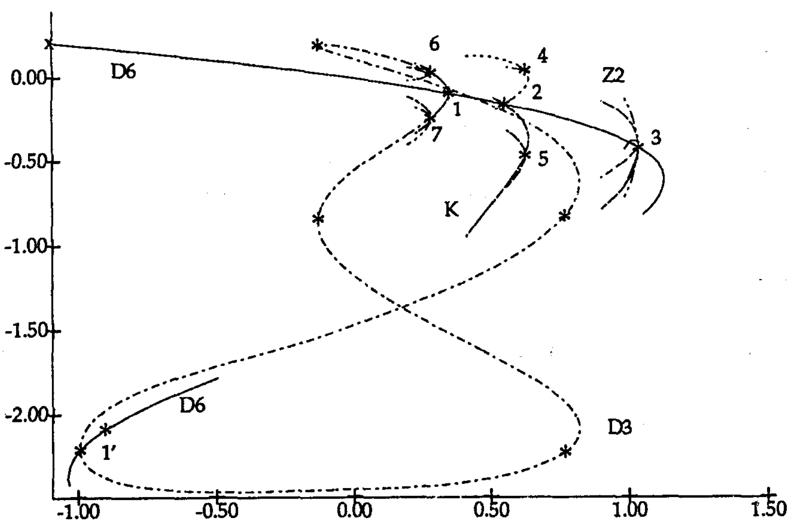

Figure 1: Part of a bifurcation diagram

1. Introduction

Figure shows some stationary solutions of a parameter-dependent nonlinear system in the form

$$\dot{x} = F(x,\lambda)\,, \quad F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n\,, \quad \text{where}$$

$$\vartheta(t)F(x,\lambda) = F(\vartheta(t)x,\lambda)\,, \quad \forall\, t \in G\,,$$

with a linear representation ϑ holds. Such *equivariant* systems have branches of stationary solutions having the same symmetry, e.g. are invariant with respect to *isotropy groups* H. As different branches may have different isotropy groups, all subgroups of G which occur as isotropy groups have to be taken into account. There may as well exist *conjugate* solutions $\vartheta(t)x$ which are given by a group operation $t \in G$. At symmetry breaking bifurcation points branches with different isotropy groups intersect such that one group is a common supergroup of the others. The subgroups fullfill the conditions of a *bifurcation subgroup*, which are arranged in the *bifurcation graph*. Of course, bifurcation points may have conjugate points (see 4 and 5 in Fig. 1 and 6 and 7). Bifurcation points may be symmetric or asymmetric depending on the isotropy groups of emanating branches.

Concentrating on one bifurcation point only the analysis may be found in GOLUBITSKY, STEWART, SCHAEFFER [8], [9], VANDERBAUWHEDE [22] while the numerical treatment was first considered by DELLNITZ, WERNER [2], and also by HEALEY [11].

The basic concept of numerical treatment is to apply the numerical pathfollowing procedure to the *symmetry reduced systems* only for one of the conjugate branches. The program SYMCON for example includes the numerical pathfollowing algorithm ALCON (DEUFLHARD, FIEDLER, KUNKEL [3]). While performing the numerical pathfollowing the bifurcation points leading to higher symmetry are detected by sign change of *symmetry monitor functions* and the points leading to smaller isotropy by sign check of determinants ([4]) or other test functions ([16] or [24]). The bifurcation points are computed by Newton's method applied to an augmented system ([4], [23], [24]).

These techniques exploite the block diagonal structure of the Jacobian (see also [12], [13], [14], [18], and [23]). Since a bifurcation point was found, the numerical pathfollowing procedure is restarted and possibly applied to a different symmetry reduced system checking determinants of a different block diagonal structure.

While the numerical mathematician concentrates on the numerical methods for determination and computation of bifurcation points we are interested in the automated preparation of examples which are equivariant with respect to different groups G. Starting with a database of irreducible representations of several finite groups (SYMMETRY [6]), different tasks and questions arise than those proposed by numerical mathematicians.

1. How are the bifurcation subgroups and thus the bifurcation graph computed automatically?

2. How are those subgroups determined which are relevant for a given problem?

3. How is the block diagonal structure with respect to a subgroup obtained if the transformation to the structure corresponding to a supergroup is known?

4. Which information is needed to organize the handling of conjugate solutions, especially conjugate bifurcation points? (How acts the group G on the bifurcation diagram?)

5. Which group theoretic computations may be saved because of conjugation?

6. How to write a REDUCE program which organizes the numerical evaluation of reduced systems and Jacobian blocks efficiently, if the representation ϑ is nearly a permutation representation?

Continueing the work in [4] the theory of linear representations (SERRE [19] or STIEFEL, FÄSSLER [20]) is used intensively, but not repeated completely. First, the analysis of equivariant systems is briefly outlined using the notion of symmetrical normal forms. The computation of the bifurcation graph and its relevant part exploiting the interrelationship between irreducible representations of a group and its subgroups is given in Section 3.

A deeper understanding of innerconnectivity of irreducible representations of G and its subgroups gives the transformation between the block diagonal structures corresponding to different groups (Section 4). In the literature this is only mentioned for the example of D_6 (IKEDA and MUROTA [13]). The handling of conjugate bifurcation points necessitates a deeper understanding of the principle of conjugacy (Section 5). Section 6 is dedicated to an overall view and the exploitation of special properties of the representation ϑ while Section 7 gives the advantages of Computer Algebra. For example the automatic code generation of the transformation matrices with the REDUCE [17] package GENTRAN [7] exploits their possible sparsity without extra implementational work as done by IKEDA and MUROTA [13], [18]. It is shown how the exploitation of occurrence of permutation representations lower the amount of produced C-code. In Section 8 the example of an hexagonal lattice dome, introduced in [11] demonstrates the success of the mixed symbolic-numeric concept.

2. Analysis of equivariant systems

The group theoretic investigations in the Sections 3-5 are derived, because we want to investigate the following problem automatically:

Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be an explicitly given function which is *G–equivariant* (VANDERBAUWHEDE [22]), i.e.

$$F(\vartheta(t)x, \lambda) = \vartheta(t)F(x, \lambda)\,, \quad \forall\, t \in G\,, \tag{1}$$

where $\vartheta : G \to GL(\mathbb{R}^n)$ is a *real orthogonal linear representation* ($\vartheta(t)\vartheta(s) = \vartheta(ts)\,\forall\, t, s \in G$). The different types of stationary solutions (x, λ), i.e.

$$F(x, \lambda) = 0\,, \tag{2}$$

are of interest. Because the system(2) is parameter dependent the solutions appear in continua. In this section the analysis of this problem class is summarized (see [2], [4], [8], and [9]).

The elements $x \in \mathbb{R}^n$ are distinguished by their symmetries. Mathematically speaking

$$G_x := \{t \in G \mid \vartheta(t)x = x\} \tag{3}$$

denotes the *isotropy group* $H = G_x$ of x. In turn x is called *H–invariant*, if $\vartheta(t)x = x, \forall\, t \in H$. The isotropy group $H = G_x$ is the maximal subgroup of G with the property, that x is H–invariant. A simple but fundamental fact is that equivariant systems have continua of solutions with the same isotropy group. Different solution paths with different isotropy groups may intersect in the so-called bifurcation points. Depending on whether different or equal groups interact they are called *symmetry breaking* or *symmetry preserving bifurcation points.* Since the Jacobian $DF(x^*, \lambda^*)$ is singular in a bifurcation point (x^*, λ^*) or has pure imaginary eigenvalues in a Hopf-point (x^+, λ^+), the consequences of the equivariance (1) for the Jacobian have to be considered.

Transformation to block diagonal structure

For $x \in \mathbb{R}^n$ with isotropy group $H = G_x$

$$\begin{aligned} \vartheta(t)D_xF(x,\lambda) &= D_xF(x,\lambda)\vartheta(t), \quad \forall\, t \in H\,, \\ \vartheta(t)D_\lambda F(x,\lambda) &= D_\lambda F(x,\lambda), \quad \forall\, t \in H\,, \end{aligned} \tag{4}$$

hold. By the theory of symmetry adapted basis (STIEFEL, FAESSLER [20]) the Jacobian $D_xF(x, \lambda)$ may be block diagonalized with an orthogonal transformation matrix $M \in \mathbb{R}^{(n,n)}$ which depends on H. For this we remind that a real irreducible representation of H has the property that it does not split into subrepresentations. Up to isomorphy a finite group has a finite number

h of real irreducible representations ϑ^i_H. One distinguishes 3 different types (real, complex, quaternonian) of real irreducible representations. The type of a real irreducible representation of complex type consists of 2 complex irreducible representations which are complex conjugate to each other. We restrict to groups which have no irreducible representations of quaternonian type.

Theorem 2.1 *([20], [23]): Let H be a group and let $\vartheta^i : H \to GL(\mathbb{R}^{n_i})$, $n_i \in \mathbb{N}$, $i = 1, \dots, h$, be its real irreducible orthogonal linear representations with corresponding characters $\chi^i : H \to \mathbb{R}$. Let $A \in \mathbb{R}^{n,n}$ and $\vartheta : H \to GL(\mathbb{R}^n)$ with $\vartheta = \sum_{i=1}^{h_H} c_i \vartheta^i$ where c_i are the multiplicities.*

If $\vartheta_t A = A \vartheta_t$ is satisfied for all $t \in H$, then there exist orthogonal matrices

$$\begin{array}{lll} M = (M_1, \dots, M_{h_H}), & M_i \in \mathbb{R}^{n, c_i \cdot n_i}, & i = 1, \dots, h, \\ M_i = (M_{i1}, \dots, M_{in_i}), & M_{ij} \in \mathbb{R}^{n, c_i}, & j = 1, \dots, n_i, \end{array}$$

with the property

$$\begin{array}{rcl} M^T A M & = & diag(B_1, \dots, B_h), \quad B_i \in \mathbb{R}^{n_i c_i \times n_i c_i}, \\ M^T \vartheta(t) M & = & diag(\tilde{\vartheta}_i(t)) . \end{array}$$

If ϑ^i is an irreducible representation of real type (absolute irreducible) of dimension n_i, then

$$B_i = M_i^T A M_i = \begin{pmatrix} A_i & & \\ & \ddots & \\ & & A_i \end{pmatrix}, \quad M_{i1}^T A M_{i1} = A_i \in \mathbb{R}^{c_i, c_i},$$

where A_i appears n_i times.

In [20] a straightforward computation of M_{ij} using projections and the Gram-Schmid process is described. But for this the irreducible representations ϑ^i and its characters $\chi^i, i = 1, \dots, h$ have to be known.

The matrix M introduces the coordinate transformation $x = Mu$.

Definition 2.2 *([4]): Let $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be G-equivariant. Let H be a subgroup of G and $M = M^H$ the transformation matrix. Then the function $g : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by*

$$g(u, \lambda) := M^T F(Mu, \lambda)$$

is called the symmetrical normal form *of F with respect to H.*

The name symmetrical normal form is justified by the fact that the Jacobian $D_u g(u,\lambda)$ has block diagonal form for every H–invariant point $x = Mu$.

In these coordinates we have in analogy to (1), (4) with $\tilde{\vartheta}(t) = M^T\vartheta(t)M$

$$\begin{array}{rcll} g(\tilde{\vartheta}(t)u,\lambda) & = & \tilde{\vartheta}(t)g(u,\lambda) & \forall\, t \in G, u \in \mathbb{R}^n, \lambda \in \mathbb{R}, \\ \tilde{\vartheta}(t)D_u g(u,\lambda) & = & D_u g(u,\lambda)\tilde{\vartheta}(t) & \forall\, t \in H, \forall H-\text{invariant } u \\ \tilde{\vartheta}(t)D_\lambda g(u,\lambda) & = & D_\lambda g(u,\lambda) & \forall\, t \in H, \forall H-\text{invariant } u. \end{array} \tag{5}$$

Symmetry reduced systems

Let ϑ^1 denote the trivial representation. Then each H–invariant $x \in \mathbb{R}^n$ corresponds to one $u = M^T x = (\tilde{u}, 0)$ with $\tilde{u} \in \mathbb{R}^{c_1}$. Because F is equivariant (1), the set of H–invariant solutions of (2) is equivalent to the solution set of the *H-reduced equations*

$$\tilde{g}(\tilde{u},\lambda) = M_1^T g(M_1\tilde{u},\lambda) = 0\,, \tag{6}$$

where $\tilde{g} : \mathbb{R}^{c_1+1} \to \mathbb{R}^{c_1}$. This is a well known fact and is often used.

Then

$$\begin{array}{rclcl} D_u g(\tilde{u},0,\lambda) & = & M^T D_x F(M_1^T\tilde{u},\lambda)M & = & \text{diag}(B_i), \\ & & M_{i1}^T D_x F(M_1^T\tilde{u},\lambda)M_{i1} & = & A_i(\tilde{u},\lambda), \\ D_\lambda g(\tilde{u},0,\lambda) & = & (D_\lambda\tilde{g}(\tilde{u},\lambda)^T, 0)^T, & & \\ D_{\tilde{u}}\tilde{g}(\tilde{u},\lambda) & = & M_1^T D_x F(M_1\tilde{u},\lambda)M_1 & = & A_1(\tilde{u},\lambda), \\ D_\lambda\tilde{g}(\tilde{u},\lambda) & = & M_1^T D_\lambda F(M_1\tilde{u},\lambda), & & \\ D\tilde{g}(\tilde{u},\lambda) & = & (A_1, D_\lambda\tilde{g}(\tilde{u},\lambda))\,. & & \end{array} \tag{7}$$

Bifurcation graph

If a block A_i ($i \neq 1$, ϑ^i of real type) becomes singular, a *symmetry breaking bifurcation point* may occur. Because the kernel of DF has dimension n_i, multidimensional irreducible representations gives raise to a multiple bifurcation point. In the *equivariant branching lemma* of VANDERBAUWHEDE [22] and CICOGNA (see also [9], p. 82 and [2]) the multiple problem is reduced to a simple bifurcation phenomena. DELLNITZ and WERNER [2] introduced the definition of a *bifurcation subgroup*.

Definition 2.3 : *Let $\vartheta^i : H \to GL(\mathbb{R}^{n_i})$ be a real irreducible representation of real type. K is called a* bifurcation subgroup *of H of type i, if* *a.)* $0 \neq v \in \mathbb{R}^{n_i}$ *exists with* $K = H_v$,

b.) *for every* K*-invariant* $w \in \mathbb{R}^{n_i}$ *exists* $a \in \mathbb{R}$ *with* $w = av$.

Definition 2.4 : *A group* H *is called a* bifurcation supergroup *of* K*, if* K *is a bifurcation subgroup of* H *of some type.*

If on a branch of solutions $(u(\lambda), \lambda) = (\tilde{u}(\lambda), 0, \lambda)$ with isotropy group H one block $A_i, i \geq 2$ becomes singular at (u^*, λ^*) then it follows from the equivariant branching lemma that generically branches of solutions emanate having the isotropy of bifurcation subgroups K of H of type i. These branches are solutions of the K-reduced problem.

A deeper result from analysis is that the blocks A_i corresponding to a real irreducible representation of complex type generically do not become singular. The main reason is that A_i is equivalent to a complex matrix consisting of to blocks which are complex conjugate to each other. A singular block A_1 indicates a turning point, a symmetry preserving bifurcation point or a symmetry breaking bifurcation point where x^* has the isotropy of a bifurcation supergroup of H. In the last case a branch with the isotropy of the bifurcation supergroup intersects in (x^*, λ^*) Definition 2.3 leads to the definition of a *bifurcation graph* showing all bifurcation subgroups and bifurcation supergroups (see Fig. 2).

The first aim of automation is the computation of the bifurcation graph. But the numerical pathfollowing applied to the reduced systems and the evaluation of Jacobian blocks is needed only for the isotropy groups of a given equivariant system (2). Thus Section 3 is devoted to the algorithmic determination of the relevant part of the bifurcation graph.

3. Computation of relevant bifurcation subgroups

In this and the next section the relation of irreducible representations of a group H with the irreducible representations of its subgroups is fundamental. We start with a technical definition. The restriction of a representation $\vartheta : H \to GL(\mathbb{R}^n)$ to a representation $\vartheta \downarrow K$ of a subgroup K of H is given by

$$[\vartheta \downarrow K](t) = \vartheta(t) \quad \forall\, t \in K.$$

Innerconnectivity multiplicities

The real irreducible representations $\vartheta_H^i, i = 1, \ldots, h_H$ of H may be restricted to representations $\vartheta_H^i \downarrow K$ of K. Then a canonical decomposition with respect to K exists, i. e. integers $d_{i\underline{j}} \in \mathbb{N}$ exists with

$$\vartheta_H^i \downarrow K = \sum_{\underline{j}=1}^{h_K} d_{i\underline{j}} \, \vartheta_K^{\underline{j}}, \quad i = 1, \ldots, h_H, \tag{8}$$

where $\vartheta_K^{\underline{j}}, \underline{j} = 1, \ldots, h_K$ are the real irreducible representations of K. For clarification indices corresponding to the subgroup K are underlined.

Definition 3.5 : *The integers $d_{i\underline{j}}, i = 1, \ldots, h_H,\ \underline{j} = 1, \ldots, h_K$, are called* innerconnectivity multiplicities.

The innerconnectivity multiplicities $d_{i\underline{j}}$ are easily computed by a formula for multiplicities (see [20]). They have two applications. For the given representation $\vartheta : G \to GL(\mathbb{R}^n)$ in (1) the multiplicities c_i^G in the canonical decomposition

$$\vartheta = \sum_{i=1}^{h_G} c_i^G \vartheta_G^i$$

are obtained by the formula mentioned above. (c_1^G is the dimension of the G-reduced system and c_i^G are the dimensions of the Jacobian blocks A_i.) Computing H-invariant solutions of (2) means consideration of

$$\vartheta \downarrow H = \sum_{\underline{j}=1}^{h_H} c_{\underline{j}}^H \, \vartheta_H^{\underline{j}} . \tag{9}$$

The multiplicities $c_{\underline{j}}^H$ for subgroups H may be obtained easily with the innerconnectivity multiplicities with respect to G and H:

$$c_{\underline{j}}^H = \sum_{i=1}^{h_G} d_{i\underline{j}} c_i^G , \quad \underline{j} = 1, \ldots, h_H . \tag{10}$$

Second the innerconnectivity multiplicities enable the computation of bifurcation subgroups.

Lemma 3.6 : *Let $\vartheta^i : H \to GL(\mathbb{R}^{n_i})$ be a non-trivial irreducible representation and $\vartheta^i \downarrow K$ its restriction to a subgroup K of H. Then K is a* bifurcation subgroup *of H* of type i, *if*

$$\vartheta^i \downarrow K = \sum_{\underline{j}=1}^{h_K} d_{i\underline{j}}\, \vartheta_K^{\underline{j}} ,$$

with $d_{i1} = 1$ and if K is maximal with this property.

Bifurcation subgroups of type i are isotropy groups of ϑ_H^i. But by this definition it is not considered whether they are isotropy groups for the given representation $\vartheta : G \to GL(\mathbb{R}^n)$ in (1).

Relevant subgroups

Definition 3.7 : *A subgroup H of G is called a* relevant bifurcation subgroup *of $\vartheta : G \to GL(\mathbb{R}^n)$ of* level *1, if*

- *a.) $c_1^G \geq 1$, where $\vartheta = \sum_{i=1}^{h_G} c_i^G\, \vartheta_G^i$*
- *b.) $i \in \{2, \ldots, h_G\}$ exists, such that H is a bifurcation subgroup of type i, and*
- *c.) $c_i^G \geq 1$.*

Definition 3.8 : *A subgroup K of G is called a* relevant bifurcation subgroup *of $\vartheta : G \to GL(\mathbb{R}^n)$ of* level ν *($\nu > 1$), if*

- *a.) a relevant bifurcation subgroup H of ϑ of level $\nu - 1$ exists and*
- *b.) $i \in \{2, \ldots, h_H\}$ exists, such that K is a bifurcation subgroup of H of type i and*
- *c.) $c_i^H \geq 1$, where $\vartheta \downarrow H = \sum_{i=1}^{h_H} c_i^H\, \vartheta_H^i$*

For finite groups G there is a level μ such that there are no relevant bifurcation subgroups of level $\nu > \mu$ for all linear representations of G.

Definition 3.9 : *The relevant bifurcation subgroups of $\vartheta : G \to GL(\mathbb{R}^n)$ of all levels $\nu \geq 1$ are called* relevant subgroups *of G with respect to ϑ.*

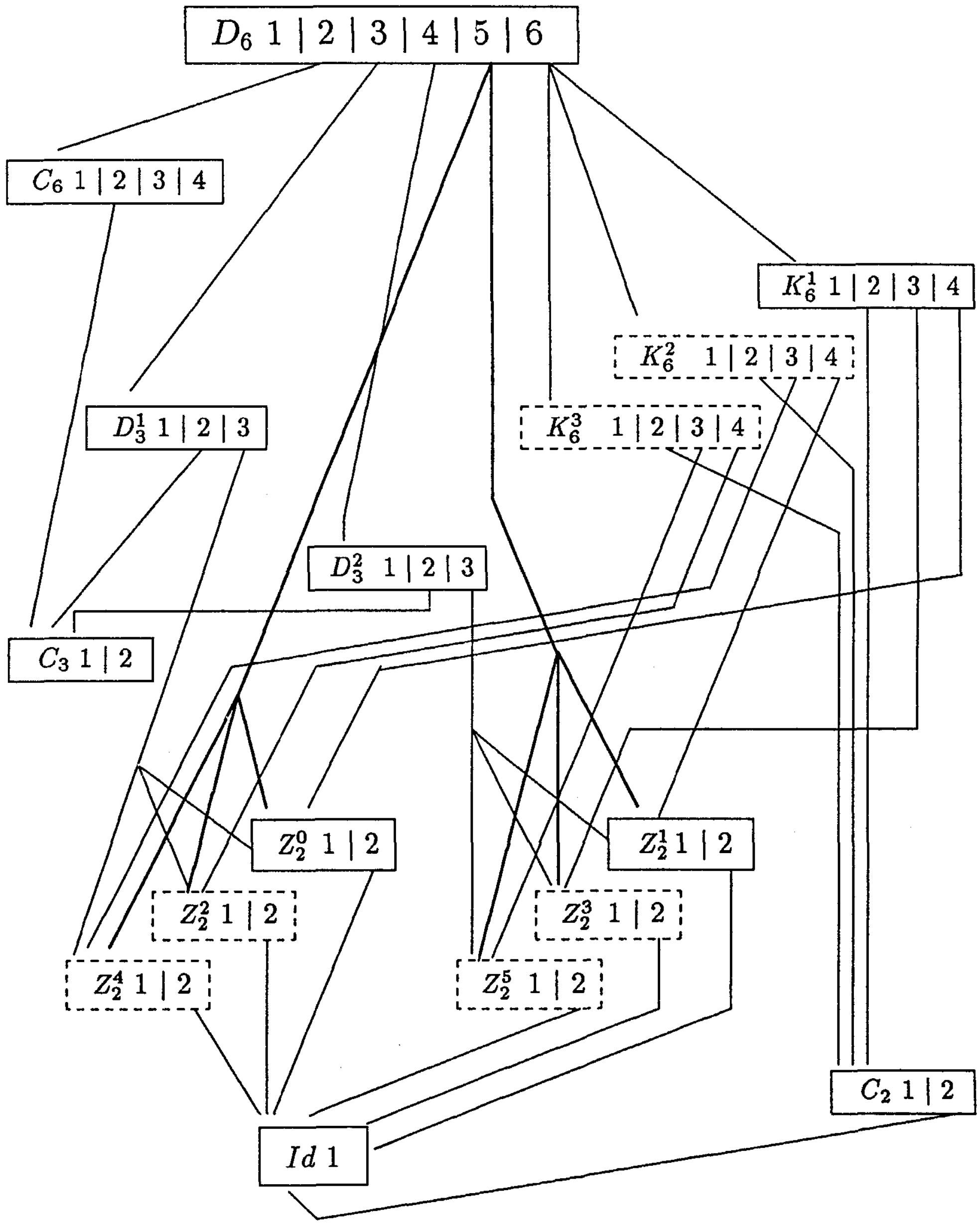

Figure 2: Bifurcation graph for D_6

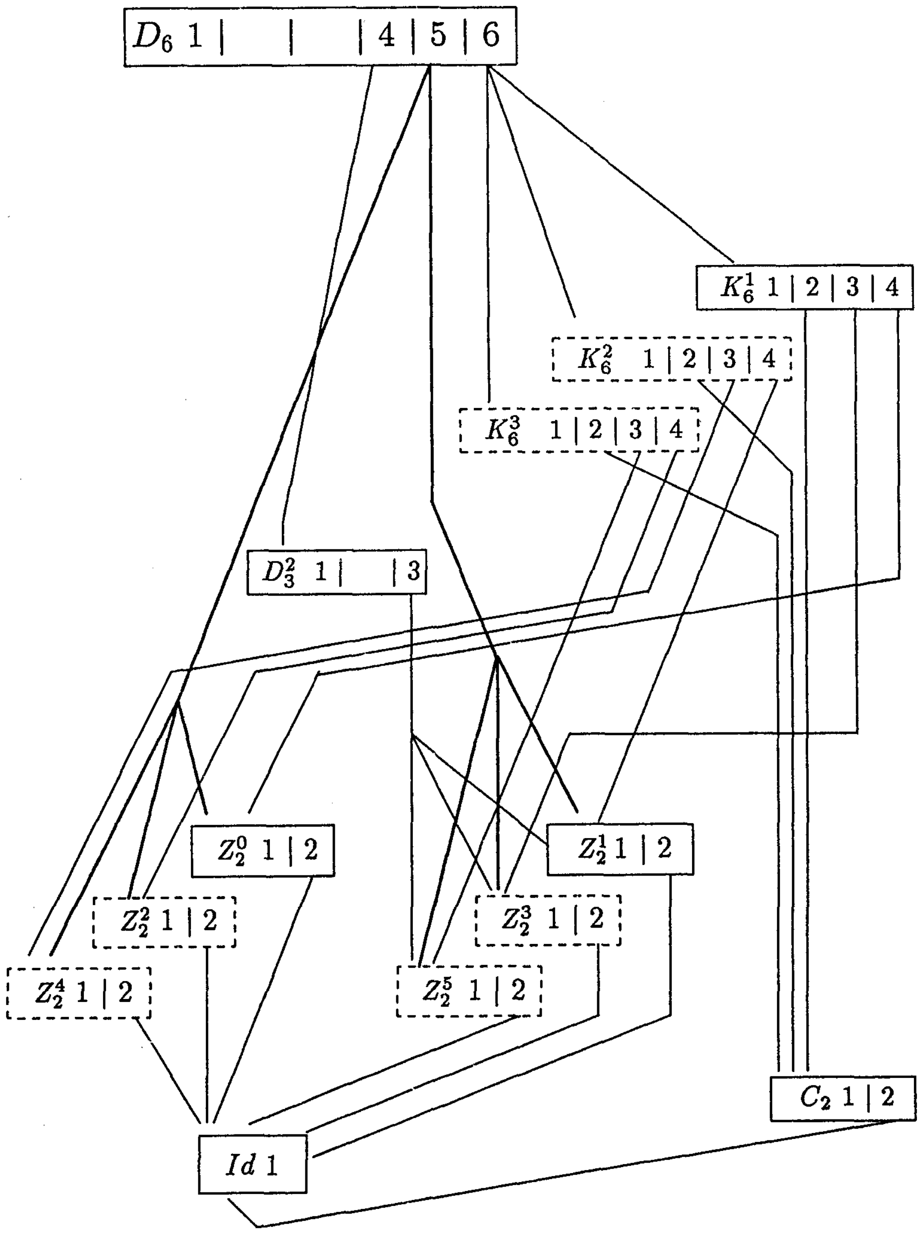

Figure 3: Relevant part of bifurcation graph for D_6-brusselator

Lemma 3.10 : *The relevant subgroups of G with respect to ϑ are isotropy groups of ϑ.*

The relevant subgroups of G give the relevant part of the bifurcation graph (compare Fig. 2 and Fig. 3) which is needed in the numerical computation of the complete bifurcation scenario of an equivariant system (2).

Remarks:

1.) The type i of a bifurcation subgroup is not unique. $\mathbb{Z}_2$ is a bifurcation subgroup of D_5 of types 3 and 4.

2.) The level of a relevant bifurcation subgroup of ϑ is not unique. $\mathbb{Z}_2$ is a relevant bifurcation subgroup of D_6 of level 1 and 2.

3.) In contrast to [2], the bifurcation graphs in [4] show the relation between bifurcation subgroups and their irreducible representations.

4. Innerconnectivity

Once the transformation matrix $M = M^G$ for the absolute supergroup G is computed by means of projections, the transformation matrices M^H, where H is a relevant subgroup of G, are easily obtained in the following way.

Recall that for each $i = 1, \ldots, h_G$ the restricted irreducible representations $\vartheta_G^i \downarrow H$ have a canonical decomposition (see (8)). Furthermore there exist coordinate transformations $C^i \in \mathrm{IR}^{n_i,n_i}$ such that $(C^i)^T \vartheta_G^i(t)\, C^i$ are simultaneously block diagonal for all $t \in H$, where the blocks consist of $d_{i\underline{j}}$ matrices $\vartheta_H^{\underline{j}}(t), \underline{j} = 1, \ldots, h_H$:

$$(C^i)^T \vartheta_G^i(t)\, C^i = \mathrm{diag}(\vartheta_H^{\underline{j}}(t)), \quad \forall\, t \in H, \quad i = 1, \ldots, h_G. \tag{11}$$

These matrices *innerconnectivity matrices* C^i are computed as usual by means of projections applied to $\vartheta_G^i \downarrow H$. Note that the columns of C^i form a symmetry adapted basis of IR^{n_i}.

Transformation to block diagonal structure wrt different groups

Based on $d_{i\underline{j}}$ and C^i a *connection matrix* $C_{GH} \in \mathrm{IR}^{n,n}$ is defined such that

$$M^H = M^G C_{GH}\,, \tag{12}$$

is a coordinate transformation matrix for H in the sense of Theorem 2.1. The computation of M^H or its parts $M^H_{i\underline{j}}$ using the connection matrix is much easier than by application of the projections, but the definition of C_{GH} is tedious to describe in detail. For this we introduce the notation

$$D := \coprod_{i=1}^{m} D_i = (D_1, \ldots, D_m), \tag{13}$$

where D_i are given $n \times j_i$ matrices and D denotes the matrix with n rows and $\sum_{i=1}^{m} j_i$ columns consisting of the collection of the matrices D_i. In this notation a decomposition exists

$$C^i = \coprod_{\substack{\underline{j}=1 \\ d_{i\underline{j}} \neq 0}}^{h_H} C^{i\underline{j}}, \qquad C^{i\underline{j}} = \coprod_{\underline{k}=1}^{n_{\underline{j}}} C^{i\underline{j}\underline{k}}, \quad i = 1, \ldots, h_G, \tag{14}$$

where $C^{i\underline{j}}$ are real $n_i \times (n_{\underline{j}} \cdot d_{i\underline{j}})$ matrices and $C^{i\underline{j}\underline{k}}$ are $n_i \times d_{i\underline{j}}$ matrices. $n_{\underline{j}}$ is the dimension of $\vartheta^{\underline{j}}_H$. $C^{i\underline{j}}$ correspond to the irreducible representations $\vartheta^{\underline{j}}_H$. Note that for $d_{i\underline{j}} > 1$ the matrix $C^{i\underline{j}}$ is not unique. Recall the partitions of M^G and M^H with respect to the irreducible representations

$$M^G = \coprod_{i=1}^{h_G} M^G_i, \quad M^G_i = \coprod_{l=1}^{n_i} M^G_{il},$$
$$M^H = \coprod_{\underline{j}=1}^{h_H} M^H_{\underline{j}}, \quad M^H_{\underline{j}} = \coprod_{\underline{k}=1}^{n_{\underline{j}}} M^H_{\underline{j}\underline{k}},$$

Then the innerconnectivity of irreducible representations of G and H (see (8), (11)) implies

$$M^H_{\underline{j}\underline{k}} = \coprod_{\substack{i=1 \\ d_{i\underline{j}} \neq 0}}^{h_G} \coprod_{\mu=1}^{d_{i\underline{j}}} \sum_{l=1}^{n_i} M^G_{il} C^{i\underline{j}\underline{k}}_{l\mu}, \tag{15}$$

which is written in(12) in compact form.

If H is a proper subgroup, the innerconnectivity matrices C^i are given, and M^G was already computed, then M^H is uniquely defined by this procedure. We prefer (15) to (12) because only the parts $M^H_{\underline{j}1}$ are needed for the computation of reduced equations and Jacobian blocks. In IKEDA and MUROTA [13] the rearrangements of M for the subgroups of D_6 are explicitly given.

$$C_{GH} =$$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c}
 & \cdots & \overbrace{\mathbf{0}}^{1} & \cdots & \overbrace{\qquad\mathbf{0}\qquad}^{\underline{k}} & \cdots & \overbrace{\mathbf{0}}^{n_{\underline{j}}} & \cdots & \\
\hline
\vdots & \cdots & & \cdots & \vdots & \cdots & & \vdots & \cdots \\
\vdots & & & & & & & & \vdots \\
\hline
\mathbf{0} & \cdots & * & \cdots & \underbrace{\;}_{1}\Big|\;\underbrace{A^{i\underline{j}\underline{k}}}_{i}\;\Big|\underbrace{\;}_{h_G} & \cdots & * & \cdots & \mathbf{0} \\
\hline
\vdots & & & & & & & & \vdots \\
\vdots & \cdots & & \cdots & \vdots & \cdots & & \vdots & \cdots \\
\hline
 & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \cdots &
\end{array}\right)
\begin{array}{l}
\}\,1 \\
\\
\\
\left.\begin{array}{c} 1 \\ \vdots \\ n_i \end{array}\right\}\, i \\
\\
\\
\}\,h_G
\end{array}$$

$$\underbrace{\quad}_{1} \qquad \underbrace{\qquad\qquad\qquad\qquad\qquad}_{\underline{j}} \qquad \underbrace{\quad}_{h_H}$$

$$A^{i\underline{j}\underline{k}} = \begin{pmatrix} C_{11}^{i\underline{j}\underline{k}} I & \cdots & C_{1d_{i\underline{j}}}^{i\underline{j}\underline{k}} I \\ \vdots & & \vdots \\ C_{n_i 1}^{i\underline{j}\underline{k}} I & \cdots & C_{n_i d_{i\underline{j}}}^{i\underline{j}\underline{k}} I \end{pmatrix} \quad \text{with} \quad \begin{array}{rcl} A^{i\underline{j}\underline{k}} & \in & \mathrm{Mat}(n_i c_i^G, d_{i\underline{j}} c_i^G) \\ C^{i\underline{j}\underline{k}} & \in & \mathrm{Mat}(n_i, d_{i\underline{j}}) \\ I & \in & \mathrm{Mat}(c_i^G, c_i^G) \end{array}$$

Figure 4: Connection matrix with respect to G and H

For a proper subgroup K of H which is a proper subgroup of G the problem arises that M^K may be computed by $M^G C_{GH} C_{HK}$ or $M^G C_{GK}$ which may happen to be different. In [18] it is stated that one should rearrange the order of irreducible representations and choose right bases for them and choose between conjugate groups such that there is a maximal chain of subgroups $G > H > K > \ldots$ with the property that the same transformation matrix $M = M^G = M^H = M^K \cdots$ is valid. But the relevant subgroups are not arrangeable in one chain in general. The counterexample of D_6 shows that it is not possible to determine one common matrix M for all non-conjugate subgroups. In SYMCON the transformation matrices $M^K = M^G C_{GK}$ are used.

Coordinate transformation

By $x = M^G u_G$ and $x = M^H u_H$ two coordinate transformations are defined. One may switch between G- and H-coordinates by

$$u_G = C_{GH}\, u_H \tag{16}$$

This coordinate change may be written more concrete avoiding the multiplications and additions with zero:

$$\begin{array}{rcll} u_G &=& (u_G^1, \ldots, u_G^{h_G}), & \\ u_G^i &=& (u_G^{i1}, \ldots, u_G^{in_i}), & i = 1, \ldots, h_G, \\[1ex] u_H &=& (u_H^1, \ldots, u_H^{h_H}), & \\ u_H^{\underline{j}} &=& (u_H^{\underline{j}1}, \ldots, u_H^{\underline{j}n_{\underline{j}}}), & \underline{j} = 1, \ldots, h_H, \\ u_H^{\underline{j}\underline{k}} &=& (u_H^{\underline{j}\underline{k}1}, \ldots, u_H^{\underline{j}\underline{k}h_G}), & \\ u_H^{\underline{j}\underline{k}i} &=& (u_H^{\underline{j}\underline{k}i1}, \ldots, u_H^{\underline{j}\underline{k}id_{i\underline{j}}}), & i = 1, \ldots, h_G, \end{array} \tag{17}$$

$$u_H^{\underline{j}\underline{k}i\nu} = \sum_{l=1}^{n_i} C_{l\nu}^{i\underline{j}\underline{k}}\, u_G^{il} \qquad \begin{array}{rcl} i &=& 1, \ldots, h_G, \\ \underline{j} &=& 1, \ldots, h_H, \\ \underline{k} &=& 1, \ldots, n_{\underline{j}}, \\ \nu &=& 1, \ldots, d_{i\underline{j}}. \end{array} \tag{18}$$

Writing

$$C_{GH} = \coprod_{\underline{j}=1}^{h_H} C_{GH}^{\underline{j}},$$

this gives

$$u_H^j = (C_{GH}^j)^T u_G . \tag{19}$$

The inverse operation is given by

$$u_G^{il} = \sum_{\underline{j}=1}^{h_H} \sum_{\underline{k}=1}^{n_{\underline{j}}} \sum_{\nu=1}^{d_{i\underline{j}}} C_{l\nu}^{i\underline{j}\underline{k}} u_H^{\underline{j}\underline{k}\,i\nu} \qquad i = 1, \ldots, h_G,\ l = 1, \ldots, n_i . \tag{20}$$

Offset directions

The formulas (18), (19) and (20) for the coordinate changes are helpful to determine the directions of emanating branches which are used for offset of the numerical pathfollowing. From the numerical determination of a bifurcation point $(\tilde{u}_H^*, \lambda^*)$ with isotropy group H and type i corresponding to an irreducible representation ϑ_H^i the kernel of $Dg^H(\tilde{u}_H^*, 0, \lambda^*)$ is known to be spanned by vectors $t_1, \ldots, t_{n_i}$ with

$$\begin{aligned} t_{\mu H}^{\bar{i}} &= 0 \qquad \forall \mu = 1, \ldots, n_i, \forall \bar{i} = 1, \ldots, h_H, \bar{i} \neq i, \\ t_{\mu H}^{il} &= 0 \qquad \forall l = 1, \ldots, n_i, l \neq \mu , \\ t_{\mu H}^{i\mu} &= z, \qquad z \in \mathbb{R}^{c_i^H} . \end{aligned} \tag{21}$$

Then a K-invariant vector in the kernel in reduced K–coordinates, where K is a bifurcation subgroup of H of type i, is given by

$$\hat{t} = t_K^1 = (C_{GK}^1)^T C_{GH} \left(\sum_{l=1}^{n_i} b_l \cdot t_i \right) , \tag{22}$$

where b_l are arbitrary numbers (for example $b_1 = 1, b_2 = 2, b_3 = 1$).

Symmetry monitor

While numerical pathfollowing a branch of solutions with isotropy K a symmetry breaking bifurcation point $(\tilde{u}^*, 0, \lambda^*)$ with the isotropy of a bifurcation supergroup H has to be detected. This is done by a *symmetry monitor function*

$$\mathrm{sm} : \mathbb{R}^{c_1^K} \to \mathbb{R}^{c_1^K - c_1^H} .$$

Let $u_K = (\tilde{u}, 0) = (u_K^1, 0)$ be a K–invariant point in K–coordinates which we have chosen. If $G = H$ then u_K is decomposed as in (17). This is not ascertained for $H \neq G$. Then

$$\hat{u}_K^1 = (C_{HK}^1)^T (C_{GH})^T C_{GK}^1 \, u_K^1 \,, \tag{23}$$

is a reduced K–invariant vector in such coordinates that

$$(\hat{u}_K^1)_l = 0, \quad \forall\, l = 1 + c_1^H, \ldots, c_1^K \,, \tag{24}$$

if this vector is H–invariant. Thus

$$\mathrm{sm}(\tilde{u}) = (\hat{u}_K^{112}, \ldots, \hat{u}_K^{11h_H}) \,. \tag{25}$$

5. Action of G on the bifurcation graph

The aim of this section is to show how symbolic and numerical computations are saved by the principle of conjugacy. Recall the linear representation $\vartheta : G \to GL(\mathrm{I\!R}^n)$, and the system $F(x, \lambda) = 0$ in (2) which is equivariant with respect to ϑ. G acts on $\mathrm{I\!R}^n$ by $\vartheta(t)$. If for a given $x \in \mathrm{I\!R}^n$ the transformed $\vartheta(t)x \neq x$, then $\vartheta(t)x$ is called a *conjugate* vector to x. If x is a solution of (2), then also its conjugates are solutions. G acts on the set of subgroups of G by tHt^{-1} which in case $tHt^{-1} \neq H$ is called *conjugate subgroup* of H. Both fit together in the sense that the isotropy group of $\vartheta(t)x$ is $tHt^{-1} = G_{\vartheta(t)x}$.

Computation of conjugates in a cycle

The conjugate elements of x form the *orbit* O_x. The order of O_x is equal to the *index* m of $H = G_x$ in G, which is the number of left cosets of H in G. Once a solution x is found numerically the conjugates are easily obtained by a cycle of group operations

$$\begin{array}{lcll} x_0 & := & x, & \\ x_\mu & := & \vartheta(r_i) x_{i-1} \,, & i = 1, \ldots, m-1 \,, \end{array} \tag{26}$$

where $s_i := r_i \cdot \ldots \cdot r_1$ are representatives of the *left cosets* of G/H and $r_m := s_{m-1}^{-1}$ gives the original vector $\vartheta(r_m)x_{m-1} = x$ (see Fig. 5).

Symmetrical subgroups

The group

$$N_G(H) := \{t \in G \mid \; tHt^{-1} = H\} \,,$$

```
symbolic procedure mk!_cycle(superg,subgid);
begin
scalar Nset, elem, ris, ri;
  Nset:=subsetminus(get!*elements(superg),get!*elements(subgid));
  elem:='id;
  ris:=nil;
  while Nset do
    << ri:=search!_ri!_coset(superg,get!*elements(superg),elem,Nset);
       ris:=append(ris, list(ri));
       elem:=get!*product(superg,ri,elem);
       Nset:=subsetminus(Nset,mk!_left!_coset(superg,elem,subgid));
    >>;
  ris:=append(ris,list(get!*inverse(superg,elem)));
  set!_cycle(subgid, ris);
end;
```

Figure 5: Function choosing some group elements which give the conjugate solutions in a cycle

is called the normalizer of H in G. For x with isotropy group $H = G_x$ the conjugate element $\vartheta(t)x$, $t \in N_G(H) - H$ has the same isotropy group $H = G_{\vartheta(t)x}$.

If K is a bifurcation subgroup of H there are only two possibilities for the normalizer.

Theorem 5.11 ([2]): *Let K be a bifurcation subgroup of H of type i. Then either*

$$N_H(K)/K \cong \mathbb{Z}_2 \quad or \quad N_H(K) = K\,.$$

The first case $N_H(K)/K \cong \mathbb{Z}_2$ corresponds to a pitchfork bifurcation point and K is thus called *symmetrical.*

These two cases are distinguished by computation of the normalizer $N_H(K)$. While the action of G on the subgroups is determined the normalizer $N_G(K)$ is derived which easily gives $N_H(K) = N_G(K) \cap H$.

Conjugate representations

The action of G on its subgroups is much more sophisticated. Let

$$\varepsilon_H(s) := \{r \in H \mid \exists s_1 \in H \text{ with } r = s_1 s s_1^{-1}\}$$

denote the *equivalence classes* of H and $\varepsilon(H)$ the set of classes. For $t \in G$ and a proper subgroup H of G a mapping

$$t^\varepsilon : \varepsilon(H) \to \varepsilon(tHt^{-1}), \quad t^\varepsilon(s) = \varepsilon_{tHt^{-1}}(tst^{-1}),$$

is induced which is a permutation of equivalence classes of H, if $t \in N_G(H)$.

Let $\chi(H)$ denote the set of *class functions* $\chi : H \to \mathrm{IR}$, $(\chi(r) = \chi(s)\ \forall r \in \varepsilon_H(s))$ then $t \in G$ induces the mapping

$$t_H^\chi : \chi(H) \to \chi(tHt^{-1}), \quad t_H^\chi(\chi(s)) = \tilde{\chi}(tst^{-1}) = \chi(s). \tag{27}$$

Because the characters χ^i of irreducible representations ϑ_H^i are special class functions this gives conjugate class functions $t_H^\chi(\chi^i) : tHt^{-1} \to \mathrm{IR}$, which are again characters of irreducible representations of the conjugate group tHt^{-1}.

Definition 5.12 *(see [15]) : Let H be a subgroup of G, $\rho : H \to GL(V)$ a linear representation, $t \in G$.*

$$\rho^t : tHt^{-1} \to GL(V), \quad \rho^t(tst^{-1}) := \rho(s), \quad \forall s \in H,$$

is called the conjugate representation *of ρ.*

Of course it may happen that ρ^t is equivalent to ρ. The group operations with this property form the *inertia group* H^ρ of ρ. For each irreducible representation ϑ_H^i the conjugate representation $(\vartheta_H^i)^t$ is an irreducible representation of the conjugate group tHt^{-1} and thus equivalent to one $\vartheta_{tHt^{-1}}^\iota$. For $t \in N_G(H)$ this is a permutation of irreducible representations. By

$$t_H^I : \{1,\dots,h_H\} \to \{1,\dots,h_{tHt^{-1}}\}, \quad \left((\vartheta_H^i)^t \sim \vartheta_{tHt^{-1}}^{t_H^I(i)}\right) \tag{28}$$

we denote the induced mapping between indices of irreducible representations.

Because it is not convenient to handle with the equivalence of representations, the mapping t^I_H is determined with the characters χ^i. For each i and ι one has to check whether

$$t^{\chi}_H(\chi^i_H) = \chi^{\iota}_{tHt^{-1}} \,.$$

If equality holds then $t^I_H(i) = \iota$.

Proposition 5.13 : *Let H be a proper subgroup of G and K a proper subgroup of H. Let $d_{i\underline{j}}$ and $C^{i\underline{j}}$ denote the innerconnectivity multiplicities and matrices with respect to H and K. Let c^H_i denote the multiplicity of ϑ^i_H of $\vartheta \downarrow H$ for a given representation $\vartheta : G \to GL(\mathbb{R}^n)$. Then*

a.) $d_{i\underline{j}} = d_{t^I_H(i) t^I_K(\underline{j})} \quad \forall t \in G \quad \forall\, i = 1, \ldots, h_H, \underline{j} = 1, \ldots, h_K$

b.) $\vartheta^i_H(t) C^{i1}$ *is for all $t \in H$ the innerconnectivity matrix with respect to H and tKt^{-1}.*

c.) $c^H_i = c^{tHt^{-1}}_{t^I_H(i)} \quad \forall\, t \in G.$

Proposition 5.14 : *If K is a bifurcation subgroup of H of type i, then $\forall\, t \in G$ the conjugate group tKt^{-1} is a bifurcation subgroup of tHt^{-1} of type $t^I_H(i)$. If H is a relevant subgroup with respect to $\vartheta : G \to GL(\mathbb{R}^n)$ then $tHt^{-1}, \forall\, t \in G$ are relevant subgroups.*

Restriction to non-conjugate groups

This principle of conjugacy has consequences for the numerical computations as well as for the preparing group theoretic computations. The numerical pathfollowing procedure is applied to the reduced systems (6) with respect to non-conjugate relevant subgroups only.

Solutions of (2) including symmetry breaking bifurcation points are conjugated by cycle elements. Because the type and bifurcation subgroups are stored together with a bifurcation point, a sophisticated administration of bifurcation points is needed. This necessitates the knowledge of t^I_H and the action of t on the subgroups of G.

Once a symmetry breaking bifurcation point with isotropy of a non-conjugate H is computed formula (22) gives the directions of emanating branches. In

Symcon the offset directions are computed for all bifurcation subgroups K which are non-conjugate with respect to H. The other directions are given by conjugation. Computation of offset directions for subgroups K which are non-conjugate in G could have the disadvantage that the bifurcation supergroup is a conjugate group of H and thus more of the innerconnectivity matrices $C^{i\underline{j}}$ are needed in the symbolic part (see below).

Pathfollowing a branch with non-conjugate isotropy the symmetry monitor functions with respect to all bifurcation supergroups have to be considered.

Consequences for group theoretic computations

Recall

$$\vartheta_H^i \downarrow K = \sum_{\underline{j}=1}^{h_K} d_{i\underline{j}} \vartheta_K^{\underline{j}}.$$

The innerconnectivity multiplicities $d_{i\underline{j}}$ are computed with respect to non-conjugate subgroups H and non-conjugate subgroups K of H (conjugacy in H!). The others are given by conjugation. Especially for $t \in N_H(K)$ the relation $d_{i\underline{j}} = d_{i\,t_K^I(\underline{j})}$ for the conjugate irreducible representations with numbers $\underline{j}$ and $t_K^I(\underline{j})$. Moreover Clifford's theory ([1], [15]) states: if K is normal in H (which means $N_H(K) = H$) then $\underline{j} \in \{1, \ldots, h_H\}$ exists with

$$\vartheta_H^i \downarrow K = d_{i\underline{j}} \sum_{t \in H/K} \vartheta_K^{t_K^I(\underline{j})}. \tag{29}$$

As few as possible innerconnectivity matrices $C^{i\underline{j}}$ should be computed. They have to be computed for G and all non-conjugate H. Additionally, some other C^{i1} are needed. Because conjugate solutions with isotropy $\tilde{H}$ are computed numerically using $M_1^{\tilde{H}}$, in the symbolic preparation of $M_1^{\tilde{H}}$ the matrices C^{i1} with respect to G and conjugate groups of H are needed. In the computation of these C^{i1} itself Proposition 5.13 is applied. Secondly, these includes matrices C^{i1} which are needed for the determination of offset directions. K being a bifurcation subgroup of a non-conjugate subgroup H and K being non-conjugate in H, the matrices C^{i1} with respect to G and K are needed. Thirdly, some C^{i1} are needed for the symmetry monitor functions. To avoid the use of $C^{i\underline{j}}, \underline{j} \geq 2$ with respect to G and conjugate subgroups $\tilde{H}$, the formulas (23) and (24) for the symmetry monitor functions have to be modified. On a branch of solutions with non-conjugate isotropy group K the monitor functions recognize $\tilde{H}$–invariant points, where $\tilde{H}$ is not known to be a member of the chosen non-conjugate relevant subgroups. Let H be a chosen nonconjugate relevant subgroup which is conjugate to $\tilde{H}$. Then $t \in G$ exists with $\tilde{H} = tHt^{-1}$ and $\tilde{K} = tKt^{-1}$.

$$\hat{u}^1_{\tilde{K}} = \left(C^1_{H\tilde{K}}\right)^T (C_{GH})^T \vartheta(t^{-1}) C^1_{GK} u^1_K \,, \tag{30}$$

gives an alternative symmetry monitor, which is equivalent to the detection of points with isotropy H on the branch with isotropy $\tilde{K}$.

For this formula the matrices C^{i1} with respect to non-conjugate H and its bifurcation subgroups K are necessary. Computing these C^{i1} itself again Propostion 5.13 b.) may be used.

During the numerical computations the action $t^I_{\tilde{H}}$ on irreducible representations (28) is needed only for some group elements t and subgroups $\tilde{H}$, if H is a non-conjugate relevant subgroup and $t = r_i$ a member of its cycle with $s = s_{i-1} = r_{i-1} \cdot \ldots \cdot r_1$ and $\tilde{H} = sHs^{-1}$.

6. Overview of the algorithm

The group theoretic computations in SYMCON simulate a mathematician who has read the analysis (equivariant braching lemma), looks up the irreducible representations of the group, and then prepares and implements a given equivariant system (2).

For the second point the SYMMETRY Package ([6]) is used which contains functions for the computation of symmetry adapted bases and a database containing irreducible representations for the small dihedral groups, the symmetric group S_4 and others.

Problem independent part

The first step is the determination of all subgroups which is implemented like the other grouptheoretic computations in RLISP. Starting with an arbitrary set of elements of fixed order including a known subgroup (e.f. { id }) elements are eliminated and new are chosen with a weighting procedure until a subgroup is found. This works fine for the small groups from the SYMMETRY Package.

A group isomorphism between these computed subgroups and a stored abstract one from the Package are constructed based on a search of equivalent generators. This enables the use of the stored irreducible representations for the subgroups.

While conjugacy is exploited the rest of the action of G on the subgroup tree is determined giving the normalizers as well. The algorithm from representation theory gives the innerconnectivity multiplicities $d_{i\underline{j}}$ and matrices $C^{i\underline{j}}$ which determines with Lemma (3.6) the bifurcation graph. The computation of cycles terminates the problem independent part which ran only once.

Organization of numerical computations

Based on some stored information ($\vartheta_G^i, d_{i\underline{j}}, C_{i\underline{j}}, t_H^I$, bifurcation graph with conjugate and symmetric groups, cycles) the equivariant systems (2) are tackled starting with determination of multiplicities c_i^G, c_i^H (10), the relevant subgroups (see 3.9) and parts $M_{i\underline{j}}^G$ of the transformation matrix.

During these computations and the following C code generation the intensive exploitation of conjugacy was the important aim.

First the group operations $\vartheta(t)$ including its action on the tree of relevant subgroups and the actions on irreducible representations t_H^I (see 28) needed for bifurcation point administration are generated with GENTRAN.

Then for each relevant subgroup coordinate transformation (15), symmetry monitors (25) and offset directions (22) are generated.

A clear distinction between a set of non-conjugate relevant subgroups and their conjugates are made as the reduced system (6) and Jacobian blocks A_i are generated for the non-conjugates only.

Their evaluation is a difficult point, see below. In the numerical part the pathfollowing of non-conjugate solutions is done with ALCON (see [3], [4]) applied to the reduced systems. The conjugate solutions are computed by group operations in a cycle.

While pathfollowing the symmetry monitor functions and determinants of Jacobian blocks are evaluated detecting bifurcation points leading to higher or smaller symmetry. The block structure is exploited as well for the computation of eigenvalues giving the stability of solutions and detection of Hopf points. The third use is the modification of step length by subcondition of the blocks A_i.

Independent on whether the detection is done with determinants or bordering, the computation of bifurcation points is done with an augmented

```
static void gD6D313(U,g)
double *U, *g;
{
   transformD6D313(U,ygID);
   ELEM_RD6(ygID,ygRD6);
   g[1]=f3(ygID);
   g[2]=(SQRT3*SQRT2*f4(ygRD6)+SQRT3*SQRT2*f4(ygID))/2.0;
   g[3]=(SQRT3*SQRT2*f6(ygRD6)+SQRT3*SQRT2*f6(ygID))/2.0;
   g[4]=(-(SQRT3*SQRT2*f4(ygRD6))+SQRT3*SQRT2*f4(ygID))/2.0;
   g[5]=(-(SQRT3*SQRT2*f6(ygRD6))+SQRT3*SQRT2*f6(ygID))/2.0;
}
```

Figure 6: D_3-reduced equations for the hexagonal lattice dome

system where A_i appears directly ([4], [23]) or indirectly in a testfunction ([24]) the use of the block structure is essential.

No extra work has to be done to avoid multiplication with zeros in M_{ij}. So the extra work in [18] is superfluous.

Exploitation of quasi-permutation representation

If the matrices $\vartheta(t)$ contain a lot of zeros, 1 and -1, then an evaluation of $\tilde{g}(\tilde{u},\lambda) = (M_1^H)^T f(M_1^H\tilde{u},\lambda)$ and $A_i(\tilde{u},\lambda) = (M_{i1}^H)^T D_x f(M_1^H\tilde{u},\lambda) M_{i1}^H$ by evaluating f and df at $(x,\lambda) = (M_1\tilde{u},\lambda)$ numerically and numerical matrix multiplications means an unnecessary computation with the complete system. Thus in the new version of SYMCON the functions $f_k(t,x,\lambda)$ and $df_{\nu\mu}(t,x,\lambda)$ are generated for a set of indices which are computed with Computer Algebra methods. For the case of a pure permutation representation see also [10].

Because the matrix multiplications with M_{ij}^H are done in REDUCE, the numerical evaluations of $\tilde{g}(\tilde{u},\lambda)$ and A_i consist of function calls of f_k and $df_{\nu\mu}$ with different conjugate vectors $\vartheta(t)M_1^H\tilde{u}$, where t are representatives of the left cosets of H in G (see Fig. 6). This is also the best way in view of minimization the amount of Code.

It remains to mention that for the trivial group instead of $M = M^G C_{G\{id\}}$ no coordinate transformation is done.

```
static void invtransformD6D312(Y,u)
double *Y,*u;
{
    u[1]=Y[3];
    u[2]=(Y[19]*SQRT3*SQRT2+Y[16]*SQRT3*SQRT2+Y[13]*SQRT3*SQRT2+Y[10]*SQRT3*
     SQRT2+Y[7]*SQRT3*SQRT2+Y[4]*SQRT3*SQRT2)/6.0;
    u[3]=(Y[21]*SQRT3*SQRT2+Y[18]*SQRT3*SQRT2+Y[15]*SQRT3*SQRT2+Y[12]*SQRT3*
     SQRT2+Y[9]*SQRT3*SQRT2+Y[6]*SQRT3*SQRT2)/6.0;
    u[4]=(-(Y[20]*SQRT3*SQRT2)+Y[17]*SQRT3*SQRT2-(Y[14]*SQRT3*SQRT2)+Y[
11]*SQRT3
     *SQRT2-(Y[8]*SQRT3*SQRT2)+Y[5]*SQRT3*SQRT2)/6.0;
    u[5]=Y[22];
}
```

Figure 7: Coordinate transformation $(\tilde{u}, \lambda) = (M_1^{D_3})^T(x, \lambda)$ for the hexagonal lattice dome

dimension degree	2	3	4
10	6600	28600	100100
20	92400	708400	4250400
50	3315000	58565000	790627500

Table 1: Number of possible terms in a Jacobian consisting of polynomials

example	REDUCE	compile	link	numeric	file
brussD3	20150 ms	9.2u sec	6.5u sec	2.3 sec	16960 Bytes
brussD4	55280 ms	19.3u sec	6.3u sec	8.38 sec	33211 Bytes
brussD6	350980 ms	73.7u sec	6.7u sec	16.8 sec	101741 Bytes
brussS4	127570 ms	43.0u sec	6.7u sec	8.28 sec	61168 Bytes
dome	52 min	333.2u sec	8.4u sec	1558.6 sec	308723 Bytes

Table 2: Performance of Symcon on a Data General

group	subgroup search	identify subgroups	bifgraph
K4	30 ms	50 ms	250 ms
D3	170 ms	90 ms	560 ms
D4	330 ms	340 ms	1370 ms
C4	50 ms	30 ms	70 ms
C5	20 ms	20 ms	200 ms
C6	110 ms	90 ms	490 ms
D6	3440 ms	850 ms	7240 ms
A4	6270 ms	420 ms	1830 ms
S4	216510 ms	9530 ms	17630 ms

Table 3: Performance of computation of bifurcation graphs on a Data General

7. The battle between REDUCE and C

For evaluation of the mixture of symbolic and numeric computations one has to take into account that the symbolic part is normally done by hand. In contrast to numerical computations the algorithmic complexity of symbolic computations is much higher (see Table 1). Nevertheless the bottleneck is not REDUCE or GENTRAN but the compiler that cannot compile expressions larger than a certain size – the assumption obviously being that hand written code of such size cannot be correct anyway.

So the compilation of the Jacobian matrix of the hexagonal lattice dome in a former version of SYMCON was only possible after modifications of GENTRAN and thus changing the way of C Code generation.

Hence our most important aim was to produce as few as code as possible, which is done in SYMCON by exploitation of the quasi-permutation structure of the representation.

The advantages of symbolic computation in this context are

- the equivariance check (safety of implementation),
- generation of matrices avoiding unnecessary operations with zero (see Fig. 6 and 7),
- symbolic differentiation,

- handling of abstract functions (see Section 6.).

Table 2 gives the computing times for the hexagonal lattice dome mentioned in Section 8 and the brusselators with respect to different groups which have been often treated. The definition is given for example in [4]. The table shows that the computing time is an important criterion for ranking of numerical algorithms for larger problems only. For smaller problems the criteria of reliability, robustness, easy implementation, and transparency are other important criteria.

8. Example: Hexagonal lattice dome

In HEALEY [11] an example of a deformation of an hexagonal lattice dome is given. It was also treated in [21]. There are seven free nodes $I \in \{A, B, C, D, E, F, G\}$ with displacement vectors $x_I \in \mathbb{R}^3$ which form the unknowns $x = (x_A, \ldots, x_G) \in \mathbb{R}^{21}$ of the system. In [5] the example is fully described and an overview of stable solutions is given. We found more than 600 symmetry breaking bifurcation points of different types which are connected by ca. 200 branches of solutions with non-conjugate isotropy groups. Figure 8 gives an impression of the complexity of this problem.

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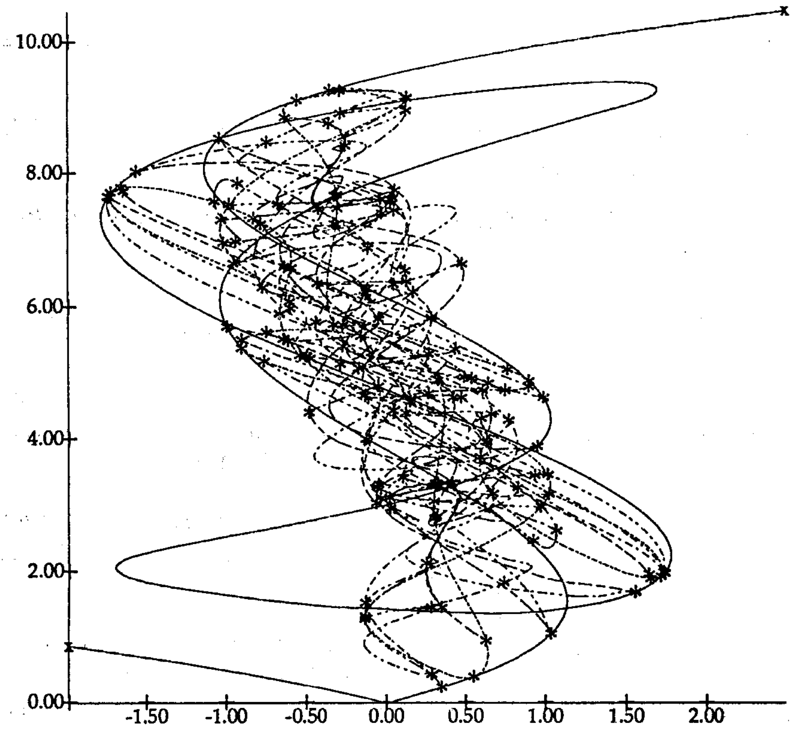

Figure 8: Hexagonal lattice dome($\|\cdot\|_2$ versus λ).

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