Zuse Institute Berlin

Takustr. 7
14195 Berlin
Germany

Berenike Masing[1]

Optimal Line Planning in the Parametric City

[1] 0000-0001-7201-2412

ZIB Report 21-26 (July 2021)

Zuse Institute Berlin
Takustr. 7
14195 Berlin
Germany

Telephone: +49 30 84185-0
Telefax: +49 30 84185-125

E-mail: bibliothek@zib.de
URL: http://www.zib.de

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

Optimal Line Planning in the Parametric City

Berenike Masing[1]

Zuse Institute Berlin
masing@zib.de

Abstract. We formulate the line planning problem in public transport as a mixed integer linear program ($MILP$), which selects both passenger and vehicle routes, such that travel demands are met with respect to minimized travel times for both operators and users. We apply $MILP$ to the *Parametric City*, a generic city model developed by Fielbaum et al. [2]. While the infrastructure graph and demand are entirely rotation symmetric, asymmetric optimal line plans can occur. Using group theory, we analyze the properties of symmetric solutions and introduce a *symmetry gap* to measure their deviation of the optimum. We also develop a $1 + \frac{1+\sqrt{2}}{g}$-approximation algorithm, depending only on the cost related parameter g. Supported by computational experiments, we conclude that in practice symmetric line plans provide good solutions for the line planning problem in the Parametric City.

Keywords: line planning, city modelling, symmetry, mixed integer programming, approximation algorithm

1 Introduction

The goal of line planning is to determine the most efficient routes, as well as frequencies of service in order to satisfy travel demands in a city. We do so with the help of a mixed integer linear programming problem ($MILP$) formulation, whose objective is to minimize both operator as well as passenger travel times combined by a scalarization parameter. It considers all circuits in a graph as potential lines and all simple paths as passenger routes. Good line plans must generally be computed — at great expense due to the model size — for each city individually, their solutions are difficult to compare and cannot be applied to other cities. Our approach is therefore use the *Parametric City*, a generic model developed by Fielbaum et al. [2] for the purpose of designing transportation services. It can be adjusted to represent the most characteristic aspects of the city, such as its geography, as well as the degree of mono-, polycentricity and dispersion. The Parametric City is entirely rotation symmetric — it is therefore natural to assume that this symmetry is reflected in the optimal line plans. However, there are cases, in which the optimal line plans are asymmetric. Our main attention is on this influence of symmetry: On the optimal solutions and how much a symmetric solution deviates from its optimum. We examine in which cases optimal solutions must be symmetric, when they can be utilized as good approximations and in which cases it is detrimental to assume symmetry in the line plans.

2 The Parametric City

We choose the Parametric City [2] as city representative, since this model balances generality and simplicity – it can represent any city and its prominent features, while remaining simple enough to be analyzable. It is comprised of a infrastructure graph $\mathcal{G} = (V, A)$ (see Fig. 1) with $2n + 1$ vertices and a demand $d_{s,t}, (s,t) \in V \times V$ (see Table 1). Table 2 gives an overview of the parameters.

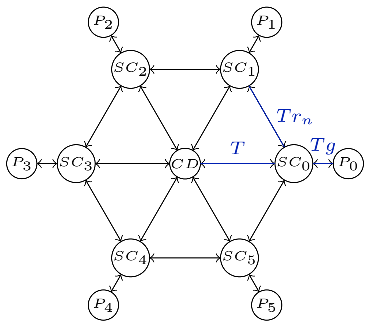

Fig. 1: Graph $\mathcal{G}$ with $n = 6$

s,t	SC_i	$SC_j, j \neq i$	CD
P_i	$\frac{aY}{n}\beta$	$\frac{aY}{n(n-1)}\gamma$	$\frac{aY}{n}\alpha$
SC_i	0	$\frac{(1-a)Y}{n(n-1)}\tilde{\gamma}$	$\frac{(1-a)Y}{n}\tilde{\alpha}$

Table 1: Demand $d_{s,t}$ (not listed vertex-pairs correspond to $d_{s,t} = 0$)

n	no. of subcenters/peripheries
T g, r_n	arc length (SC_i, CD) factors for arc length $(SC_i, P_i), (SC_i, SC_{i\pm1})$
Y a	total patronage f.o.t. from P_i
$\alpha\,(\tilde{\alpha})$ β $\gamma\,(\tilde{\gamma})$	f.o.t. from $P_i\,(SC_i)$ to CD f.o.t. from P_i to SC_i f.o.t. from $P_i\,(SC_i)$ to SC_j, $i \neq j$
	$\alpha + \beta + \gamma = 1,\ \tilde{\alpha} + \tilde{\gamma} = 1,$ $\alpha/\gamma = \tilde{\alpha}/\tilde{\gamma}$

Table 2: Parameters in the Parametric City, f.o.t. = fraction of travelers $\in [0, 1]$

2.1 Rotation Symmetry

The graph $\mathcal{G}$ is evidently rotation symmetric. While the demand matrix is not symmetric in the usual sense, the demand itself is rotation symmetric as well: E.g., the demand from a periphery P_i to the central business district CD is the same as that of any other periphery P_j to CD. This notion of symmetry can be precisely defined with the help of group actions:

The group $G = \mathbb{Z}/n\mathbb{Z}$ acts on the vertices of $\mathcal{G}$ by rotation around CD. This action extends to any tuple of vertices, in particular arcs, path, and lines. We denote by $G \cdot x$ the group orbit of a vertex (tuple) x. E.g., $G \cdot SC_0$ corresponds to the set of all subcenters, and $G \cdot CD = \{CD\}$. With this group action, we describe the rotation symmetry of the demand by the property of $d_{s,t} = d_{s',t'}$ for all $(s', t') \in G \cdot (s,t)$ for any vertex tuple $(s,t) \in V \times V$.

3 The Line Planning Problem

We formulate the line planning problem $(MILP)$ as a mixed integer program using two types of variables: $y_p \in \mathbb{R}$ for the passenger flow on path $p \in P$, and

$f_l \in \mathbb{N}$ for the frequency of line $l \in L$. P is the set of all simple paths, while L is the line pool consisting of all simple directed cycles in $\mathcal{G}$. We denote the sets P_a and L_a as the set of paths and lines which use arc $a \in A$. Further, $P_{s \to t}$ is the set of all s-t-paths. This follows [1], with a few minor changes: We restrict to only one mode of transport, do not include line-activation costs, and expand the line pool to include all circular, not only bidirectional lines.

$$
\begin{array}{lll l}
(MILP) & (MILP_A) & & (1) \\
\min \mu \sum_{l \in L} c_l f_l + (1-\mu) \sum_{p \in P} c_p y_p & \min \mu \sum_{a \in A} c_a F_a + (1-\mu) \sum_{p \in P} c_p y_p & & (2) \\
\text{s.t.} \sum_{p \in P_{s \to t}} y_p = d_{s,t} & \text{s.t.} \sum_{p \in P_{s \to t}} y_p = d_{s,t} & \forall (s,t) \in V \times V & (3) \\
\sum_{p \in P_a} y_p - \sum_{l \in L_a} f_l K \le 0 & \sum_{p \in P_a} y_p - F_a K \le 0 & \forall a \in A & (4) \\
\sum_{l \in L_a} f_l \le \Lambda & F_a \le \Lambda & \forall a \in A & (5) \\
 & \sum_{a \in \delta_v^+} F_a - \sum_{a \in \delta_v^-} F_a = 0 & \forall v \in V & (6) \\
f_l \in \mathbb{N} \quad \forall l \in L & F_a \in \mathbb{N} & \forall a \in A & (7) \\
y_p \ge 0 & y_p \ge 0 & \forall p \in P & (8)
\end{array}
$$

For a solution (f, y) of $MILP$, $f = (f_l)_{l \in L}$ is called the line plan and $y = (y_p)_{p \in P}$ the passenger flow. A line l is part of the line plan if and only if $f_l > 0$, analogously for the passenger flow.

We refer to [5] for an explanation of the constraints. The objective is a combination of operator and user costs respectively and are scalarized by parameter $\mu \in [0, 1]$. The larger μ, the more focus lies on the minimization of operator costs, while a small μ aims at user-friendly line plans. We consider the running and travel times as the total length of a line or path, i.e., $c_l = \sum_{a \in l} c_a$ and $c_p = \sum_{a \in p} c_a$, where c_a is the length of arc $a \in A$ as defined in the Parametric City, cf. Fig. 1.

As costs depend on the arc-lengths of the routes only, we can reformulate and hence reduce the model size significantly: Instead of considering the large line pool as variables, one can consider the frequencies of all aggregated lines traversing an arc, i.e., by considering $F_a := \sum_{l \in L_a} f_l$. To model the circulations of the lines, we can impose standard flow conservation constraints (6), where δ_v^+ and δ_v^- denote the set of out- and incoming arcs at node v respectively. The entire arc-based mixed integer linear programming problem $MILP_A$ can be found on the right of Definition 3. In fact, the following holds:

Lemma 1. *$MILP$ and $MILP_A$ are equivalent, in the sense that for a feasible solution (F, y) to $MILP_A$ there exists a feasible solution (f, y) to $MILP$ with $cost_A(F, y) = cost(f, y)$ and vice versa.*

4 Symmetry

Definition 1 (Symmetric Solution). *Consider a solution (f, y) to MILP and the equivalent solution (F, y) to $MILP_A$. Then*

1. *(f, y) is line-symmetric if $f_l = f_{l'}$ for all $l' \in G \cdot l, l \in L$,*
2. *(f, y) or (F, y) is path-symmetric if $y_p = y_{p'}$ for all $p' \in G \cdot p, p \in P$,*
3. *(f, y) is arc-symmetric if $\sum_{l \in L_a} f_l = \sum_{l \in L_{a'}} f_l$ for all $a' \in G \cdot a, a \in A$,*
4. *(F, y) is arc-symmetric if $F_a = F_{a'}$ for all $a' \in G \cdot a, a \in A$.*

The solution (f, y) is symmetric if it is line- and path-symmetric, while (F, y) is symmetric if it is arc- and path-symmetric.

Proposition 1 (Sufficient condition for symmetry). *A line-symmetric, arc-symmetric or path-symmetric optimal solution is sufficient for the existence of an entirely symmetric optimal solution.*

Thus, to determine an optimal value of symmetric solutions, it is enough to impose symmetric arc-frequency conditions (Def. 1,4) on $MILP_A$. We denote the model as $MILP_{sym}$. As we have only six orbits on the set of arcs in the Parametric City, $MILP_{sym}$ reduces to a problem with a fixed number of variables. Due to further geometric properties, this number gets reduced even further to only three variables. This has significant consequences: As was proven by Lenstra [4], a mixed integer programming problem with a fixed number of variables can be solved in polynomial time. Hence:

Proposition 2. *The symmetric line planning problem $MILP_{sym}$ is solvable in polynomial time.*

Given a feasible general solution (F, y), it is always possible to construct a feasible symmetric solution (F^s, y^s) by taking (rounded) averages per orbit. This allows for an estimate of how much the optimal solution deviates from a symmetric one at most:

$$cost(F^s, y^s) - cost(F, y) \leq \mu 2T(1 + r_n)(n - 1). \tag{9}$$

Consequently, if we optimize for user comfort only, i.e., if $\mu = 0$, the existence of a symmetric optimal line plan is guaranteed. However, for other values of μ we introduce the *symmetry gap* $\Gamma := \frac{OptVal(MILP_{sym})}{OptVal(MILP_A)}$. This gives us the means to measure the quality of a symmetric solution compared to an asymmetric one.

Proposition 3. *The relative symmetry gap Γ is bound by:*

$$\Gamma \leq C_n(\alpha, \gamma) \leq C_n \leq 1 + \frac{(1 + \sqrt{2})}{g}.$$

The values of $C_n(\alpha, \gamma)$ and C_n can be computed efficiently by using the bound (9) and the optimal value of the LP-relaxation, which can be computed analytically; a detailed description can be found in [5]. Due to $MILP_{sym}$ being solvable in polynomial time and in combination with Propoisiton 3, we can formulate the following proposition:

Proposition 4. *The algorithm that arises from restricting the line planning problem to symmetric solutions is a* $1 + \frac{(1+\sqrt{2})}{g}$*-approximation algorithm for the Parametric City if the cost related parameter* g *is fixed.*

To ascertain how the gap behaves when g goes to 0, consider the following instance: Set $g = 1/n, \mu = 1$ and choose a very large capacity, e.g., set $K = Y$. In other words, the entire patronage can fit into a single vehicle. Regardless of any other parameter choice, the optimal frequency plans of $MILP_A$ and $MILP_{sym}$ are the ones displayed in Fig. 2 on the left and right respectively. The corresponding gap becomes arbitrarily large for $n \to \infty$. However, this extreme example is

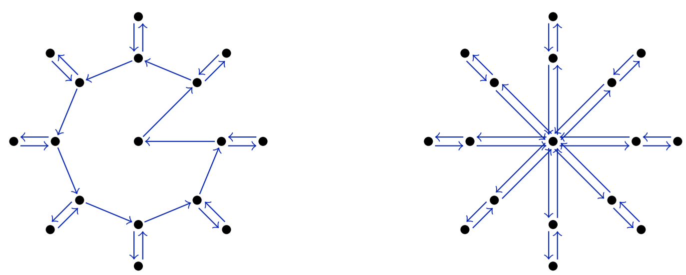

Fig. 2: Optimal frequency plans: general vs. symmetric model

constructed and does not reflect realistic input data. To assess whether and how often in practice purely asymmetric solutions occur, and how large the gap becomes, we solved multiple large batches of Parametric City instances for realistic parameter choices.

4.1 Computational Results

We performed multiple computations, choosing various geometry-related parameters in the Parametric City, as well as different values of μ. For each choice, we computed the optimal solution for all demand parameters $\alpha, \gamma \in [0.025, 0.95]$ and a step size of 0.025. Each problem was solved with Gurobi 9 [3] to optimality (with a tolerance of 10^{-4} in the relative MIP gap)in three variations: the standard $MILP_A$, its symmetric version $MILP_{sym}$, as well as the restriction to both rotation and reflection symmetric solutions by imposing additional reflection symmetry constraints. See below a representative example for the choices of $n = 8, g = 1/3, Y = 24000, K = 100, \mu = 1, a = 0.8$. Surprisingly, even with realistic results, purely asymmetric solutions can be found, as becomes evident on the left of Fig. 3. When comparing different values of μ, one can also observe that their number increases with μ. For a more in-depth comparison see [5]. Also noticeable is the fact that reflection symmetry occurs in the rotation symmetric solutions roughly as often as not. This observation is unexpected, since the demand is also reflection symmetric. When looking at the symmetry gap Γ how-

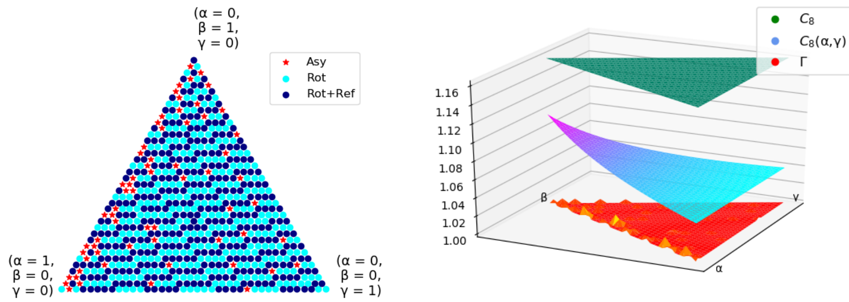

Fig. 3: Realistic input data

ever, it becomes evident that the difference in costs is extremely small. For this instance, the actually computed deviation of symmetric solutions from the optimal is less than 2%, as becomes evident on the right of Fig. 3 and is significantly smaller than the theoretical upper bounds also depicted (cf. Proposition 3).

We conclude that in practice, city planners are justified in assuming symmetric solutions: The line plans can either be considered as optimal, due to the somewhat idealized underlying city model, or one can use them as good and easy to compute approximations. With the exception of when operator costs are ignored, the possibility of asymmetric solutions should be kept in mind however, as unfortunate parameter choices can lead to a large deviation in costs.

References

1. Borndörfer, R., Grötschel, M., Pfetsch, M.E.: A column-generation approach to line planning in public transport. Transportation Science **41**(1), 123–132 (2007). doi: 10.1287/trsc.1060.0161
2. Fielbaum, A., Jara-Díaz, S., Gschwender, A.: A parametric description of cities for the normative analysis of transport systems. Networks and Spatial Economics **17** (2016). doi: 10.1007/s11067-016-9329-7
3. Gurobi Optimization, LLC: Gurobi optimizer reference manual (2020). http://www.gurobi.com
4. Lenstra, H.W.: Integer programming with a fixed number of variables. Mathematics of Operations Research **8**(4), 538–548 (1983). http://www.jstor.org/stable/3689168
5. Masing, B.: Optimal line planning in the parametric city. Master's thesis, Technische Universität Berlin (2020)