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Abstract: In this paper, two classes of second order accurate high resolution schemes are presented on regular triangular meshes for initial value problem of two dimensional conservation laws. The first class are called Runge-Kutta-FVM MmB (locally Maximum- minimum Bounds preserving) schemes, which are first discretized by (FVM) finite volume method in space direction and modifying numerical fluxes, and then by Runge-Kutta methods in time direction; The second class, constructed by Taylor expansion in time, and then by FVM methods and making modifications to fluxes, are called Taylor- FVM MmB schemes. MmB properties of both schemes are proved for 2-D scalar conservation law. Numerical results are given for Riemann problems of 2-D scalar conservation law and 2-D gas dynamics systems and some comparisons are made between the two classes of the schemes.

Key words and phrases: MmB schemes, 2-D, conservation laws, gas dynamics systems, Runge-Kutta-FVM, Taylor-FVM.

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1. Introduction

It is well known that many efficient difference schemes have been produced to solve one dimensional conservation laws, especially, TVD schemes [1][2][3][4][5][6][7], which are first presented by Harten [1]; MUSCL schemes by van Leer [8][9], ENO schemes [10][11]. For 2-D conservation laws, splitting schemes by 1-D TVD schemes have been used to solve practical problems [12][13][14]. Unfortunately, from [15], we know that any 2-D conservative TVD scheme is at most first order accurate, although numerical results have shown that the splitting methods using second order accurate 1-D TVD schemes seem to work quite well for practical problems in fluid dynamics systems. Henece it is neccesery to present a new concept beyond TVD in two dimensions.

In [16][17], nonsplitting upwind difference schemes were presented for two dimensional Euler equations. In [18] [19], a class of second order accurate high resolution and nonoscillatory schemes, which were called local Maximum and minimum Bounds preserving (MmB) schemes, were derived for 1-D and 2-D. From the schemes we can see that MmB and TVD schemes of the form [18][19] are almost identical in 1-D, but there are second order accurate high resolution 2-D MmB schemes for 2-D scalar conservation law. Numerical results were obtained by using unsplitting second order accurate MmB schemes for Riemann problems of 2-D scalar conservation law [20], 2×2 nonlinear hyperbolic systems in conservation laws [21] and 2-D gas dynamics systems [22][23].

All the papers mentioned above for 2-D conservation laws were considered on rectangular meshes. For triangular meshes, finite volume methods on general triangular meshes were presented by Jameson in [24] [25]. In his papers, the schemes were modified by using pressure for 2-D Euler equations from the experiences of numerical computations, and several first order accurate upwind schemes called Godunov schemes were listed on triangular meshes [26][27]. A class of second order accurate MmB schemes were presented on regular triangular meshes for 2-D scalar conservation law in [28].

In the paper, using the experiences of constructing MmB schemes on rectangular meshes [18][19], we present two classes of second order accurate high resolution schemes on regular triangular meshes for 2-D scalar conservation law in section 2 and 3, respectively. The first class of the schemes called Runge-Kutta -FVM MmB schemes are derived, and the second class of the schemes which are called Taylor-FVM MmB schemes. In section 4 we generalize the two classes of the schemes to 2-D sysytems in conservation laws; In the last section numerical results are given for Riemann problems in three pieces of 2-D scalar conservation law and gas dynamics systems and in four pieces for gas dynamics systems.

Before the detail discriptions to the methods, it is necessary to recall the properties of initial value problem for 2-D scalar conservation law from [18].

Consider the initial value problem for 2-D scalar conservation law,

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 \\ u(x,y,t)|_{t=0} = u_0(x,y) \end{cases} \tag{1.1}$$

where $u_0(x,y)$ is a piecewise smooth function.

From [18], let w(p) be the neiborhood of point p in the plane $t = t_0$. By the characteristic relation of (1.1), consider the values of the solution u in w(p) depend on the values of the interval I(w(p)), then the following inequalities are satisfied

$$\begin{array}{ccccc} inf u(Q) & \le & u(Q') & \le & sup u(Q) \\ Q \in I(w(p)) & & Q' \in w(p') & & Q \in I(w(p)) \end{array} \tag{1.2}$$

where p' is a point in the plane $t = t_1$. (1.2) are basic properties of (1.1).

2. Runge-Kutta-FVM MmB schemes for 2-D scalar conservation law

Consider the initial value problem for 2-D scalar conservation law,

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \\ u(x,y,t)|_{t=0} = u_0(x,y) \end{cases} \tag{2.1}$$

where $u_0(x,y)$ is a piecewise smooth function.

First, according to Jameson [24], we divid R^2 into regular triangular meshes as follows,

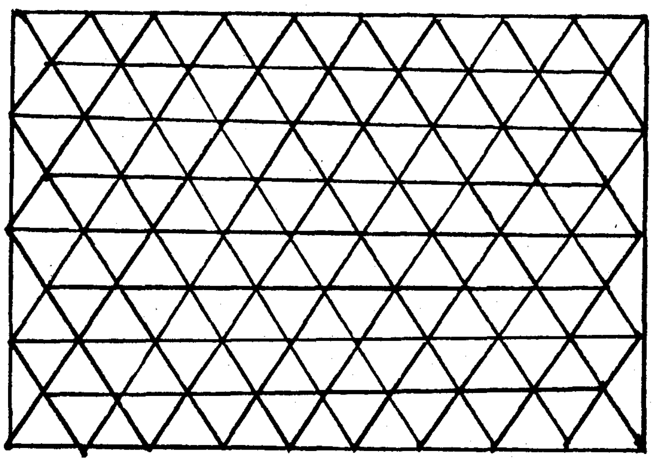

Fig. 2.1

We choosè a set of local meshes for neighborhood of point (x_i, y_i),

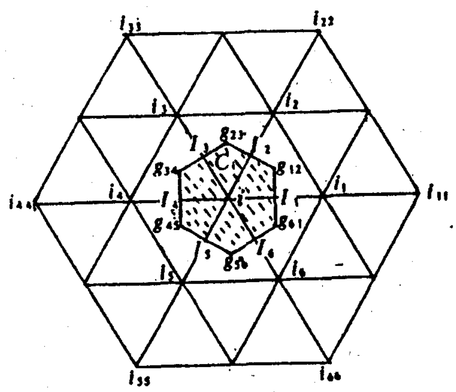

Fig. 2.2

Definition 2.1, A semidiscrete scheme,

$$\frac{\partial u_i}{\partial t} = C_i u_i + \sum_{j=1}^{6} C_{i_j} u_{i_j}$$

is called a semidiscrete MmB scheme if

$$C_i \le 0, \qquad C_{i_j} \ge 0, \qquad j = 1, ..., 6$$

and

$$C_i + \sum_{j=1}^{6} C_{i_j} = 0$$

.

From the theory of ordinary differential systems, if

$$\frac{\partial U}{\partial t} = AU$$

where A is a M-matrix[33], then the systems are contractive. Definition 2.1 gives us a rule how to construct a semidiscrete scheme for 2-D conservation law so that we can get full discrete MmB schemes in this way.

Definition 2.2, A scheme

$$u_i^{n+1} = L_h u_i^n$$

is called a MmB scheme if

$$min(u_i, u_{i_1}, ..., u_{i_6}) \le u_i^{n+1} \le max(u_i, u_{i_1}, ..., u_{i_6}) \tag{2.4}$$

Condition (2.4) is equivalent to the form

$$u_i^{n+1} = C_i u_i^n + \sum_{j=1}^{6} C_{i_j} u_{i_j}, \quad C_i \ge 0, \quad C_{i_j} \ge 0 \tag{2.5}$$

and

$$C_i + \sum_{j=1}^{6} C_{i_j} = 1 \tag{2.6}$$

2.1 semidiscrete MmB schemes

Integrate (2.1) on C_i, due to Green's formula, we have,

$$\int\int_{C_i} u_t ds + \int_l (f\nu^x + g\nu^y) dl = 0 \tag{2.7}$$

where $l = \overline{g_{i_1} g_{i_2} \dots g_{i_j} g_{i_1}}$, $\nu = (\nu^x, \nu^y)$ is a unit out normal vector.

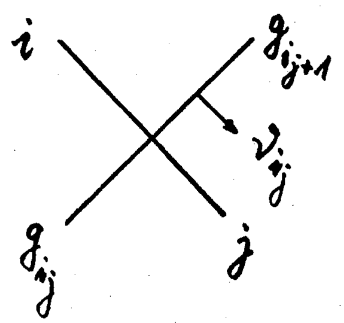

Fig. 2.3

From (2.7), a semidiscrete scheme is derived,

$$Ar(C_i)\frac{\partial u_i}{\partial t} + \sum_{j=1}^{6}[f(u_{I_j})|g_{i_j} g_{i_{j+1}}|\nu^x_{i_j} + g(u_{I_j})|g_{i_j} g_{i_{j+1}}|\nu^y_{i_j}] = 0 \tag{2.8}$$

where $|g_{i_j} g_{i_{j+1}}|$ is the distance of points g_{i_j} and $g_{i_{j+1}}$, and

$$Ar(C_i) = \int\int_{C_i} ds$$

Choose $f(u_{I_j})$ to $\frac{1}{2}(f(u_i) + f(u_{i_j}))$ and rewrite the formula. A second order accurate semidiscrete scheme becomes

$$Ar(C_i)\frac{\partial u_i}{\partial t} + \frac{1}{2}\sum_{j=1}^{3}[(f(u_{i_j}) - f(u_{i_{j+3}})\nu^x_{i_j} + (g(u_{i_j}) - g(u_{i_{i+3}}))\nu^y_{i_j}]|g_{i_j} g_{i_{j+1}}| = 0 \tag{2.9}$$

When scheme (2.9) is explicitly discretized in time direction, the scheme obtained is simple but not stable in L_2 and produces oscillation near discontinuities. In order to get stable and nonoscillatory scheme, here we modify scheme (2.9) so that the modified scheme is MmB.

Using the experiences of constructing MmB schemes on rectangular meshes [18][19], we give the following modified semidiscrete schemes,

$$\begin{aligned}\frac{\partial u_i}{\partial t} = & -S_i\sum_{j=1}^{3}[a^+_{i_{j+3}}(u_i-u_{i_{j+3}})+\tfrac{1}{2}a^+_{i_j}Q^+_{i_j}(u_{i_j}-u_i)-\tfrac{1}{2}a^+_{i_{j+3}}Q^+_{i_{j+3}}(u_i-u_{i_{j+3}})\\ & +a^-_{i_j}(u_{i_j}-u_i)-\tfrac{1}{2}a^-_{i_j}Q^-_{i_j}(u_{i_j}-u_i)+\tfrac{1}{2}a^-_{i_{j+3}}Q^-_{i_{j+3}}(u_i-u_{i_{j+3}})]\end{aligned} \tag{2.10}$$

where

$$S_i=|g_{i_j}g_{i_{j+1}}|/Ar(C_i), \qquad |g_{i_j}g_{i_{j+1}}|=h_i, \quad \forall j, \quad j=1,...,6$$

$$a^\pm=\tfrac{1}{2}[a\pm|a|], \qquad Q^\pm_{i_j}=Q(r^\pm_{i_j}), \qquad Q^\pm_{i_{j+3}}=Q(r^\pm_{i_{j+3}})$$

$$r^+_{i_j}=\frac{a^+_{i_{j+3}}(u_i-u_{i_{j+3}})}{a^+_{i_j}(u_{i_j}-u_i)} \qquad r^-_{i_j}=\frac{a^-_{i_{j_j}}(u_{i_{j_j}}-u_i)}{a^-_{i_j}(u_{i_j}-u_i)}$$

$$r^+_{i_{j+3}}=\frac{a^+_{i_{j+3_{j+3}}}(u_{i_{j+3}}-u_{i_{j+3_{j+3}}})}{a^+_{i_{j+3}}(u_i-u_{i_{j+3}})} \qquad r^-_{i_{j+3}}=\frac{a^-_{i_j}(u_{i_j}-u_i)}{a^-_{i_{j+3}}(u_i-u_{i_{j+3}})}$$

$$a_{i_j}=\begin{cases}[(f_{i_j}-f_i)\nu^x_{i_j}+(g_{i_j}-g_i)\nu^y_{i_j}]/(u_{i_j}-u_i), & u_{i_j}\neq u_i\\ \frac{\partial f}{\partial u}\nu^x_{i_j}+\frac{\partial g}{\partial u}\nu^y_{i_j}|_{u_i}, & u_{i_j}=u_i\end{cases}$$

$$a_{i_{j+3}}=\begin{cases}[(f_i-f_{i_{j+3}})\nu^x_{i_{j+3}}+(g_i-g_{i_{j+3}})\nu^y_{i_{j+3}}]/(u_i-u_{i_{j+3}}), & u_{i_{j+3}}\neq u_i\\ \frac{\partial f}{\partial u}\nu^x_{i_{j+3}}+\frac{\partial g}{\partial u}\nu^y_{i_{j+3}}|_{u_i}, & u_{i_{j+3}}=u_i\end{cases}$$

Rewrite scheme (2.10), we have,

$$\begin{aligned}\frac{\partial u_i}{\partial t}=-S_i\sum_{j=1}^{3} & [(1+\tfrac{1}{2}Q^+_{i_j}/r^+_{i_j}-\tfrac{1}{2}Q^+_{i_{j+3}})a^+_{i_{j+3}}(u_i-u_{i_{j+3}})\\ & +(1-\tfrac{1}{2}Q^-_{i_j}+\tfrac{1}{2}Q^-_{i_{j+3}}/r^-_{i_{j+3}})a^-_{i_{j+3}}(u_{i_j}-u_i)]\end{aligned} \tag{2.11}$$

By the definition to semidiscrete MmB scheme, we know that: if

$$\begin{cases}(1+\tfrac{1}{2}Q^+_{i_j}/r^+_{i_j}-\tfrac{1}{2}Q^+_{i_{j+3}})a^+_{i_{j+3}}\geq 0,\\ (1-\tfrac{1}{2}Q^-_{i_j}+\tfrac{1}{2}Q^-_{i_{j+3}}/r^-_{i_{j+3}})a^-_{i_j}\leq 0\end{cases}$$

scheme (2.10) is MmB.

Theorem 2.1, Let $Q(r) \geq 0$ and $Q(r) \equiv 0, r \leq 0$, then scheme (2.10) is a semidiscrete MmB scheme, if

$$Q(r) \leq 2$$

The proof of Theorem 2.1 is almost the same as the proof of the following Theorem 2.2.

2.2 Euler forward MmB schemes

For the sake of simplicity, here we mention that the condition (2.4) is also equivalent to the following form

$$u_i^{n+1} = u_i^n - \sum_{j=1}^{6} C_{ij}(u_i^n - u_{i_j}^n), \quad C_{ij} \geq 0, \quad \sum_{j=1}^{6} C_{ij} \leq 1 \tag{2.12}$$

Here we use the Euler forward method to discretize equations (2.10),

$$\begin{aligned} u_i^{n+1} = u_i^n - \quad & \lambda_i \sum_{j=1}^{3} [a_{i_{j+3}}^+ (u_i - u_{i_{j+3}}) + a_{i_j}^- (u_{i_j} - u_i) \\ & + \tfrac{1}{2} a_{i_j}^+ Q_{i_j}^+ (u_{i_j} - u_i) - \tfrac{1}{2} a_{i_{j+3}}^+ Q_{i_{j+3}}^+ (u_i - u_{i_{j+3}}) \\ & - \tfrac{1}{2} a_{i_j}^- Q_{i_j}^- (u_{i_j} - u_i) + \tfrac{1}{2} a_{i_{j+3}}^- Q_{i_{j+3}}^- (u_i - u_{i_{j+3}})] \end{aligned} \tag{2.13}$$

Theorem 2.2, Let $Q(r) \equiv 0$, if $r \leq 0$ and $Q(r) \geq 0$, then if

$$Q(r)/r \leq 2., \qquad Q(r) \leq 2$$

under the condition $\max_{i,j} |a_{i_j}| \lambda_i \leq \frac{1}{12}$, scheme (2.13) is a full discrete MmB scheme.

Proof: Rewrite (2.13) in the following form,

$$\begin{aligned} u_i^{n+1} = u_i - \lambda_i \sum_{j=1}^{3} \quad & [(1 + \tfrac{1}{2} Q_{i_j}^+ / r_{i_j}^+ - \tfrac{1}{2} Q_{i_{j+3}}^+) a_{i_{j+3}}^+ (u_i - u_{i_{j+3}}) \\ & + (1 - \tfrac{1}{2} Q_{i_j}^- + \tfrac{1}{2} Q_{i_{j+3}}^- / r_{i_{j+3}}^-) a_{i_j}^- (u_{i_j} - u_i) \end{aligned}$$

By condition (2.12), we let

$$
\begin{aligned}
&\lambda_i(1+\tfrac{1}{2}Q^+_{i_j}/r^+_{i_j}-\tfrac{1}{2}Q^+_{i_{j+3}})a^+_{i_{j+3}} \geq 0\\
&-\lambda_i(1-\tfrac{1}{2}Q^-_{i_j}+\tfrac{1}{2}Q^-_{i_{j+3}}/r^-_{i_{j+3}})a^-_{i_j} \geq 0
\end{aligned}
$$

and

$$
\lambda_i\sum_{j=1}^{3}[(1+\frac{1}{2}Q^+_{i_j}/r^+_{i_j}-\frac{1}{2}Q^+_{i_{j+3}})a^+_{i_{j+3}}-(1-\frac{1}{2}Q^-_{i_j}+\frac{1}{2}Q^-_{i_{j+3}}/r^-_{i_{j+3}})a^-_{i_j}] \leq 1
$$

then by signals of $a^+_{i_{j+3}}$ and $a^-_{i_j}$, we have

$$
\begin{aligned}
&1+\tfrac{1}{2}Q^+_{i_j}/r^+_{i_j}-\tfrac{1}{2}Q^+_{i_{j+3}} \geq 0\\
&1-\tfrac{1}{2}Q^-_{i_j}+\tfrac{1}{2}Q^-_{i_{j+3}}/r^-_{i_{j+3}} \geq 0
\end{aligned}
$$

and

$$
\begin{aligned}
&1+\tfrac{1}{2}Q^+_{i_j}/r^+_{i_j}-\tfrac{1}{2}Q^+_{i_{j+3}} \leq \frac{1}{6\lambda_i a^+_{i_{j+3}}}\\
&1-\tfrac{1}{2}Q^-_{i_j}+\tfrac{1}{2}Q^-_{i_{j+3}}/r^-_{i_{j+3}} \leq \frac{1}{6\lambda_i|a^-_{i_j}|}
\end{aligned}
$$

Under the assumption $Q(r) \geq 0$, Q(r)$\equiv$ 0, when $r \leq 0$ and $\max\limits_{i,j}|a_{i_j}|\lambda_i \leq \frac{1}{12}$, scheme (2.13) is MmB if the inequalities are satisfied,

$$
Q^+_{i_j}/r^+_{i_j} \leq 2, \qquad Q^-_{i_{j+3}}/r^-_{i_{j+3}} \leq 2
$$

and

$$
Q^+_{i_{j+3}} \leq 2 \qquad Q^-_{i_j} \leq 2
$$

.

2.3 Runge-Kutta MmB schemes

We know that schemes (2.13) obtained by Euler forward method are first order accurate in time direction. To get high order accurate schemes in time, we use Runge-Kutta methods to discretize (2.10). Write (2.10) in the following form.

$$
\frac{\partial u_i}{\partial t} = R(u)_i
$$

then Runge-Kutta MmB schemes are constructed

$$
\begin{aligned}
&u_i^{(0)} = u_i^n \\
&u_i^{(1)} = u_i^{(0)} - \alpha_1 \Delta t R_i^{(0)} \\
&\cdots \\
&u_i^{(m-1)} = u^{(0)} - \alpha_{m-1} \Delta t R_i^{(m-2)} \\
&u_i^{(m)} = u_i^{(0)} - \Delta t R_i^{(m-1)} \\
&u_i^{n+1} = u^{(m)}
\end{aligned}
\tag{2.14}
$$

where

$$
R^{(q)} = \sum_{r=1}^{q} \beta_{q,r} R^{(q)}, \qquad \sum_{r=1}^{q} \beta_{q,r} = 1
$$

From the structure of scheme (2.14), the MmB properties can be proved stage by stage.

3. Taylor-FVM MmB schemes for 2-D scalar conservation law.

In this section, first using Taylor expansion, we discretize (2.1) in time direction; and then derive full discrete schemes by FVM.

By Taylor expansion, we get the approximate equation which is second order accurate in time direction,

$$
\begin{aligned}
u^{n+1} &= u^n + \Delta t u_t^n + \tfrac{1}{2}\Delta t^2 u_{tt}^n \\
&= u^n - \Delta t (f_x + g_y) + \tfrac{1}{2}\Delta t^2 [(f_u(f_x + g_y))_x + (g_u(f_x + g_y))_y]
\end{aligned}
\tag{3.1}
$$

Integrate (3.1) on C_i (in section 2.1), we have

$$
\int\int_{C_i} u^{n+1} ds = \int\int_{C_i} u^n ds - \Delta t \int_l (f\nu^x + g\nu^y) dl + \frac{1}{2}\Delta t^2 \int_l (f_u \nu^x + g_u \nu^y)(f_x + g_y) dl
\tag{3.2}
$$

In (3.2), take u_i as the integral average u on C_i, that is

$$
u_i = \frac{1}{Ar(c_i)} \int\int_{C_i} u ds \tag{3.3}
$$

To discretize $\int_l (f_u \nu^x + g_u \nu^y)(f_x + g_y) dl$, now make transformation of the coordinates,

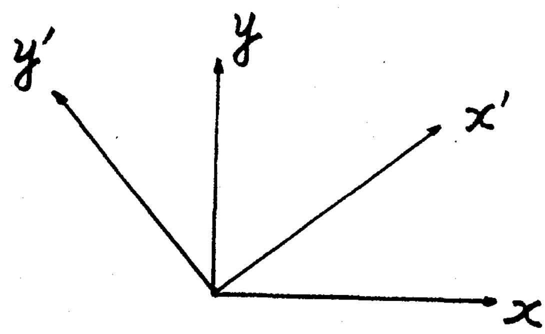

Fig. 3.1

$$\begin{cases} x' = \nu^x x + \nu^y y \\ y' = \nu^x y - \nu^y x \end{cases}$$

then

$$\begin{cases} f_x = f_{x'}\nu^x - f_{y'}\nu^y \\ g_y = g_{x'}\nu^y + g_{y'}\nu^x \end{cases}$$

and

$$f_x + g_y = f'_{x'} + g'_{y'}$$

where

$$f' = f\nu^x + g\nu^y, \qquad g' = g\nu^x - f\nu^y$$

(3.2) becomes

$$\begin{aligned} u_i^{n+1} = u_i^n - & \frac{\Delta t}{Ar(C_i)} \sum_{j=1}^{6} (f_{I_j}\nu_{i_j}^x + g_{I_j}\nu_{i_j}^y)|g_{i_j}g_{i_{j+1}}| \\ & + \frac{1}{2}\frac{\Delta t^2}{Ar(C_i)} \sum_{j=1}^{6} [(f_u\nu_{i_j}^x + g_u\nu_{i_j}^y)(f'_{x'} + g'_{y'})]_{I_j}|g_{i_j}g_{i_{j+1}}| \end{aligned} \tag{3.4}$$

According to (3.3), let

$$\begin{aligned} (g'_{y'})_{I_j} &= 0 \\ A_{i_j} &= (f_u\nu_{i_j}^x + g_u\nu_{i_j}^y)_{I_j} \\ &= \begin{cases} \dfrac{(f_{i_j} - f_i)\nu_{i_j}^x + (g_{i_j} - g_i)\nu_{i_j}^y}{u_{i_j} - u_i} & if \quad u_{i_j} \neq u_i \\ (f_u\nu_{i_j}^x + g_u\nu_{i_j})|_{u_i}, & if \quad u_{i_j} = u_i \end{cases} \\ (f'_{x'})_{I_j} &= \frac{1}{h}(f'_{i_j} - f'_i) \end{aligned}$$

then we get a second order accurate scheme which is like Lax-Wendroff scheme .

$$\begin{aligned} u_i^{n+1} = u_i^n & -\tfrac{1}{2}\frac{\Delta t|g_{i_1}g_{i_2}|}{Ar(C_i)}\sum_{j=1}^{3}[(f_{i_j}-f_{i_{j+3}})\nu_{i_j}^x+(g_{i_j}-g_{i_{j+3}})\nu_{i_j}^y)] \\ & +\tfrac{1}{2}\frac{\Delta t^2|g_{i_1}g_{i_2}|}{Ar(C_i)h}\sum_{j=1}^{3}[a_{i_j}^2(u_{i_j}-u_i)-a_{i_{j+3}}^2(u_i-u_{i_{j+3}})] \end{aligned} \tag{3.5}$$

By constructing MmB schemes on rectangular meshes [19], modified schemes of (3.5) are derived in the following form,

$$\begin{aligned} u_i^{n+1} = u_i^n & -S_i\sum_{j=1}^{3}[a_{i_{j+3}}^+(u_i-u_{i_{j+3}})+a_{i_j}^-(u_{i_j}-u_i)] \\ & -\tfrac{1}{2}S_i\sum_{j=1}^{3}[a_{i_j}^+(1-a_{i_j}^+\lambda)Q_{i_j}^+(u_{i_j}-u_i)-a_{i_{j+3}}^+(1-a_{i_{j+3}}^+)Q_{i_{j+3}}^+(u_i-u_{i_{j+3}}) \\ & -a_{i_j}^-(1+a_{i_j}^-\lambda)Q_{i_j}^-(u_{i_j}-u_i)+a_{i_{j+3}}^-(1+a_{i_{j+3}}^-\lambda)Q_{i_{j+3}}^-(u_i-u_{i_{j+3}})] \end{aligned} \tag{3.6}$$

where S_i, $Q_{i_j}^{\pm}$ and $Q_{i_{j+3}}^{\pm}$ are chosen as in section 2, and $\lambda=\frac{\Delta t}{h}$.

Theorem 3.1, Let $Q(r)\geq 0$, $Q(r)\equiv 0$ $(r\leq 0)$, if

$$Q_{i_j}^+/r_{i_j}^+ \leq \frac{2}{1-\lambda a_{i_j}^+} \qquad Q_{i_{j+3}}^-/r_{i_{j+3}}^- \leq \frac{2}{1+\lambda a_{i_{j+3}}^-}$$

and

$$Q_{i_{j+3}}^+ \leq \frac{2}{1-\lambda a_{i_{j+3}}^+} \qquad Q_{i_j}^- \leq \frac{2}{1+\lambda a_{i_j}^-}$$

under the condition $\max_{i,j}\lambda|a_{i_j}|\leq\frac{1}{12}$, scheme (3.6) is a MmB scheme.

Proof: Rewrite (3.6) in the form,

$$\begin{aligned} u_i^{n+1} = \ & u_i^n - S_i\sum_{j=1}^{3}[(1+\tfrac{1}{2}(1-a_{i_j}^+\lambda)Q_{i_j}^+/r_{i_j}^+ - \tfrac{1}{2}(1-a_{i_{j+3}}^+\lambda)Q_{i_{j+3}}^+)a_{i_{j+3}}^+(u_i-u_{i_{j+3}}) \\ & +(1-\tfrac{1}{2}(1+a_{i_j}^-\lambda)Q_{i_j}^- + \tfrac{1}{2}(1+a_{i_{j+3}}^-\lambda)Q_{i_{j+3}}^-/r_{i_{j+3}}^-)a_{i_{j+3}}^-(u_{i_j}-u_i)] \end{aligned} \tag{3.7}$$

From (2.12), scheme (3.7) is MmB if

$$(1+\tfrac{1}{2}(1-a_{i_j}^+\lambda)Q_{i_j}^+/r_{i_j}^+ - \tfrac{1}{2}(1-a_{i_{j+3}}^+\lambda)Q_{i_{j+3}}^+)a_{i_{j+3}}^+ \geq 0$$
$$(1-\tfrac{1}{2}(1+a_{i_j}^-\lambda)Q_{i_j}^- + \tfrac{1}{2}(1+a_{i_{j+3}}^-\lambda)Q_{i_{j+3}}^-/r_{i_{j+3}}^-)a_{i_{j+3}}^- \leq 0$$

and

$$S_i \sum_{j=1}^{3} [(1 + \tfrac{1}{2}(1 - a_{i_j}^+ \lambda) Q_{i_j}^+ / r_{i_j}^+ - \tfrac{1}{2}(1 - a_{i_{j+3}}^+ \lambda) Q_{i_{j+3}}^+) a_{i_{j+3}}^+$$
$$-(1 - \tfrac{1}{2}(1 + a_{i_j}^- \lambda) Q_{i_j}^- + \tfrac{1}{2}(1 + a_{i_{j+3}}^- \lambda) Q_{i_{j+3}}^- / r_{i_{j+3}}^-) a_{i_{j+3}}^-] \leq 1$$

then due to the signals of $a_{i_{j+3}}^+$ and $a_{i_j}^-$, we have

$$1 + \tfrac{1}{2}(1 - a_{i_j}^+ \lambda) Q_{i_j}^+ / r_{i_j}^+ - \tfrac{1}{2}(1 - a_{i_{j+3}}^+ \lambda) Q_{i_{j+3}}^+ \geq 0$$
$$1 - \tfrac{1}{2}(1 + a_{i_j}^- \lambda) Q_{i_j}^- + \tfrac{1}{2}(1 + a_{i_{j+3}}^- \lambda) Q_{i_{j+3}}^- / r_{i_{j+3}}^- \geq 0$$

and

$$1 + \tfrac{1}{2}(1 - a_{i_j}^+ \lambda) Q_{i_j}^+ / r_{i_j}^+ - \tfrac{1}{2}(1 - a_{i_{j+3}}^+ \lambda) Q_{i_{j+3}}^+ \leq \frac{1}{6 S_i a_{i_{j+3}}^+}$$
$$1 - \tfrac{1}{2}(1 + a_{i_j}^- \lambda) Q_{i_j}^- + \tfrac{1}{2}(1 + a_{i_{j+3}}^- \lambda) Q_{i_{j+3}}^- / r_{i_{j+3}}^- \leq \frac{1}{6 S_i a_{i_{j+3}}^-}$$

Under the assumption $Q(r) \geq 0$, $Q(r) \equiv 0$, when $r \leq 0$ and $\max\limits_{i,j} |a_{i_j}| \lambda \leq \frac{1}{12}$, scheme (3.6) is MmB sufficiently,

$$Q_{i_j}^+ / r_{i_j}^+ \leq 2/(1 - a_{i_j}^+ \lambda), \qquad Q_{i_{j+3}}^- / r_{i_{j+3}}^- \leq 2/(1 + a_{i_{j+3}}^- \lambda)$$

and

$$Q_{i_j}^- \leq 2/(1 + a_{i_j}^- \lambda), \qquad Q_{i_{j+3}}^+ \leq 2/(1 - a_{i_{j+3}}^+ \lambda)$$

4. Generalizations of the two classes of the schemes to systems

Consider the initial value problem for 2-D systems in conservation laws,

$$\begin{cases} U_t + F(U)_x + G(U)_y = 0 \\ U(x,y,t)|_{t=0} = U_0(x,y) \end{cases} \tag{4.1}$$

where $U_0(x,y)$ is a piecewise smooth vector function and $U = (u_1, ..., u_n)^T$,$F(U) = (f_1(U), ..., f_n(U))^T$ and $G(U) = (g_1(U), ..., g_n(U))^T$.

Here the genaralized schemes of (2.10), (2.13) and (3.6) to (2.14) are in the following forms,

(i) semidiscrete schemes

$$\frac{\partial U_i}{\partial t} = -S_i \quad \sum_{j=1}^{3}[A^+_{i_{j+3}}(U_i - U_{i_{j+3}}) + \tfrac{1}{2}R_{i_j}(\Lambda^+_{i_j}Q^+_{i_j} - \Lambda^-_{i_j}Q^-_{i_j})R^{-1}_{i_j}(U_{i_j} - U_i)$$
$$-\tfrac{1}{2}R_{i_{j+3}}(\Lambda^+_{i_{j+3}}Q^+_{i_{j+3}} - \Lambda^-_{i_{j+3}}Q^-_{i_{j+3}})R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}}) \tag{4.2}$$

(ii) full discrete schemes

$$U_i^{n+1} = U_i - S_i \quad \sum_{j=1}^{3}[A^+_{i_{j+3}}(U_i - U_{i_{j+3}}) + \tfrac{1}{2}R_{i_j}(\Lambda^+_{i_j}Q^+_{i_j} - \Lambda^-_{i_j}Q^-_{i_j})R^{-1}_{i_j}(U_{i_j} - U_i)$$
$$-\tfrac{1}{2}R_{i_{j+3}}(\Lambda^+_{i_{j+3}}Q^+_{i_{j+3}} - \Lambda^-_{i_{j+3}}Q^-_{i_{j+3}})R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}})] \tag{4.3}$$

and

$$U_i^{n+1} = U_i^n - S_i \sum_{j=1}^{3}[A^+_{i_{j+3}}(U_i - U_{i_{j+3}}) + A^-_{i_j}(U_{i_j} - U_i)$$
$$+\tfrac{1}{2}R_{i_j}(\Lambda^+_{i_j}(I - \Lambda^+_{i_j}\lambda)Q^+_{i_j} - \Lambda^-_{i_j}(I + \Lambda^-_{i_j}\lambda)Q^-_{i_j})R^{-1}_{i_j}(U_{i_j} - U_i)$$
$$-\tfrac{1}{2}R_{i_{j+3}}(\Lambda^+_{i_{j+3}}(I - \Lambda^+_{i_{j+3}})Q^+_{i_{j+3}} - \Lambda^-_{i_{j+3}}(I + \Lambda^-_{i_{j+3}}\lambda)Q^-_{i_{j+3}})R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}})] \tag{4.4}$$

where

$$Q^{\pm} = diag(Q^{1,\pm}, ..., Q^{n,\pm}), \qquad Q^{k,\pm}_{i_j} = Q(r^{k,\pm}_{i_j}), \qquad Q^{k,\pm}_{i_{j+3}} = Q(r^{k,\pm}_{i_{j+3}})$$

$$r^{k,+}_{i_j} = \frac{(R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}}))^k}{(R^{-1}_{i_j}(U_{i_j} - U_i))^k} \qquad r^{k,-}_{i_j} = \frac{(R^{-1}_{i_{j_j}}(U_{i_{j_j}} - U_{i_j}))^k}{(R^{-1}_{i_j}(U_{i_j} - U_i))^k}$$

$$r^{k,+}_{i_{j+3}} = \frac{(R^{-1}_{i_{j+3_{j+3}}}(U_{i_{j+3}} - U_{i_{j+3_{j+3}}}))^k}{(R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}}))^k} \qquad r^{k,-}_{i_{j+3}} = \frac{(R^{-1}_{i_j}(U_{i_j} - U_i))^k}{(R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}}))^k}$$

$$k = 1, .., n$$

$$A^{\pm} = R\Lambda^{\pm}R^{-1}, \qquad \Lambda^{\pm} = \tfrac{1}{2}(\Lambda \pm |\Lambda|)$$

$$A_{i_j} = R_{i_j}\Lambda_{i_j}R^{-1}_{i_j} \qquad A_{i_{j+3}} = R_{i_{j+3}}\Lambda_{i_{j+3}}R^{-1}_{i_{j+3}}$$

and as Roe[29], we let

$$(F(U_{i_j}) - F(U_i))\nu^x_{i_j} + (G(U_{i_j}) - G(U_i))\nu^y_{i_j} = A_{i_j}(U_{i_j} - U_i)$$

and

$$(F(U_i) - F(U_{i_{j+3}}))\nu^x_{i_{j+3}} + (G(U_i) - G(U_{i_{j+3}}))\nu^y_{i_{j+3}} = A_{i_{j+3}}(U_i - U_{i_{j+3}})$$

5. Numerical Experiments

In this section, we give numerical solutions of Riemann problems in three pieces for scalar conservation law and in three and four pieces 2-D gas dynamics systems and choose the limiter in the both of the schemes as Roe's Superbee[3]:

$$Q(r) = max(0, min(2,r), min(1,2r))$$

5.1 Riemann problem in three pieces for 2-D scalar conservation law.

Consider the Riemann problem in three pieces for the following conservation law,

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial u^3}{\partial y} = 0 \\ u(x,y,t)|_{t=0} = \begin{cases} u_1, & x > 0, y \geq -\sqrt{3}x \\ u_2, & x < 0, y \geq \sqrt{3}x \\ u_3, & elsewhere \end{cases} \end{cases} \tag{5.1}$$

The theoretical solutions of (5.1) have been studied in [30], here we give the numerical solutions for four cases by the two classes of MmB schemes.

In the following results of this subsection, we show the contour lines of the numerical solutions if no special additional words in the following figures.

(1) three shock waves — non rarefaction wave.

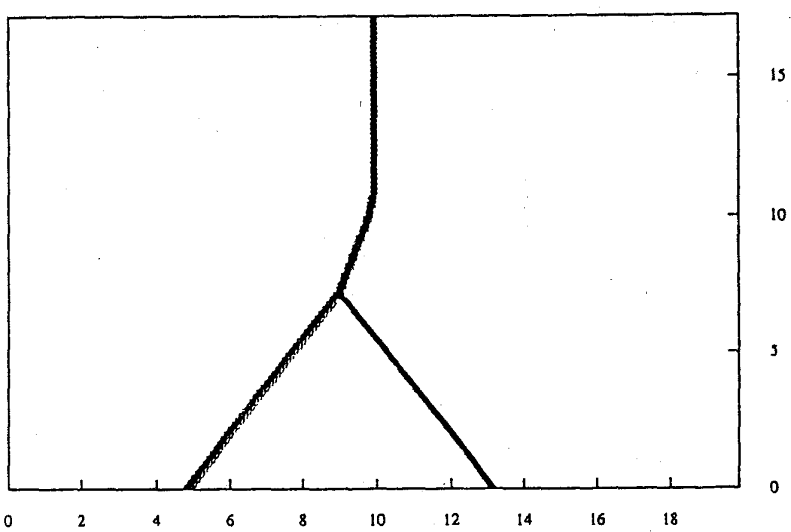

Fig. 5.1-a Runge Kutta-FVM MmB Scheme

mesh points 201×201, $\lambda = 0.2$, time steps n=350

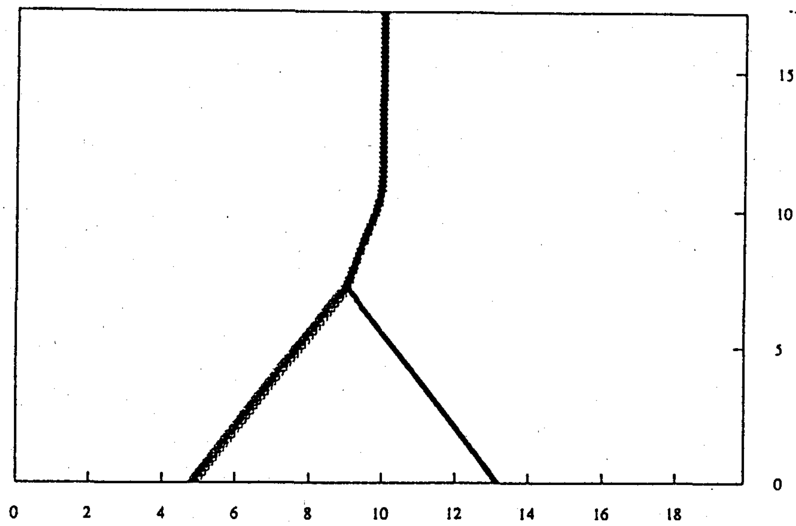

Fig. 5.1-b Taylor-FVM MmB Scheme

mesh points 201 × 201, $\lambda = 0.2$, time steps n=350

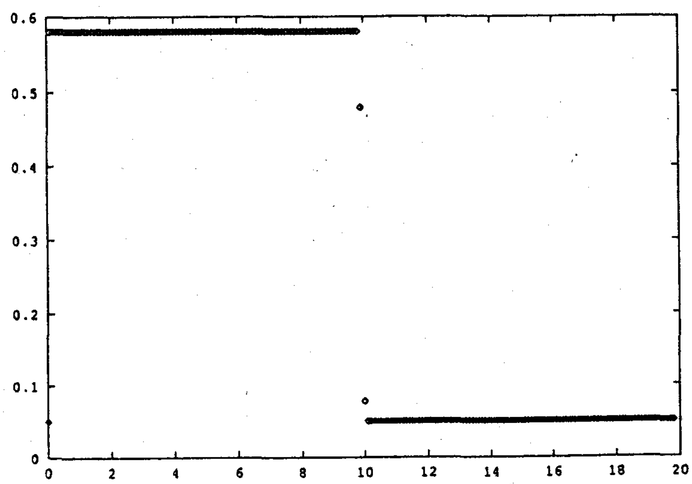

Fig. 5.1-c Runge Kutta-FVM MmB scheme, Curve at y=16

mesh points 201 × 201, $\lambda = 0.2$, time steps n=350

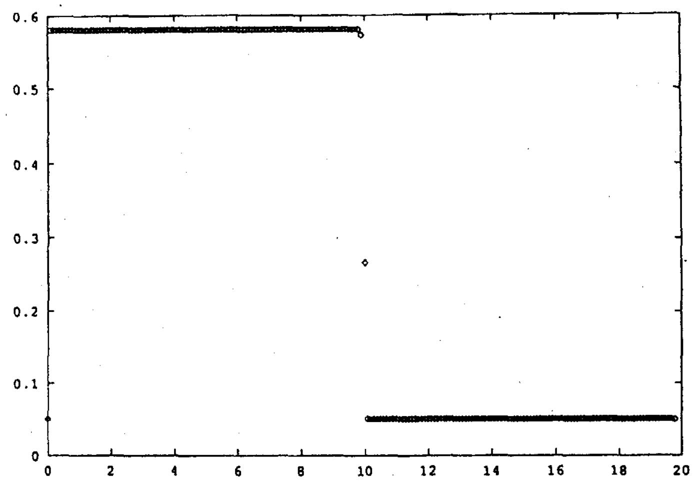

Fig. 5.1-d Taylor-FVM MmB Schemes, Curve at y=16

mesh points 201×201, $\lambda = 0.2$, time steps n=350

In this case, the initial data are u1=0.05, u2=0.32 and u3=0.58, From Fig. 5.1-c and 5.1-d, we may know that both of the schemes are nonoscillatory and have two points in the shock region.

(2) two shock waves and one rarefaction wave

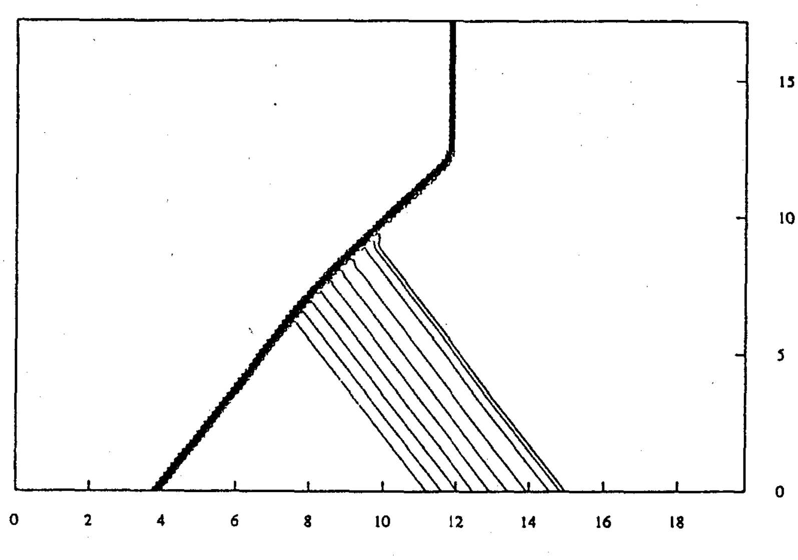

Fig. 5.2-a Runge Kutta-FVM MmB Scheme

mesh points 201×201, $\lambda = 0.2$, time steps n=350

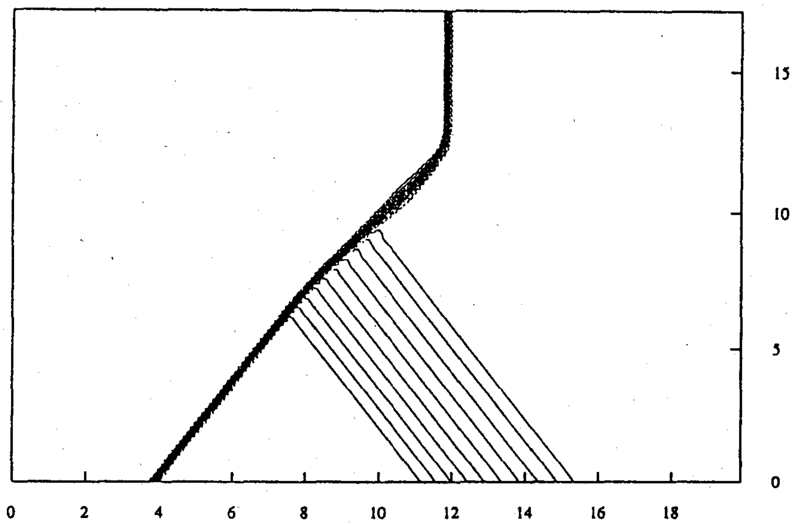

Fig. 5.2-b Taylor-FVM MmB Scheme

mesh points 201 × 201, $\lambda = 0.2$, time steps n=350

Choose u1=0.32, u2=0.58, u3=0.05,

(3) one shock wave and two rarefaction waves

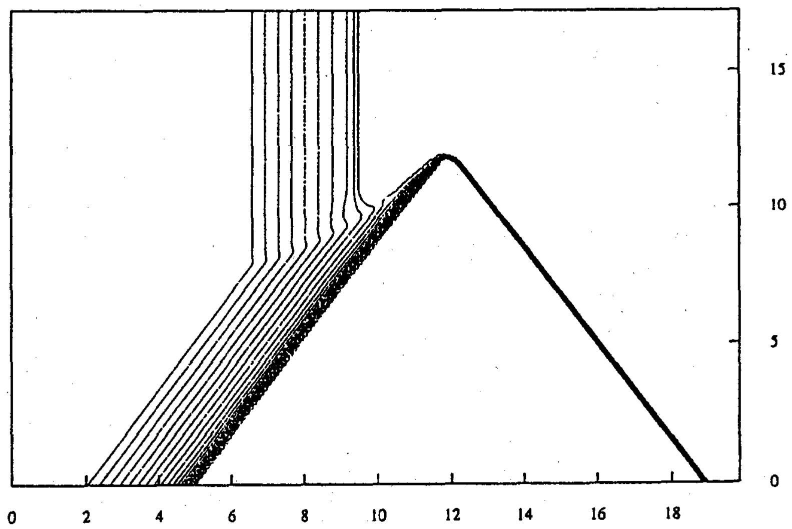

Fig. 5.3-a Runge Kutta-FVM MmB Scheme

mesh points 201 × 201, $\lambda = 0.2$, time steps n=350

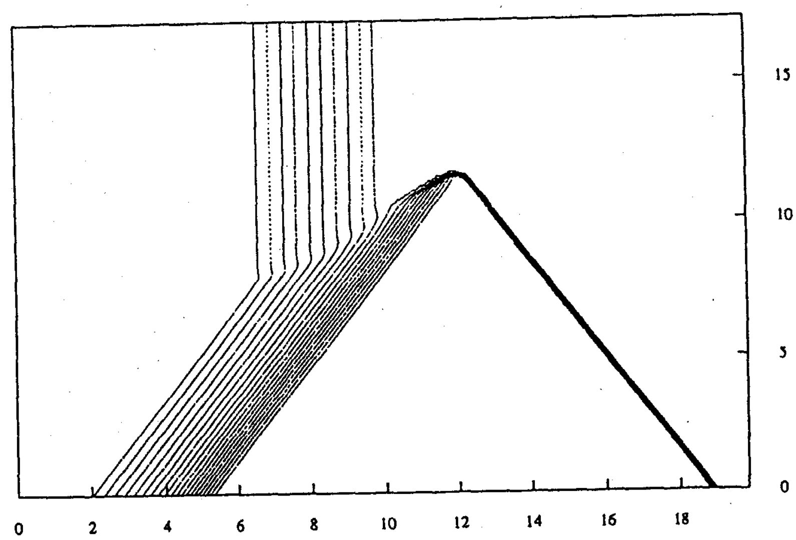

Fig. 5.3-b Taylor-FVM MmB Scheme

mesh points 201 × 201, $\lambda = 0.2$, time steps n=350

u1=0.32, u2=0.05, u3=0.58.

(4) three rarefaction waves — non shock wave.

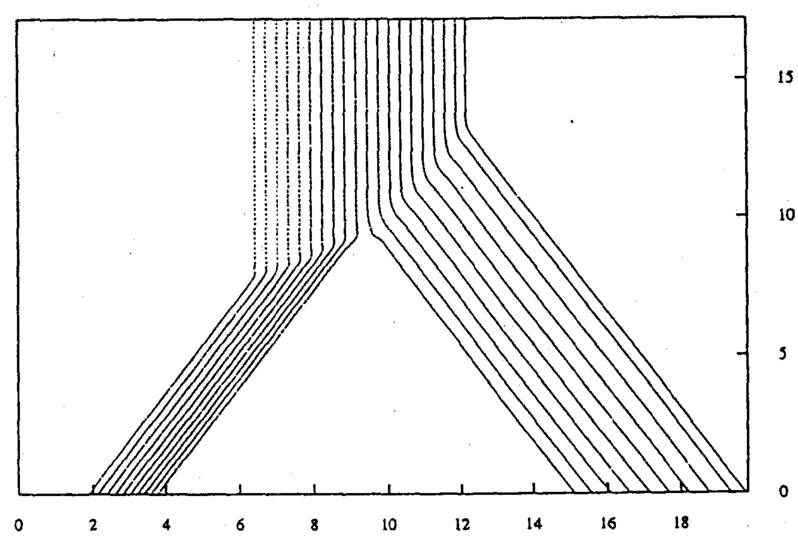

Fig. 5.4 Taylor-FVM MmB Scheme

mesh points 201 × 201, $\lambda = 0.2$, time steps n=350

Here u1=0.58, u2=0.05, u3=0.32.

From Fig. 5.2 and 5.3, for slope shock, Runge Kutta-FVM MmB scheme is sharper than Taylor-FVM MmB scheme and for rarefaction wave Taylor-FVM MmB scheme is better than Runge kutta-FVM MmB scheme.

5.2 Riemann problems in three pieces for 2-D gas dynamics systems.

Consider the system of adiabatic flow

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0 \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0 \\ (\rho(e + \frac{u^2+v^2}{2}))_t + (\rho u(h + \frac{u^2+v^2}{2}))_x + (\rho v(h + \frac{u^2+v^2}{2}))_y = 0 \end{cases} \tag{5.2}$$

$$e = \frac{p}{(\gamma-1)\rho}, \qquad h = e + \frac{p}{\rho}$$

where ρ, (u,v) and p denote density, velocity and pressure, respectively. In [31], Solutions of Riemann problems in four pieces have studied by using characteristic method. From the paper, we know that the characteristics in direction (μ, ν) called θ direction are written to the form,

$$\lambda_0 = u_\theta, \quad flow \quad characteristic$$
$$\lambda_\pm = u_\theta \pm c, \quad wave \quad characteristics$$

where $u_\theta = \mu u + \nu v$ is a velocity in θ direction and c is sound speed, $c = \sqrt{\gamma p/\rho}$.

The Riemann data in three pieces are discribed as follows:

$$(\rho, p, u, v)|_{t=0} = ST_i, \qquad i = 1,2,3 \tag{5.3}$$

where ST_i (i=1,2,3) are constant states. See Fig. 5.5

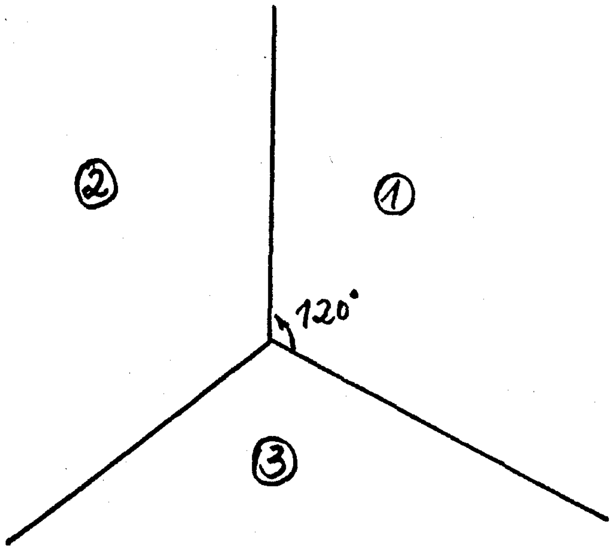

Fig. 5.5 Distribution of Riemann Initial Data

From [31], by Rankine-Hugoniot condition and under the assumption that each jump in initial data outside of the origin projects exactly one planar wave of slip planes, so the condition of contact discontinuity (J) along direction (μ,ν) is,

$$\begin{cases} u_{\mu,1} \neq u_{\mu,2} \\ u_{\nu,1} = u_{\nu,2} \\ \rho_1 \neq \rho_2 \\ p_1 = p_2 \end{cases}$$

see Fig. 5.6

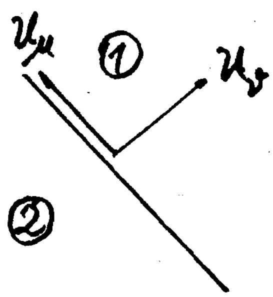

Fig. 5.6

Due to the signals of $Curl(u,v) = v_x - u_y$, we divid contact discontinuity into two classes, which are defined by Zhang [23].

$$J^+, \quad if \quad Curl(u,v) = +\infty; \qquad J^-, \quad if \quad Curl(u,v) = -\infty$$

Known the structure of wave from initial data, we show the following two configrations by numerical results.

(a) Curl(u,v)=$-\infty$

Initial data:

$$\begin{array}{llll} \rho_1 = 0.5, & p_1 = 2.0, & u_1 = \sqrt{3}/2, & v_1 = -1.0 \\ \rho_2 = 1.0, & p_2 = 2.0, & u_2 = \sqrt{3}/2, & v_2 = 1.0 \\ \rho_3 = 1.5, & p_3 = 2.0, & u_3 = -\sqrt{3}/2, & v_3 = 0.0 \end{array}$$

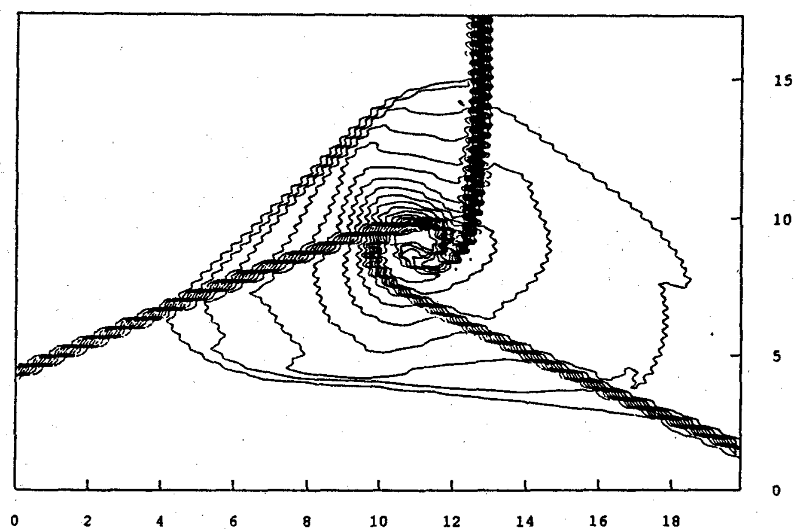

Fig.5.7-a Runge Kutta - FVM MmB Schemes Density Contour Lines

mesh points 101×101, $\lambda = 0.1$, time steps n=150

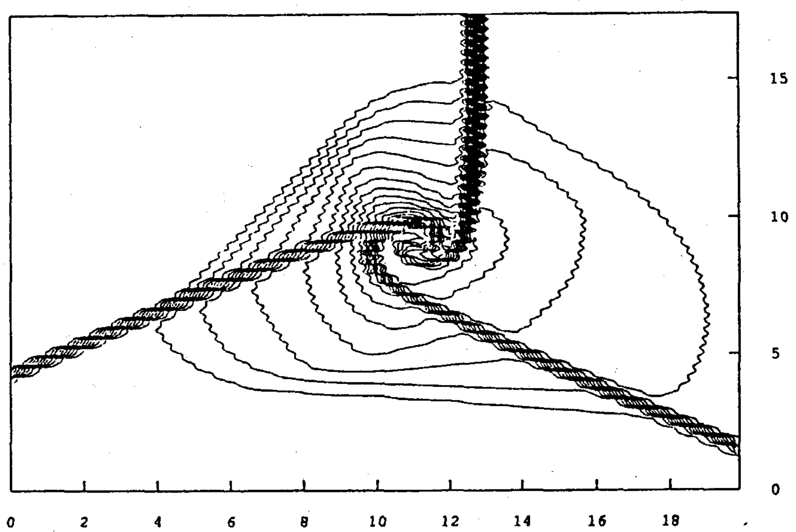

Fig.5.7-b Taylor - FVM MmB Schemes Density Contour Lines

mesh points 101×101, $\lambda = 0.1$, time steps n=150

and

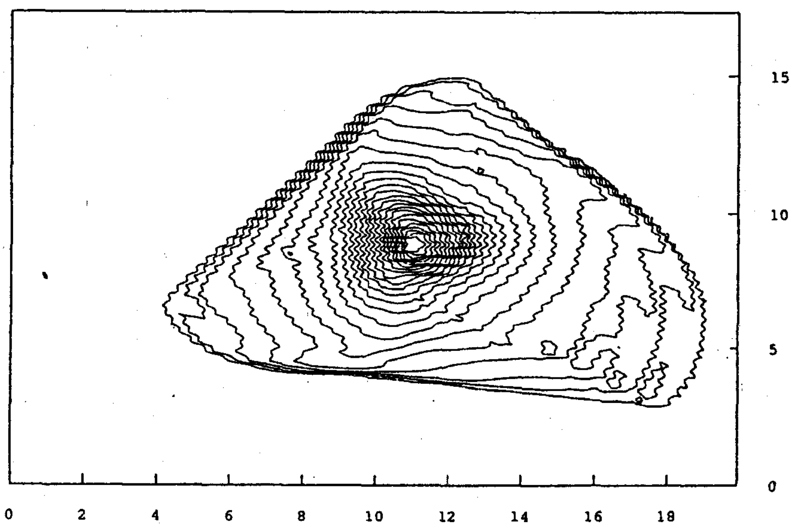

Fig.5.7-c Runge Kutta - FVM MmB Schemes Pressure Contour Lines

mesh points 101×101, $\lambda = 0.1$, time steps n=150

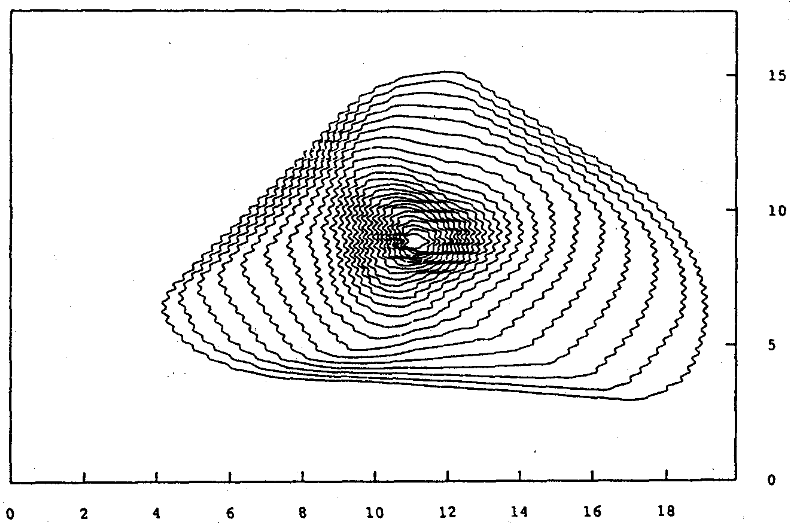

Fig.5.7-d Taylor - FVM MmB Schemes Pressure Contour Lines

mesh points 101 $\times$ 101, $\lambda = 0.1$, time steps n=150

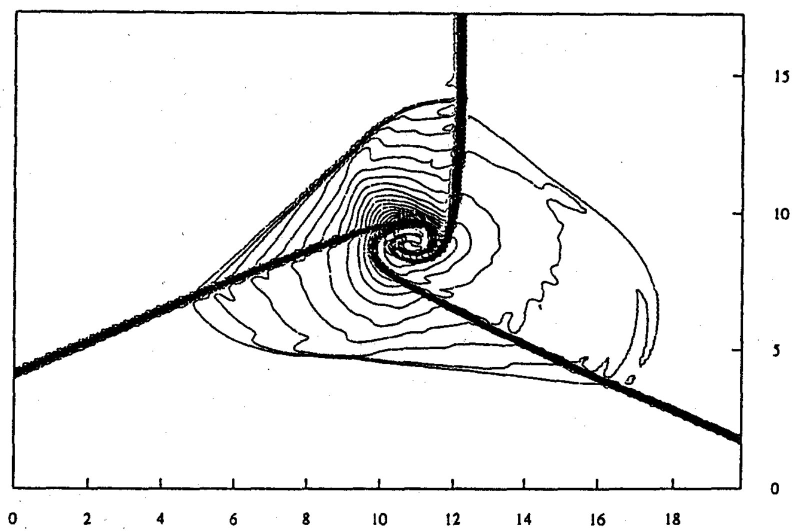

Fig. 5.7-a' Runge Kutta-FVM MmB Scheme Density contour lines

mesh points 201 $\times$ 201, $\lambda = 0.1$, time steps n=250

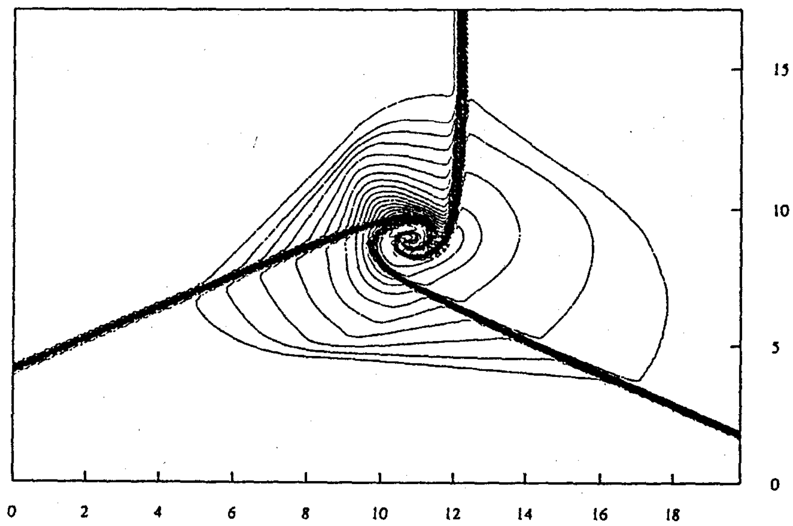

Fig. 5.7-b' Taylor-FVM MmB Scheme Density contour lines

mesh points 201×201, $\lambda = 0.1$, time steps n=250

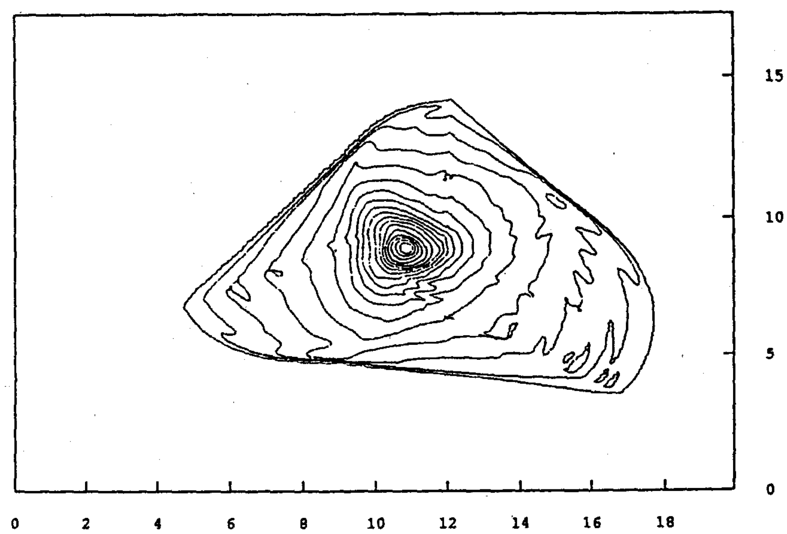

Fig. 5.7-c' Runge Kutta-FVM MmB Scheme Pressure contour lines

mesh points 201×201, $\lambda = 0.1$, time steps n=250

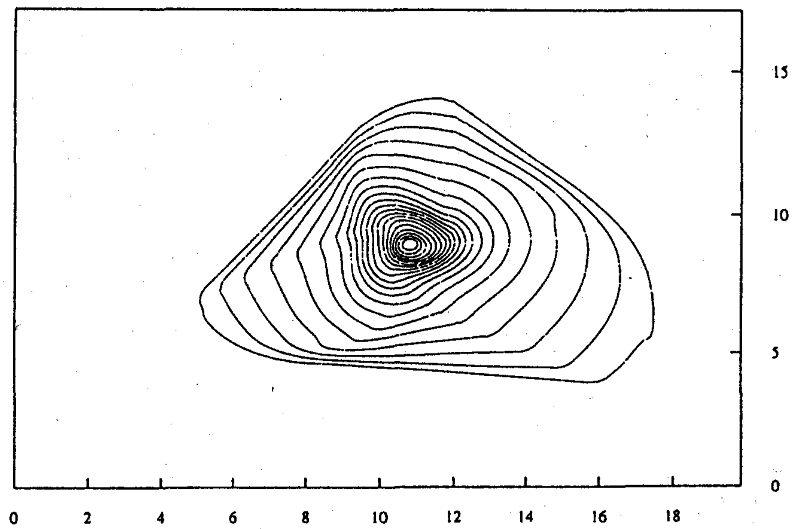

Fig. 5.7-d' Taylor-FVM MmB Scheme Pressure contour lines

mesh points 201×201, $\lambda = 0.1$, time steps n=250

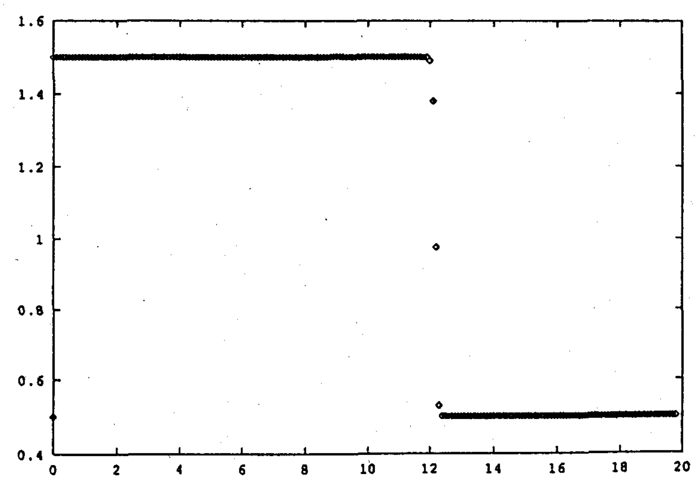

Fig. 5.7-e Runge Kutta-FVM MmB Scheme Curve Density at y=16

mesh points 201×201, $\lambda = 0.1$, time steps n=250

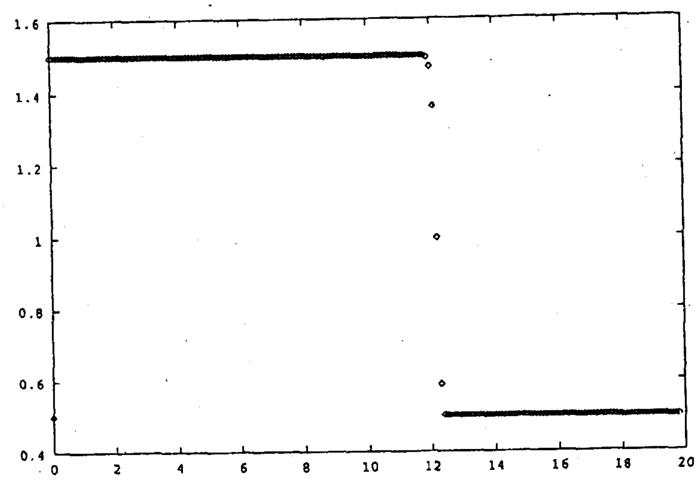

Fig. 5.7-f Taylor-FVM MmB Scheme Curve Density at y=16

mesh points 201 $\times$ 201, $\lambda = 0.1$, time steps n=250

In Fig. 5.7-e and 5.7-f, we can see that there are three or four points and nonoscillations in the region of contact discontinuity; In Fig. 5.7-a and 5.7-b, there is a spiral in the psuado-subsonic region defined in [31] and from the figures of contour lines, the numerical solution by Taylor-FVM MmB scheme is better than the solution by Runge Kutta-FVM MmB scheme.

(b) Curl(u,v)=$+\infty$

Initial data,

$$\begin{array}{llll}
\rho_1 = 0.5, & p_1 = 2.0, & u_1 = -\sqrt{3}/2, & v_1 = 1.0 \\
\rho_2 = 1.0, & p_2 = 2.0, & u_2 = -\sqrt{3}/2, & v_2 = -1.0 \\
\rho_3 = 1.5, & p_3 = 2.0, & u_3 = \sqrt{3}/2, & v_3 = 0.0
\end{array}$$

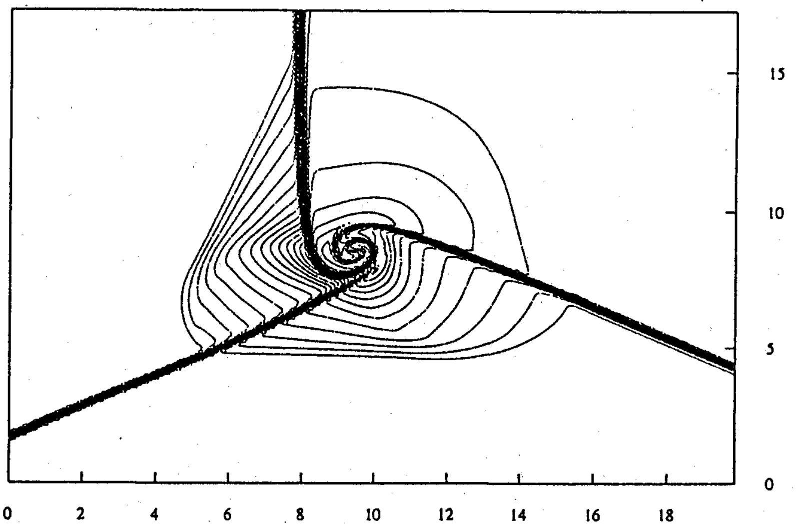

Fig. 5.8-a Taylor-FVM MmB Scheme Density contour lines

mesh points 201 $\times$ 201, $\lambda = 0.1$ and time steps n=250

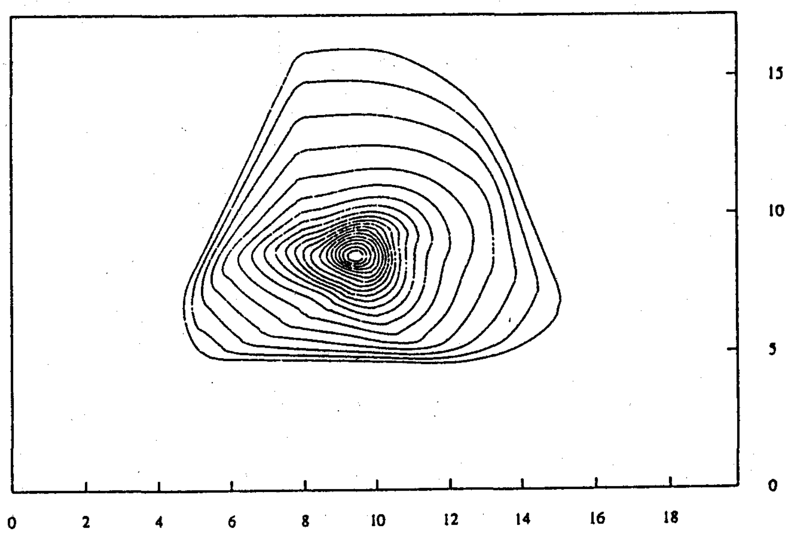

Fig. 5.8-b Taylor-FVM MmB Scheme Pressure contour lines

mesh points 201 $\times$ 201, $\lambda = 0.1$ and time steps n=250

From the above figures in this subsection, we can see that the direction of the rotation of the spiral depends on the signal of Curl(u,v). Also see [23].

5.3 Riemann problems in four pieces for 2-D gas dynamics systems.

Here two cases of the numerical Riemann solutions in four pieces for only containing contact discontinuities for (5.2) are listed by Taylor-FVM MmB scheme. The distribution of initial data is discribed as,

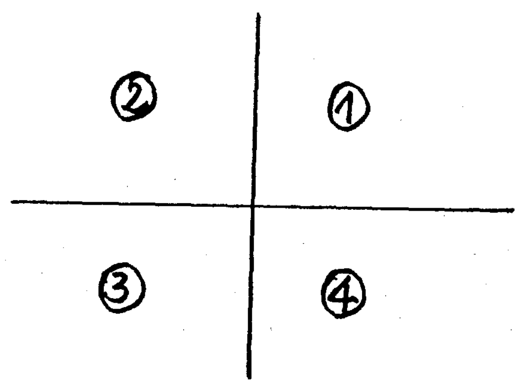

Fig. 5.9 Distribution of initial data

(i) counter-clockwise

Initial data,

$$
\begin{array}{llll}
\rho_1 = 2.5, & p_1 = 2.0, & u_1 = 1.0, & v_1 = -1.0 \\
\rho_2 = 1.5, & p_2 = 2.0, & u_2 = 1.0, & v_2 = 1.0 \\
\rho_3 = 0.5, & p_3 = 2.0, & u_3 = -1.0, & v_3 = 1.0 \\
\rho_4 = 1.5, & p_4 = 2.0, & u_4 = -1.0, & v_4 = -1.0
\end{array}
$$

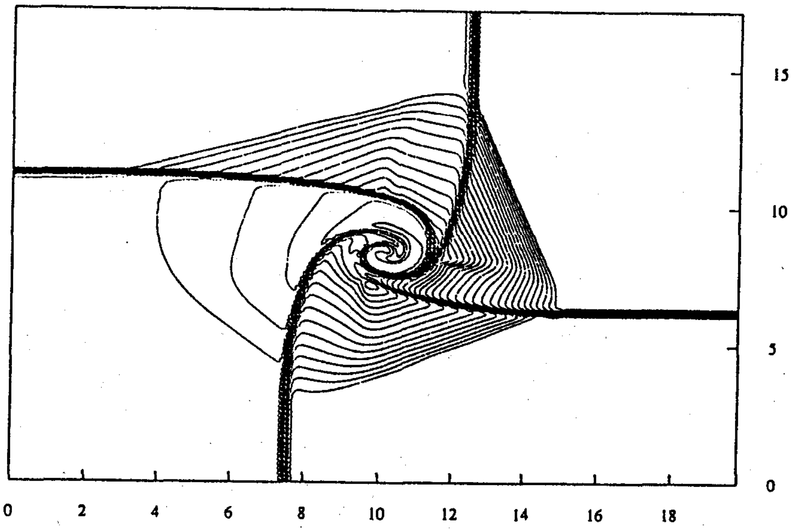

Fig. 5.10-a Density contour lines

mesh points 201 × 201, $\lambda = 0.1$ and time steps n=250

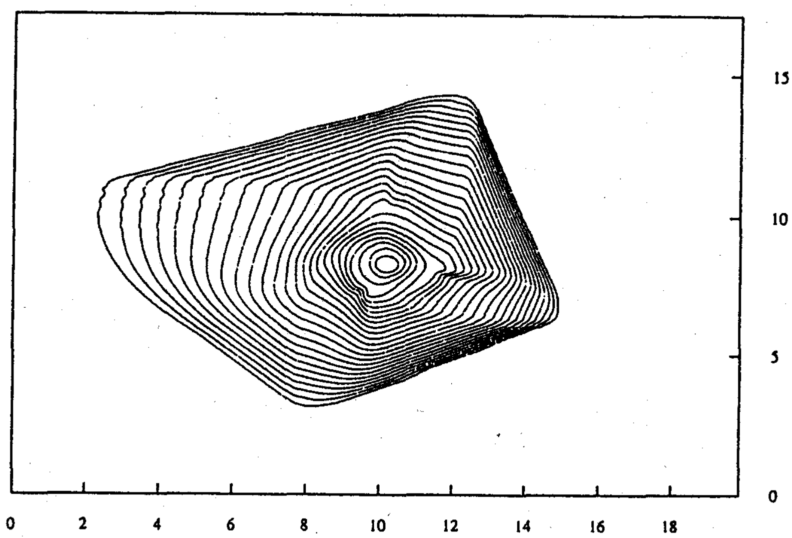

Fig. 5.10-b Pressure contour lines

mesh points 201 × 201, $\lambda = 0.1$ and time steps n=250

(ii) clockwise

Initial data,

$$\begin{array}{llll}\rho_1 = 2.5, & p_1 = 1.0, & u_1 = -1.0, & v_1 = -1.0 \\ \rho_2 = 1.5, & p_2 = 1.0, & u_2 = -1.0, & v_2 = 1.0 \\ \rho_3 = 0.5, & p_3 = 1.0, & u_3 = 1.0, & v_3 = 1.0 \\ \rho_4 = 1.5, & p_4 = 1.0, & u_4 = 1.0, & v_4 = -1.0\end{array}$$

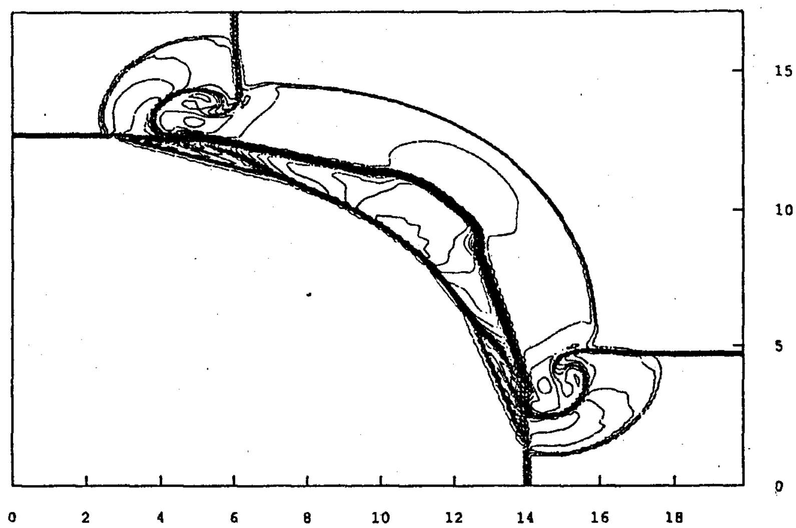

Fig. 5.11-a Density contour lines

mesh points 201 $\times$ 201, $\lambda = 0.1$ and time steps n=200

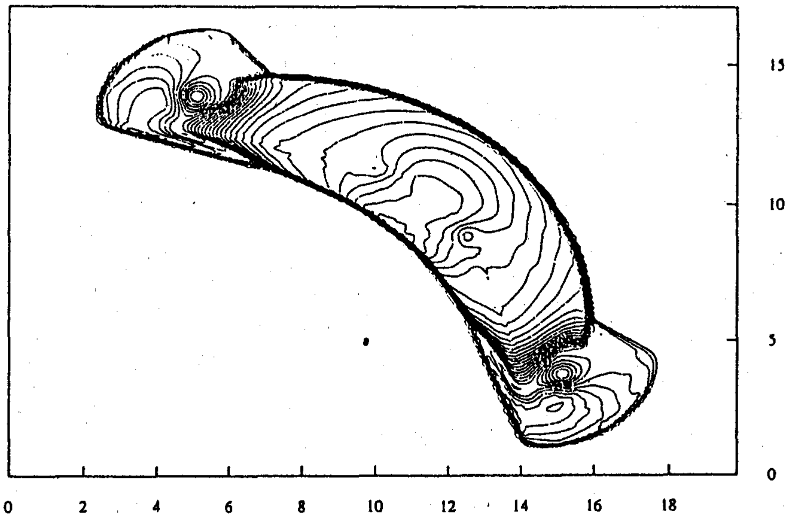

Fig. 5.11-b Pressure contour lines

mesh points 201 $\times$ 201, $\lambda = 0.1$ and time steps n=200

In the paper, we give two classes of explicit MmB schemes on regular triangular meshes for 2-D conservation laws, Numerical results show the high resolution and nonoscillation. In [32] we will present implicit second order accurate MmB schemes on general triangular meshes.

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