Zuse Institute Berlin

Takustr. 7
14195 Berlin
Germany

Ralf Borndörfer[1], Fabian Danecker[2], Martin Weiser[3]

Newton's Method for Global Free Flight Trajectory Optimization[4]

[1] 0000-0001-7223-9174
[2] 0000-0002-8953-808X
[3] 0000-0002-1071-0044
[4] to appear in: Public Transport Optimization: From Theory to Practice

ZIB Report 23-08 (January 2023)

Zuse Institute Berlin
Takustr. 7
14195 Berlin
Germany

Telephone: +49 30 84185-0
Telefax: +49 30 84185-125

E-mail: bibliothek@zib.de
URL: http://www.zib.de

ZIB-Report (Print) ISSN 1438-0064
ZIB-Report (Internet) ISSN 2192-7782

Newton's Method for Global Free Flight Trajectory Optimization

Ralf Borndörfer[1], Fabian Danecker[2], Martin Weiser[3]

9. Februar 2023

Abstract Globally optimal free flight trajectory optimization can be achieved with a combination of discrete and continuous optimization. A key requirement is that Newton's method for continuous optimization converges in a sufficiently large neighborhood around a minimizer. We show in this paper that, under certain assumptions, this is the case.

Keywords shortest path, flight planning, free flight, optimal control, discrete optimization, global optimization, newton's method

MSC Classification 49M15, 49M37, 65L10, 65L70, 90C26

1 Introduction

The Free Flight Trajectory Optimization Problem - finding the time-optimal route from A to B with respect to the wind conditions - is usually solved using direct or indirect methods from Optimal Control. These are highly efficient, but suffer from one key drawback: They only converge locally and are thus dependent on a sufficiently good starting point. This makes such methods, used as a standalone, incapable of meeting airlines' high expectations regarding the global optimality of routes.

In [1]–[3] a deterministic two-stage algorithm was proposed that combines discrete and continuous optimization in order to find a globally optimal solution to the free flight trajectory optimization problem. With this approach the exponential complexity of other branch and bound based algorithms is circumvented.

The discrete optimization stage involves the creation of a locally densely connected digraph of certain density characterized by node spacing h and connectivity length ℓ and the enumeration of shortest simple paths on this graph using Yen's algorithm [4]. Doing so, the space of feasible trajectories is sampled evenly and analyzed efficiently. Promising paths serve as initial guesses for a subsequent refinement stage in which a continuous solution to the problem is calculated up to the desired accuracy.

[1] 0000-0001-7223-9174
[2] 0000-0002-8953-808X
[3] 0000-0002-1071-0044

The present paper is concerned with the second stage. One way to generate a continuous solution is to apply Newton's method to the first order necessary conditions (the KKT-conditions) – an approach commonly referred to as Newton-KKT or Sequential Quadratic Programming (SQP) (see e.g., [5]). It is now shown that there is a quantifiable domain around a global optimum such that Newton-KKT converges if initialized accordingly.

Since the computational effort of the first graph-searching stage depends exclusively on the problem instance, i.e., the wind conditions, the algorithm asymptotically inherits the super fast convergence rates of the Newton-KKT method.

The paper is structured as follows. After defining the problem and introducing a formulation that is convenient for the analytical discussion in Abschnitt 2, we formally state the necessary and sufficient conditions as well as the Newton-KKT approach in Abschnitt 3. The proof of convergence is provided in Abschnitt 4 followed by a conclusion emphasizing the impact on previous and future work.

2 The Free Flight Trajectory Optimization Problem

Neglecting any traffic flight restrictions, we consider Lipschitz-continuous flight paths $\xi \in C^{0,1}(]0,1[,\mathbb{R}^2)$ connecting origin $\xi(0) = x_O$ and destination $\xi(1) = x_D$. By Rademacher's theorem, such paths are almost everywhere differentiable, and moreover contained in the Sobolev space $W^{1,\infty}(]0,1[,\mathbb{R}^2)$.

A short calculation reveals that an aircraft travelling along such a path ξ with constant airspeed $\overline{v}$ through a three times continuously differentiable wind field $w \in C^3(\mathbb{R}^2,\mathbb{R}^2)$ with bounded magnitude $\|w(x)\| < \overline{v}$ reaches the destination after a flight duration

$$T(\xi) = \int_0^1 f\big(\xi(\tau),\xi_\tau(\tau)\big)\,d\tau \tag{1}$$

with ξ_τ denoting the time derivative of ξ and

$$f(\xi,\xi_\tau) := t_\tau = \frac{-\xi_\tau^T w + \sqrt{(\xi_\tau^T w)^2 + (\overline{v}^2 - w^T w)(\xi_\tau^T \xi_\tau)}}{\overline{v}^2 - w^T w}, \tag{2}$$

see [1]–[3].

Among these paths ξ, we need to find one with minimal flight duration $T(\xi)$, since that is essentially proportional to fuel consumption [6]. This classic of optimal control is known as Zermelo's navigation problem [7]. It can easily be shown that in case of bounded wind speed, the optimal trajectory cannot be arbitrarily longer than the straight connection of origin and destination. Hence, every global minimizer is contained in an ellipse $\Omega \subset \mathbb{R}^2$ with focal points x_O and x_D.

Since the flight duration T as defined in (1) is based on a time reparametrization from actual flight time $t \in [0,T]$ to pseudo-time $\tau \in [0,1]$ according to the actual flight trajectory $x(t) = \xi(\tau(t))$ such that $\|x_t(t) - w(x(t))\| = \overline{v}$, the actual parametrization of ξ in terms of pseudo-time τ is irrelevant for the value of T. Calling two paths $\xi, \tilde{\xi}$ equivalent if there exists a Lipschitz-continuous bijection

$r :]0,1[\to]0,1[$ such that $\xi(r(\tau)) = \tilde{\xi}(\tau)$, we can restrict the optimization to equivalence classes. Moreover, every equivalence class contains a representative with constant ground speed $\|\xi_\tau(\tau)\| = L$ for almost all τ, that can be obtained from any $\tilde{\xi}$ with $\|\tilde{\xi}_\tau(\tau)\| \neq 0 \,\forall \tau$ via

$$\xi(\tau) := L \int_0^\tau \frac{\tilde{\xi}_\tau(t)}{\|\tilde{\xi}_\tau(t)\|} dt, \quad L := \int_0^1 \|\tilde{\xi}_\tau(\tau)\| d\tau. \tag{3}$$

Hence, we will subsequently consider the equivalent constrained minimization problem

$$\min_{\xi \in X,\, L \in \mathbb{R}} T(\xi), \quad \text{s.t.} \quad \|\xi_\tau(\tau)\|^2 = L^2 \quad \text{for a.a. } \tau \in]0,1[. \tag{4}$$

Here, the admissible set X is the affine space

$$X = \{\xi \in W^{1,\infty}(]0,1[, \mathbb{R}^2) \;\mid\; \xi(0) = x_O,\ \xi(1) = x_D\}. \tag{5}$$

Note that, if the constraint in (4) is satisfied, L also represents the path length, since

$$\int_0^1 \|\xi_\tau\| d\tau = L. \tag{6}$$

3 Continuous Optimization: Newton-KKT

In order to find a continuous solution to the free flight optimization problem (4) we apply Newton's method to the first order necessary conditions (the KKT-conditions), which is also known as sequential quadratic programming (SQP). Before we formally introduce Newton's method, we discuss the necessary and sufficient conditions for optimality, which also defines the goal of the presented algorithm.

3.1 Optimality Conditions

3.1.1 Necessary Conditions

Let us first consider the unconstrained problem analogous to (4),

$$\min_{\xi \in X} T. \tag{7}$$

Any global minimizer $\tilde{\xi}^{\star\star}$ of (7) is clearly non-isolated due to possible reparametrizations of the time. Let $\xi^{\star\star}$ denote the equivalent trajectory with constant ground speed, $\|\xi_\tau^{\star\star}(\tau)\| = L^{\star\star}$ for almost all τ.

Note that $T : X \to \mathbb{R}$ is Fréchet differentiable with respect to the corresponding linear space

$$\delta X := W_0^{1,\infty}(]0,1[, \mathbb{R}^2) \tag{8}$$

of directions $\delta\xi$ with zero boundary values, that consequently do not change origin and destination, equipped with the norm

$$\|\delta\xi\|_{X^\infty} = \|\delta\xi\|_{L^\infty(]0,1[)} + \|\delta\xi_\tau\|_{L^\infty(]0,1[)}. \tag{9}$$

Both solutions $\tilde{\xi}^{\star\star}, \xi^{\star\star}$ satisfy the first order necessary condition

$$0 = T'(\xi^{\star\star})[\delta\xi] \quad \forall \delta\xi \in \delta X. \tag{10}$$

We now turn back to the constrained problem (4) and express the constant ground speed requirement by means of a constraint $h(z) = 0$, where $z := (L, \xi) \in Z := \mathbb{R} \times X$ and

$$h : Z \to \Lambda := L^\infty(]0,1[, \mathbb{R}), \quad z \mapsto \xi_\tau^T \xi_\tau - L^2. \tag{11}$$

Further we define the linear space

$$\delta Z := \mathbb{R} \times \delta X \tag{12}$$

and equip the spaces Z and δZ with the norms

$$\|z\|_{Z^\infty} = |L| + \|\xi\|_{L^\infty(]0,1[)} + \|\xi_\tau\|_{L^\infty(]0,1[)}, \quad \text{and} \tag{13a}$$
$$\|z\|_{Z^2} = |L| + \|\xi\|_{L^2(]0,1[)} + \|\xi_\tau\|_{L^2(]0,1[)}. \tag{13b}$$

If not stated otherwise, we assume $\|\cdot\|$ to denote the l^2-norm. Accordingly, we use the following quantitative definition of the L^∞-norm in terms of the l^2-norm.

Definition 1. *Let $f : \Omega \mapsto \mathbb{R}^n$. Then we define*

$$\|f\|_{L^\infty(\Omega)} := \inf\{C \geq 0 : \|f(x)\|_2 \leq C \textit{ for a.a. } x \in \Omega\}. \tag{14}$$

The goal of the present paper is to find an isolated globally optimal solution $\xi^{\star\star}$ to (4) that satisfies $T(\xi^{\star\star}) \leq T(\xi)\ \forall \xi \in X$, contrary to a local optimizer $\xi^\star$ that is only superior to trajectories in a certain neighborhood, $T(\xi^\star) \leq T(\xi)\ \forall \xi \in \mathcal{N}(\xi^\star) \subseteq X$. An isolated global minimizer satisfies the necessary Karush-Kuhn-Tucker (KKT) optimality conditions [8] given that it is a regular point, which is always the case, as confirmed by the following Theorem.

Theorem 1. *Let $z = (L, \xi) \in Z$ with $L > 0$ and assume there is a direction $u \in \mathbb{R}^2$ and $c > 0$ such that $\xi_\tau^T u \geq c$ almost everywhere. Then, $h'(z) : \delta Z \to L^\infty(]0,1[)$ is surjective, i.e., z is regular.*

Beweis. Let $f \in L^\infty(]0,1[)$ be given and $b := \xi_\tau^T u \geq c$. We set

$$\delta L = -\frac{\int_0^1 b^{-1} f/2 \, d\tau}{L \int_0^1 b^{-1} \, d\tau}$$

and

$$g = b^{-1}\left(f/2 + L\delta L\right), \quad \delta\xi_\tau = gu.$$

Due to $b \geq c$ almost everywhere, b^{-1} is bounded and hence $g, \xi_\tau \in L^\infty(]0,1[)$. By construction, $\int_0^1 \delta\xi_\tau \, d\tau = 0$ holds, such that $\delta z = (\delta L, \delta\xi) \in \delta Z$.

Now we obtain

$$\begin{aligned} h'(z)[\delta z] &= 2\xi_\tau^T \delta\xi_\tau - 2L\delta L \\ &= 2bg - 2L\delta L \\ &= 2(f/2 + L\delta L) - 2L\delta L \\ &= f, \end{aligned}$$

and thus the claim. $\square$

For $\lambda \in \Lambda^*$, the Lagrangian is defined as

$$\mathcal{L}(z, \lambda) := T(\xi) + \langle \lambda, h(z) \rangle. \tag{15}$$

The KKT-conditions guarantee for a regular minimizer $z^{\star\star}$ the existence of a Lagrange multiplier $\lambda^{\star\star} \in \Lambda^*$, such that

$$\begin{aligned} 0 &= \mathcal{L}_z(z^{\star\star}, \lambda^{\star\star})[\delta z] && \forall\ \delta z \in \delta Z, \\ 0 &= \langle \delta\lambda, h(z^{\star\star}) \rangle && \forall\ \delta\lambda \in \Lambda^* \end{aligned}$$

hold, where $\delta z := (\delta L, \delta\xi) \in \delta Z$. In our case, these necessary conditions read

$$0 = \underbrace{T'(\xi^{\star\star})[\delta\xi]}_{=0\ (10)} + 2\int_0^1 \lambda^{\star\star} \left(\delta\xi_\tau^T \xi_\tau^{\star\star} - \delta L\, L^{\star\star}\right) d\tau \qquad \forall\ \delta z \in \delta Z, \tag{16a}$$

$$0 = \int_0^1 \delta\lambda \left((\xi_\tau^{\star\star})^T \xi_\tau^{\star\star} - (L^{\star\star})^2\right)\ d\tau \qquad \forall\ \delta\lambda \in \Lambda^*. \tag{16b}$$

Consider once more the unconstrained problem (7) and a global minimizer $\tilde{\xi}^{\star\star}$ thereof. As discussed before, there is an equivalent route $\xi^{\star\star}$ that satisfies the constraint and hence – together with $L^{\star\star}$ from (6) – is a global minimizer of the constrained problem, which indicates that the ground-speed-constraint (11) is only weakly active. We confirm this by showing that the corresponding Lagrange multipliers $\lambda^{\star\star}$ vanish.

Lemma 2. *Let* $(\xi^{\star\star}, L^{\star\star})$ *be a global minimizer of* (4)*. Then, this solution together with*

$$\lambda^{\star\star} = 0 \tag{17}$$

satisfies the necessary conditions (16)*.*

Beweis. Since $\xi^{\star\star}$ is also a global minimizer of the unconstrained problem, the necessary condition (10) states that $T'(\xi^{\star\star})\delta\xi = 0$. The term $\int_0^1 \lambda^{\star\star}\left(\delta\xi_\tau^T\xi_\tau^{\star\star} - \delta L\, L^{\star\star}\right) d\tau$ of (16a) vanishes for $\lambda^{\star\star} = 0$. (16b) is satisfied because $\|\xi_\tau^{\star\star}\| = L^{\star\star}$ for almost all $\tau \in]0, 1[$. □

3.1.2 Sufficient Conditions

Now we turn to the second order sufficient conditions for optimality. In general, a stationary point $(z^\star, \lambda^\star)$ is a strict minimizer, if, in addition to the necessary conditions above, the well known Ladyzhenskaya–Babuška–Brezzi (LBB) conditions (e.g., [9]) are satisfied, which comprise a) the so called *inf-sup* condition and b) the requirement that the Lagrangian's Hessian regarding z, $\mathcal{L}_{zz}$, need be positive definite on the kernel of h'.

The *inf-sup* condition states that for the minimizer $z^\star$ there is a $\kappa > 0$ such that

$$\inf_{\delta\lambda \neq 0 \in L^2(]0,1[)} \sup_{\delta z \in \delta Z^2} \frac{\langle \delta\lambda, h'(z^\star)[\delta z] \rangle}{\|\delta\lambda\|_{L^2(]0,1[)} \|\delta z\|_{Z^2}} \geq \kappa. \tag{18}$$

Formally, the second part of the LBB-conditions requires that there is a $\underline{\mathcal{B}} > 0$ such that

$$\mathcal{L}_{zz}(z^\star)[\delta z]^2 \geq \underline{\mathcal{B}}\ \|\delta z\|_{Z^2}^2$$

for any $\delta z \in \delta Z$ that satisfies

$$\langle \delta\lambda, h'(z^\star)[\delta z]\rangle = 0 \quad \forall\, \delta\lambda \in L^2(]0,1[).$$

In the present case, this reads

$$T''(\xi^\star)[\delta\xi]^2 + 2\int_0^1 \lambda^\star(\delta\xi_\tau^T\delta\xi_\tau - \delta L^2)d\tau \geq \underline{\mathcal{B}}\|\delta z\|_{Z^2}^2 \tag{19}$$

for any $\delta z \in \delta Z$ such that

$$\int_0^1 \delta\lambda \left(\delta\xi_\tau^T \xi_\tau^\star - \delta L\, L^\star\right) d\tau = 0 \quad \forall\, \delta\lambda \in L^2(]0,1[).$$

In case of a global minimizer $z^{\star\star} = (\xi^{\star\star}, L^{\star\star})$, this can be reduced using $\lambda^{\star\star} = 0$ from Lemma 2. Moreover, the constraint is equivalent to requiring that $\delta\xi_\tau^T\xi_\tau^{\star\star} = \delta L\, L^{\star\star}$ almost everywhere. With this, we conclude that for any isolated global minimizer $z^{\star\star}$ of (4) that satisfies the *inf-sup* condition, there exists a $\underline{\mathcal{B}} > 0$ such that

$$T''(\xi^{\star\star})[\delta\xi, \delta\xi_\tau]^2 \geq \underline{\mathcal{B}}\|\delta z\|_{Z^2}^2 \tag{20}$$

for any $\delta z \in \delta Z$ such that $\delta\xi_\tau^T\xi_\tau^{\star\star} = \delta L\, L^{\star\star}$ almost everywhere.

It is important to note that the second order sufficient conditions are formulated in a L^2-setting, while differentiability only holds in L^∞. This is known as *two-norm-discrepancy* [10].

3.2 Newton's Method

In order to provide a more compact notation, we use $\chi = (z, \lambda) \in Z \times \Lambda^* =: Y$ in this context and define F as the total derivative of the Lagrangian,

$$F : Z \times \Lambda^* \mapsto \delta Z^* \times \Lambda, \qquad F(\chi) := \mathcal{L}'(z, \lambda). \tag{21}$$

On Y we define the following norms,

$$\|\chi\|_{Y^\infty} = \|z\|_{Z^\infty} + \|\lambda\|_{L^\infty(]0,1[)} \quad \text{and} \tag{22a}$$

$$\|\chi\|_{Y^2} = \|z\|_{Z^2} + \|\lambda\|_{L^2(]0,1[)}. \tag{22b}$$

The problem is now to find a $\chi^{\star\star}$ such that the first order necessary conditions for optimality as stated in (16) are satisfied, which translates to

$$F(\chi^{\star\star}) = 0. \tag{23}$$

Applying Newton's method, we iteratively solve

$$F'(\chi^k)[\Delta\chi^k] = -F(\chi^k) \tag{24}$$

for $\Delta\chi^k$ and proceed with $\chi^{k+1} \leftarrow \chi^k + \Delta\chi^k$, starting with some initial value χ^0. In other words, in every iteration we need to find $(\Delta z^k, \Delta\lambda^k)$ such that

$$\begin{aligned} T''(\xi^k)[\delta\xi][\Delta\xi^k] + \langle\lambda^k, h''(z^k)[\delta z][\Delta z^k]\rangle + \langle\Delta\lambda^k, h'(z^k)[\delta z]\rangle \\ = -T'(\xi^k)[\delta\xi] - \langle\lambda^k, h'(z^k)[\delta z]\rangle \qquad \forall\delta z \in \delta Z, \end{aligned} \tag{25a}$$

$$\langle\delta\lambda, h'(z^k)[\Delta z^k]\rangle = -\langle\delta\lambda, h(z^k)\rangle \qquad \forall\delta\lambda \in \Lambda^*. \tag{25b}$$

4 Proof of Convergence

On the way to prove the existence of a non-empty domain $\mathcal{B}(\chi^{\star\star}, R)$ such that Newton's method as defined in Abschnitt 3.2 converges to the corresponding global minimizer $\chi^{\star\star}$, if initialized with a starting point within this neighborhood, we first prove that the KKT-operator F' is invertible and that the Newton step $\Delta\chi^k$ is always well defined. Essentially, this is the case if the LBB-conditions as given in (18) and (20) are satisfied. Hence, we will show that there is a $R > 0$ such that the *inf-sup* condition is satisfied and that the Lagrangian is positive definite on the kernel of the constraints for any $\chi \in \mathcal{B}(\chi^{\star\star}, R)$. Further, we show that an affine covariant Lipschitz condition holds, which finally helps to complete the proof.

Before we get there, we recall the following Lemma from [2, Lemma 7] which provides a bound for the path length of a global minimizer.

Lemma 3. *Let $z^{\star\star} = (L^{\star\star}, \xi^{\star\star})$ be a global minimizer of* (4)*, let $\|w\|_{L^\infty(\Omega)} \leq \overline{c}_0$, and define $\tilde{L} = \|x_D - x_O\|$. Then it holds that*

$$\tilde{L} \leq L^{\star\star} \leq \frac{\overline{v} + \overline{c}_0}{\overline{v} - \overline{c}_0} \tilde{L}. \tag{26}$$

As most of the subsequent results hold in a L^∞-neighborhood of a minimizer, we introduce the following notation.

Definition 2. *We call the L^∞-neighborhood of a point $z \in Z$ or $x \in Y$,*

$$\mathcal{B}(z, R) := \{\tilde{z} \in Z : \|\tilde{z} - z\|_{Z^\infty} \leq R\} \quad \textit{or} \tag{27a}$$

$$\mathcal{B}(\chi, R) := \{\tilde{\chi} \in Y : \|\tilde{\chi} - \chi\|_{Y^\infty} \leq R\}, \tag{27b}$$

respectively.

Moreover, we provide three simple yet useful bounds that hold in such a L^∞-neighborhood of a minimizer.

Lemma 4. *Let $\chi^{\star\star} = (z^{\star\star}, \lambda^{\star\star})$ be a global minimizer of* (4) *and the corresponding Lagrange multipliers. Then for every $\chi \in \mathcal{B}(\chi^{\star\star}, R)$ it holds that*

$$L^{\star\star} - R \leq L \leq L^{\star\star} + R, \tag{28a}$$

$$L^{\star\star} - R \leq \|\xi_\tau\|_{L^\infty(]0,1[)} \leq L^{\star\star} + R, \tag{28b}$$

$$0 \leq \|\lambda\|_{L^\infty(]0,1[)} \leq R. \tag{28c}$$

Beweis. The first two inequalities follow immediately, since a global minimizer satisfies the constraint from (4). The latter two are a direct consequence of Lemma 2. □

4.1 Inf-Sup Condition

We now show that the *inf-sup* condition, introduced in (18), holds in a certain neighborhood around a global minimizer. First, however, we point out that deviations $\delta\xi$ and $\delta\xi_\tau$ from a trajectory are inherently related and that the former is always bounded by the latter.

Theorem 5 (Wirtinger's inequality)**.** *Let $\delta\xi \in H_0^1(]0,1[)$. Then*

$$\|\delta\xi\|^2_{L^2(]0,1[)} \leq \frac{1}{\pi}\|\delta\xi_\tau\|^2_{L^2(]0,1[)} \tag{29}$$

holds.

Theorem 6. *Let $z^{\star\star}$ be a global minimizer of* (4)*. Further, let there be a constant $c > 0$ and some direction $u \in \mathbb{R}^2$ with $\|u\| = 1$ such that $u^T\xi_\tau^{\star\star} \geq c$ for almost all $\tau \in]0,1[$. Then for any $z = (L,\xi) \in \mathcal{B}(z^{\star\star}, R)$ with $R < c$ there is some $\kappa > 0$ such that*

$$\inf_{\lambda\neq 0\in L^2(]0,1[)} \sup_{\delta z\in\delta Z} \frac{\langle\lambda, h'(z)[\delta z]\rangle}{\|\lambda\|_{L^2(]0,1[)}\|\delta z\|_{Z^2}} \geq \kappa$$

with

$$\kappa(R) = (c-R)\left[\frac{3}{8} + 2\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)^2\right]^{-1/2}.$$

Beweis. For $f \in L^2(]0,1[)$ we define

$$\overline{f} := \int_0^1 f\,d\tau \in \mathbb{R} \quad \text{and} \quad \tilde{f} = f - \overline{f},$$

respectively, such that $(\overline{f},\tilde{f})_{L^2(0,1)} = 0$ and

$$\|f\|^2_{L^2(]0,1[)} = \|\tilde{f}+\overline{f}\|^2_{L^2(]0,1[)} = \|\tilde{f}\|^2_{L^2(]0,1[)} + \overline{f}^2.$$

With

$$\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0}\tilde{L} + R \underset{(26)}{\geq} L^{\star\star} + R \geq b := \xi_\tau^T u \geq c - R \tag{30}$$

we choose $\delta\xi_\tau = \frac{1}{2}\tilde{\lambda}u$ and $\delta L = \frac{1}{2L}\left(\overline{b\tilde{\lambda}} - (c-R)\overline{\lambda}\right)$. Note that $\delta\xi \in \delta X$ holds. For this choice, we obtain for $\delta z = (\delta L, \delta\xi)$

$$\begin{aligned}
\langle\lambda, h'(z)[\delta z]\rangle &= \int_0^1 (2\xi_\tau^T\delta\xi_\tau\lambda - 2L\delta L\lambda)\,d\tau \\
&= \int_0^1 b\tilde{\lambda}\lambda\,d\tau - 2L\delta L\overline{\lambda} \\
&= \int_0^1 (b\tilde{\lambda}^2 + b\tilde{\lambda}\overline{\lambda})\,d\tau - 2L\delta L\overline{\lambda} \\
&\underset{(30)}{\geq} (c-R)\|\tilde{\lambda}\|^2_{L^2(]0,1[)} + \left(\int_0^1 b\tilde{\lambda}\,d\tau - 2L\delta L\right)\overline{\lambda} \\
&= (c-R)\|\tilde{\lambda}\|^2_{L^2(]0,1[)} + \left(\int_0^1 b\tilde{\lambda}\,d\tau - \overline{b\tilde{\lambda}} + (c-R)\overline{\lambda}\right)\overline{\lambda} \\
&= (c-R)\left(\|\tilde{\lambda}\|^2_{L^2(]0,1[)} + \overline{\lambda}^2\right) \\
&= (c-R)\|\lambda\|^2_{L^2(]0,1[)}.
\end{aligned}$$

Moreover, we have

$$\|\delta\xi_\tau\|_{L^2(]0,1[)} \le \frac{1}{2}\|\tilde{\lambda}\|_{L^2(]0,1[)}$$

and, since clearly $c \le L^{\star\star}$,

$$\begin{aligned}
|\delta L| &\le \frac{1}{2L}\left(\|b\|_{L^2(]0,1[)}\|\tilde{\lambda}\|_{L^2(]0,1[)} + (c-R)|\overline{\lambda}|\right) \\
&\underset{(30)}{\le} \frac{1}{\tilde{L}}\left((L^{\star\star}+R)\|\tilde{\lambda}\|_{L^2(]0,1[)} + (c-R)|\overline{\lambda}|\right) \\
&\le \left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)\left(\|\tilde{\lambda}\|_{L^2(]0,1[)} + |\overline{\lambda}|\right),
\end{aligned}$$

which implies

$$\begin{aligned}
\|\delta z\|_{Z^2}^2 &\underset{(13b)}{=} \|\delta\xi\|_{L^2(]0,1[)}^2 + \|\delta\xi_\tau\|_{L^2(]0,1[)}^2 + \delta L^2 \\
&\underset{(29)}{\le} \frac{3}{2}\|\delta\xi_\tau\|_{L^2(]0,1[)}^2 + \delta L^2 \\
&\le \left(\frac{3}{8}\|\tilde{\lambda}\|_{L^2(]0,1[)}^2 + \left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)^2\left(\|\tilde{\lambda}\|_{L^2(]0,1[)} + \overline{\lambda}\right)^2\right) \\
&\le \left(\frac{3}{8}\|\tilde{\lambda}\|_{L^2(]0,1[)}^2 + 2\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)^2\|\tilde{\lambda}\|_{L^2(]0,1[)}^2 + 2\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)^2\overline{\lambda}^2\right) \\
&\le \left[\frac{3}{8} + 2\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)^2\right]\left(\|\tilde{\lambda}\|_{L^2(]0,1[)}^2 + \overline{\lambda}^2\right) \\
&= \left[\frac{3}{8} + 2\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)^2\right]\|\lambda\|_{L^2(]0,1[)}^2.
\end{aligned}$$

Consequently,

$$\langle \lambda, h'(z)[\delta z]\rangle \ge (c-R)\left[\frac{3}{8} + 2\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0} + \frac{R}{\tilde{L}}\right)^2\right]^{-1/2}\|\lambda\|_{L^2(]0,1[)}\|\delta z\|_{Z^2}$$

yields the claim. □

4.2 Positive Definiteness of the Lagrangian

The next step in order prove invertibility of the KKT-operator $F'(\chi)$, (24), is to show that the second partial derivative of the Lagrangian $\mathcal{L}(\chi)$, (15), with respect to the state z is positive definite on the kernel of the linearized constraints. On the way we derive a similar result for the objective $T(\xi)$, (1) for which we first derive an upper bound for its third derivative.

Lemma 7. *Let $\|w\|_{L^\infty(\Omega)} \le \overline{c}_0 \le \overline{v}/\sqrt{5}$, $\|w_x\|_{L^\infty(\Omega)} \le \overline{c}_1$, $\|w_{xx}\|_{L^\infty(\Omega)} \le \overline{c}_2$, and $\|w_{xxx}\|_{L^\infty(\Omega)} \le \overline{c}_3$ and define $\underline{v}^2 := \overline{v}^2 - \overline{c}_0^2$. Then, for any $\xi \in X$, the third* *directional derivative of f as given in* (2) *is bounded by*

$$
\begin{aligned}
&|f'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau]| \\
&\qquad\le \left(\overline{\gamma}_0\|\xi_\tau\|\|\delta\xi\|^2+\overline{\gamma}_2\|\delta\xi\|\|\delta\xi_\tau\|+\frac{\overline{\gamma}_4}{\|\xi_\tau\|}\|\delta\xi_\tau\|^2\right)\ \|\Delta\xi\| \\
&\qquad\quad+\left(\overline{\gamma}_1\|\delta\xi\|^2+\frac{\overline{\gamma}_3}{\|\xi_\tau\|}\|\delta\xi\|\|\delta\xi_\tau\|+\frac{\overline{\gamma}_5}{\|\xi_\tau\|^2}\|\delta\xi_\tau\|^2\right)\ \|\Delta\xi_\tau\| \qquad (31)
\end{aligned}
$$

with $\overline{\gamma}_i\ge 0$, $i\in 0,\dots,5$, given as

$$
\begin{aligned}
\overline{\gamma}_0 &= \frac{2}{\underline{v}^4}\left(37\overline{c}_1^3+21\overline{c}_1\overline{c}_2\underline{v}+2\overline{c}_3\underline{v}^2\right), & \overline{\gamma}_3 &= 40\frac{\overline{c}_1}{\underline{v}^2}, \\
\overline{\gamma}_1 &= \frac{1}{\underline{v}^3}\left(29\overline{c}_1^2+7\underline{v}\overline{c}_2\right), & \overline{\gamma}_4 &= 20\frac{\overline{c}_1}{\underline{v}^2}, \\
\overline{\gamma}_2 &= \frac{1}{\underline{v}^3}(57\overline{c}_1^2+13\underline{v}\overline{c}_2), & \overline{\gamma}_5 &= 18\frac{1}{\underline{v}}. \qquad (32)
\end{aligned}
$$

The proof can again be found in the appendix. With this result we can derive a bound for the third directional derivative of T.

Theorem 8. *Let $(L^{\star\star},\xi^{\star\star})$ be a global minimizer of* (4) *and define $\tilde{L}:=\|x_D-x_O\|$ and $\Delta\xi:=\xi-\xi^{\star\star}$. Moreover, let $\|w(p)\|\le\overline{c}_0\le\overline{v}/\sqrt{5}$, $\|w_x(p)\|\le\overline{c}_1$, $\|w_{xx}(p)\|\le\overline{c}_2$, and $\|w_{xxx}(p)\|\le\overline{c}_3$ for every $p\in\Omega$. Then, for any $\xi\in X$ with $\|\Delta\xi\|_{X^\infty}\le R<\tilde{L}$, it holds that*

$$
|T'''(\xi)[\delta\xi]^2[\Delta\xi]| \;\le\; \overline{\Gamma}\left(\|\delta\xi\|^2_{L^2(]0,1[)}+\|\delta\xi_\tau\|^2_{L^2(]0,1[)}\right)\ \|\Delta\xi\|_{C^{0,1}(]0,1[)}. \qquad (33)
$$

with $\|\Delta\xi\|_{C^{0,1}(]0,1[)}=\|\Delta\xi\|_{L^\infty(]0,1[)}+\|\Delta\xi_\tau\|_{L^\infty(]0,1[)}$ and

$$
\begin{aligned}
\overline{\Gamma}:=\max\Bigg\{&\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0}\tilde{L}+R\right)\overline{\gamma}_0+\frac{\overline{\gamma}_2}{2},\quad \frac{\overline{\gamma}_4}{\tilde{L}-R}+\frac{\overline{\gamma}_2}{2}, \\
&\overline{\gamma}_1+\frac{\overline{\gamma}_3}{2(\tilde{L}-R)},\quad \frac{\overline{\gamma}_3}{2(\tilde{L}-R)}+\frac{\overline{\gamma}_5}{(\tilde{L}-R)^2}\Bigg\} \qquad (34)
\end{aligned}
$$

and $\overline{\gamma}_0,\dots,\overline{\gamma}_5$ as given in Lemma 7 above.

Beweis. From the definition of T in (1), we know that

$$
T'''(\xi)[\delta\xi]^2[\Delta\xi]=\int_0^1 f'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau]d\tau.
$$

Inserting the bound from Lemmata 4 und 7 above and using Young's inequality yields

$$
\begin{aligned}
&|T'''(\xi)[\delta\xi]^2[\Delta\xi]| \\
&\leq \int_0^1 \left(\overline{\gamma}_0\|\xi_\tau\|\|\delta\xi\|^2 + \overline{\gamma}_2\|\delta\xi\|\|\delta\xi_\tau\| + \frac{\overline{\gamma}_4}{\|\xi_\tau\|}\|\delta\xi_\tau\|^2\right)\ \|\Delta\xi\| \\
&\qquad + \left(\overline{\gamma}_1\|\delta\xi\|^2 + \frac{\overline{\gamma}_3}{\|\xi_\tau\|}\|\delta\xi\|\|\delta\xi_\tau\| + \frac{\overline{\gamma}_5}{\|\xi_\tau\|^2}\|\delta\xi_\tau\|^2\right)\ \|\Delta\xi_\tau\|\ d\tau. \\
&\leq \|\Delta\xi\|_{L^\infty}\int_0^1 \overline{\gamma}_0\|\xi_\tau\|\|\delta\xi\|^2 + \overline{\gamma}_2\|\delta\xi\|\|\delta\xi_\tau\| + \frac{\overline{\gamma}_4}{\|\xi_\tau\|}\|\delta\xi_\tau\|^2 d\tau \\
&\quad + \|\Delta\xi_\tau\|_{L^\infty}\int_0^1 \overline{\gamma}_1\|\delta\xi\|^2 + \frac{\overline{\gamma}_3}{\|\xi_\tau\|}\|\delta\xi\|\|\delta\xi_\tau\| + \frac{\overline{\gamma}_5}{\|\xi_\tau\|^2}\|\delta\xi_\tau\|^2 d\tau \\
&\underset{(28)}{\leq} \|\Delta\xi\|_{L^\infty}\int_0^1 \left(\frac{\overline{v}+\bar{c}_0}{\overline{v}-\bar{c}_0}\tilde{L}+R\right)\overline{\gamma}_0\|\delta\xi\|^2 + \overline{\gamma}_2\|\delta\xi\|\|\delta\xi_\tau\| + \frac{\overline{\gamma}_4}{\tilde{L}-R}\|\delta\xi_\tau\|^2 d\tau \\
&\quad + \|\Delta\xi_\tau\|_{L^\infty}\int_0^1 \overline{\gamma}_1\|\delta\xi\|^2 + \frac{\overline{\gamma}_3}{\tilde{L}-R}\|\delta\xi\|\|\delta\xi_\tau\| + \frac{\overline{\gamma}_5}{(\tilde{L}-R)^2}\|\delta\xi_\tau\|^2 d\tau \\
&\underset{(\mathrm{Y})}{\leq} \|\Delta\xi\|_{L^\infty}\left[\left(\left(\frac{\overline{v}+\bar{c}_0}{\overline{v}-\bar{c}_0}\tilde{L}+R\right)\overline{\gamma}_0 + \frac{\overline{\gamma}_2}{2}\right)\|\delta\xi\|_{L^2}^2 + \left(\frac{\overline{\gamma}_4}{\tilde{L}-R} + \frac{\overline{\gamma}_2}{2}\right)\|\delta\xi_\tau\|_{L^2}^2\right] \\
&\quad + \|\Delta\xi_\tau\|_{L^\infty}\left[\left(\overline{\gamma}_1 + \frac{\overline{\gamma}_3}{2(\tilde{L}-R)}\right)\|\delta\xi\|_{L^2}^2 + \left(\frac{\overline{\gamma}_3}{2(\tilde{L}-R)} + \frac{\overline{\gamma}_5}{(\tilde{L}-R)^2}\right)\|\delta\xi_\tau\|_{L^2}^2\right] \\
&\underset{(34)}{\leq} \overline{\Gamma}\left(\|\delta\xi\|_{L^2(]0,1[)}^2 + \|\delta\xi_\tau\|_{L^2(]0,1[)}^2\right)\ \|\Delta\xi\|_{C^{0,1}(]0,1[)}.
\end{aligned}
$$

□

Having bounded the third derivative of T, we can estimate the potential decay of T'' and thus derive a lower bound for the size of this neighborhood. Similarly, we can bound h'' and hence $\mathcal{L}_{zz}$.

Theorem 9. *Let $\|w\|_{L^\infty(\Omega)} \leq \bar{c}_0 < \overline{v}/\sqrt{5}$, $\|w_x\|_{L^\infty(\Omega)} \leq \bar{c}_1$, $\|w_{xx}\|_{L^\infty(\Omega)} \leq \bar{c}_2$, and $\|w_{xxx}\|_{L^\infty(\Omega)} \leq \bar{c}_3$ and define $\tilde{L} := \|x_D - x_O\|$. Moreover, let $\chi^{\star\star} := (z^{\star\star}, \lambda^{\star\star})$ be a globally optimal solution to problem* (4)*, that satisfies the necessary and sufficient conditions* (16)*,* (18)*, and* (20) *with $\underline{\mathcal{B}} > 0$. Then there is a $0 < R < \min\left\{\frac{\underline{\mathcal{B}}}{2\overline{\Gamma}},\ \frac{\underline{\mathcal{B}}}{40},\ \frac{\tilde{L}}{2}\right\}$ with $\overline{\Gamma}$ from Theorem 8 such that*

$$
\mathcal{L}_{zz}(\chi)[\delta z]^2 \geq \frac{\underline{\mathcal{B}}}{4}\|\delta z\|_{Z^2}^2 \tag{35}
$$

holds for any $\chi \in \mathcal{B}(\chi^{\star\star}, R)$ and any $\delta z \in \delta Z$ such that $\xi_\tau^T\delta\xi_\tau = L\delta L$ holds almost everywhere.

Beweis. Let $\Delta\xi := \xi - \xi^{\star\star}$ and note that $\|\Delta\xi\|_{L^\infty(]0,1[)} \leq \|\Delta z\|_{Z^\infty} \leq R < \frac{\underline{\mathcal{B}}}{2\overline{\Gamma}}$.

Then we obtain

$$
\begin{aligned}
T''(\xi)[\delta\xi, \delta\xi_\tau]^2 &= T''(\xi^{\star\star})[\delta\xi, \delta\xi_\tau]^2 + \int_0^1 T'''(\xi + \nu\Delta\xi)[\delta\xi, \delta\xi_\tau]^2[\Delta\xi, \Delta\xi_\tau]\, d\nu \\
&\underset{(20)}{\geq} \underline{\mathcal{B}}\|\delta z\|_{Z^2}^2 + \int_0^1 T'''(\xi + \nu\Delta\xi)[\delta\xi, \delta\xi_\tau]^2[\Delta\xi, \Delta\xi_\tau]\, d\nu \\
&\underset{(33)}{\geq} \underline{\mathcal{B}}\|\delta z\|_{Z^2}^2 - \overline{\Gamma}(\|\delta\xi\|_{L^2(]0,1[)}^2 + \|\delta\xi_\tau\|_{L^2(]0,1[)}^2)\ \|\Delta z\|_{Z^\infty} \\
&\underset{(13b)}{\geq} \underline{\mathcal{B}}\|\delta z\|_{Z^2}^2 - \overline{\Gamma}\|\delta z\|_{Z^2}^2\ \|\Delta z\|_{Z^\infty}, \\
&\geq \frac{\underline{\mathcal{B}}}{2}\|\delta z\|_{Z^2}^2.
\end{aligned}
$$

Further, we point out that

$$
R \leq \frac{\tilde{L}}{2} \leq \frac{L^{\star\star}}{2}, \tag{36}
$$

which together with the bounds from Lemma 4 yields

$$
\begin{aligned}
\langle h''(z)[\delta z]^2\rangle &= \int_0^1 \lambda\left(\delta\xi_\tau^T\delta\xi_\tau - \delta L^2\right) d\tau \\
&= \int_0^1 \lambda\left(\|\delta\xi_\tau\|^2 - \left(\frac{\xi_\tau^T\delta\xi_\tau}{L}\right)^2\right) d\tau \\
&\geq -\|\lambda\|_{L^\infty(]0,1[)}\left(\|\delta\xi_\tau\|_{L^2(]0,1[)}^2 + \int_0^1 \frac{\|\xi_\tau\|^2\|\delta\xi_\tau\|^2}{L^2}\ d\tau\right) \\
&\geq -\|\lambda\|_{L^\infty(]0,1[)}\left(\|\delta\xi_\tau\|_{L^2(]0,1[)}^2 + \frac{\|\xi_\tau\|_{L^\infty(]0,1[)}^2}{L^2}\int_0^1 \|\delta\xi_\tau\|^2\ d\tau\right) \\
&\underset{(28)}{\geq} -R\left(\|\delta\xi_\tau\|_{L^2(]0,1[)}^2 + \frac{(L^{\star\star}+R)^2}{(L^{\star\star}-R)^2}\|\delta\xi_\tau\|_{L^2(]0,1[)}^2\right) \\
&\geq -R\left(1 + \frac{(L^{\star\star}+R)^2}{(L^{\star\star}-R)^2}\right)\|\delta\xi_\tau\|_{L^2(]0,1[)}^2 \\
&\underset{(36)}{\geq} -10R\|\delta\xi_\tau\|_{L^2(]0,1[)}^2 \\
&\geq -\frac{\underline{\mathcal{B}}}{4}\|\delta\xi_\tau\|_{L^2(]0,1[)}^2 \\
&\underset{(13b)}{\geq} -\frac{\underline{\mathcal{B}}}{4}\|\delta z\|_{Z^2}^2.
\end{aligned}
$$

Together, these bounds yield the claim with

$$
\begin{aligned}
\mathcal{L}_{zz}(\chi)[\delta z]^2 &= T''(\xi)[\delta\xi]^2 + \langle h''(z)[\delta z]^2\rangle \\
&\geq \frac{\underline{\mathcal{B}}}{2}\|\delta z\|_{Z^2}^2 - \frac{\underline{\mathcal{B}}}{4}\|\delta z\|_{Z^2}^2 \\
&\geq \frac{\underline{\mathcal{B}}}{4}\|\delta z\|_{Z^2}^2.
\end{aligned}
$$

□

4.3 Upper Bound for the Lagrangian

As a counterpart to the previous Lemma, we also derive an upper bound for L_{zz} close to a minimizer. Again we start with the underlying function f in order to bound the error in the objective function T.

Lemma 10. *Let $\|w\|_{L^\infty(\Omega)} \le \overline{c}_0 \le \overline{v}/\sqrt{5}$, $\|w_x\|_{L^\infty(\Omega)} \le \overline{c}_1$, and $\|w_{xx}\|_{L^\infty(\Omega)} \le \overline{c}_2$. Moreover, let $\underline{v}^2 := \overline{v}^2 - \overline{c}_0^2$. Then, for any $\xi \in X$, the second directional derivative of f as given in (2) is bounded by*

$$\begin{aligned} |f''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi}_\tau]| \le \; & \overline{\beta}_0\|\xi_\tau\|\|\delta\xi\|\|\tilde{\delta\xi}\| \\ & +\overline{\beta}_1\left(\|\delta\xi\|\|\tilde{\delta\xi}_\tau\| + \|\delta\xi_\tau\|\|\tilde{\delta\xi}\|\right) \\ & +\overline{\beta}_2\|\xi_\tau\|^{-1}\|\delta\xi_\tau\|\|\tilde{\delta\xi}_\tau\| \end{aligned} \tag{37}$$

with

$$\overline{\beta}_0 = 14\frac{\overline{c}_1^2}{\underline{v}^3} + 4\frac{\overline{c}_2}{\underline{v}^2}, \qquad \overline{\beta}_1 = 7\frac{\overline{c}_1}{\underline{v}^2}, \text{ and } \qquad \overline{\beta}_2 = \frac{4}{\underline{v}}. \tag{38}$$

The proof can be found in the appendix.

Theorem 11. *Let $z^{\star\star} = (L^{\star\star}, \xi^{\star\star})$ be a global minimizer of (4) and $\Delta z := z - z^{\star\star}$. Moreover, let $\|w\|_{L^\infty(\Omega)} \le \overline{c}_0 \le \overline{v}/\sqrt{5}$, $\|w_x\|_{L^\infty(\Omega)} \le \overline{c}_1$, and $\|w_{xx}\|_{L^\infty(\Omega)} \le \overline{c}_2$. Also define $\underline{v}^2 := \overline{v}^2 - \overline{c}_0^2$ and $\tilde{L} := \|x_D - x_O\|$. Then, for any $z \in \mathcal{B}(z^{\star\star}, R)$, the second directional derivative of T as defined in (1) is bounded by*

$$|T''(\xi)[\Delta\xi]^2| \le \overline{\mathcal{B}}\|\Delta z\|_{Z^2}^2 \tag{39}$$

with $\overline{\mathcal{B}} := \overline{\beta}_1 + \max\left\{\left(\frac{\overline{v}+\overline{c}_0}{\overline{v}-\overline{c}_0}\tilde{L} + R\right)\overline{\beta}_0,\ \frac{\overline{\beta}_2}{\tilde{L}+R}\right\}$ and $\overline{\beta}_0, \overline{\beta}_1, \overline{\beta}_2$ as defined in Lemma 10.

Beweis. From the definition of T in (1) we know that

$$T''(\xi)[\Delta\xi, \Delta\xi_\tau]^2 = \int_0^1 f''[\Delta\xi, \Delta\xi_\tau]^2 d\tau,$$

which, together with the bounds from Lemmata 4 und 10 as well as Young's inequality, then leads to

$$
\begin{aligned}
|T''(\xi)[\Delta\xi, \Delta\xi_\tau]^2| \leq & \int_0^1 \left(\overline{\beta}_0 \|\xi_\tau\| \|\Delta\xi\|^2 + 2\overline{\beta}_1 \|\Delta\xi\| \|\Delta\xi_\tau\| + \frac{\overline{\beta}_2}{\|\xi_\tau\|} \|\Delta\xi_\tau\|^2\right) d\tau \\
& \underset{(28)}{\leq} \overline{\beta}_0 (L^{\star\star} + R) \int_0^1 \|\Delta\xi\|^2 d\tau \\
& \quad + 2\overline{\beta}_1 \int_0^1 \|\Delta\xi\| \|\Delta\xi_\tau\| d\tau \\
& \quad + \frac{\overline{\beta}_2}{L^{\star\star} + R} \int_0^1 \|\Delta\xi_\tau\|^2 d\tau \\
& \underset{(\mathrm{Y})}{\leq} \left((L^{\star\star} + R)\overline{\beta}_0 + \overline{\beta}_1\right) \|\Delta\xi\|^2_{L^2(]0,1[)} \\
& \quad + \left(\overline{\beta}_1 + \frac{\overline{\beta}_2}{L^{\star\star} + R}\right) \|\Delta\xi_\tau\|^2_{L^2(]0,1[)} \\
& \underset{(26)}{\leq} \left(\left(\frac{\overline{v} + \overline{c}_0}{\overline{v} - \overline{c}_0} \tilde{L} + R\right) \overline{\beta}_0 + \overline{\beta}_1\right) \|\Delta\xi\|^2_{L^2(]0,1[)} \\
& \quad + \left(\overline{\beta}_1 + \frac{\overline{\beta}_2}{\tilde{L} + R}\right) \|\Delta\xi_\tau\|^2_{L^2(]0,1[)} \\
& \leq \overline{\mathcal{B}} \left(\|\Delta\xi\|^2_{L^2(]0,1[)} + \|\Delta\xi_\tau\|^2_{L^2(]0,1[)}\right) \\
& \underset{(13b)}{\leq} \overline{\mathcal{B}} \|\Delta z\|^2_{Z^2}.
\end{aligned}
$$

□

Theorem 12. *Let $\chi^{\star\star} = (z^{\star\star}, \lambda^{\star\star})$ be a global minimizer of (4) and the corresponding Lagrange multipliers. Then for every $\chi \in \mathcal{B}(\chi^{\star\star}, R)$ and every $\delta z \in \delta Z$ it holds that*

$$
|\mathcal{L}_{zz}(\chi)[\delta z]^2| \leq \left(\overline{\mathcal{B}} + R\right) \|\delta z\|^2_{Z^2} \tag{40}
$$

with $\overline{\mathcal{B}}(R)$ from Theorem 11.

Beweis. Using the bound from Theorem 11 and Young's inequality, we get

$$
\begin{aligned}
|\mathcal{L}_{zz}(\chi)[\delta z]^2| = & |T''(\xi)[\delta\xi]^2 + \langle h''(z)[\delta z]^2 \rangle| \\
& \underset{(39)}{\leq} \overline{\mathcal{B}} \|dz\|^2_{Z^2} + \int_0^1 |\lambda \left(\delta\xi_\tau^T \delta\xi_\tau - \delta L^2\right)| \, d\tau \\
& \leq \overline{\mathcal{B}} \|dz\|^2_{Z^2} + \|\lambda\|_{L^\infty(]0,1[)} \left(\|\delta\xi_\tau\|^2_{L^2(]0,1[)} + \delta L^2\right) \\
& \underset{(28)}{\leq} \overline{\mathcal{B}} \|dz\|^2_{Z^2} + R\left(\|\delta\xi_\tau\|^2_{L^2(]0,1[)} + \delta L^2\right) \\
& \underset{(13b)}{\leq} \left(\overline{\mathcal{B}} + R\right) \|dz\|^2_{Z^2}.
\end{aligned}
$$

□

4.4 Invertibility of the KKT-Operator

Using the previous three results, which together state the existence of a neighborhood around a minimizer such that the LBB-conditions are satisfied, we are now ready to prove that the KKT-operator F' is invertible.

Lemma 13. *Let $\chi^{\star\star} = (z^{\star\star}, \lambda^{\star\star})$ be a global minimizer of (4), that satisfies the first and second order conditions for optimality with some $\underline{\mathcal{B}} > 0$, and the corresponding Lagrange multipliers. Further, let there be a u with $\|u\| = 1$ such that $u^T \xi_\tau^{\star\star} \geq c > 0$ for almost all $\tau \in]0, 1[$. Then for F as given in (21) it holds that*

$$\|F'(\chi)^{-1}\|_{Y^2} \leq \omega_1 \tag{41}$$

for every $\chi = (z, \lambda) \in \mathcal{B}(\chi^{\star\star}, R)$ and

$$\omega_1 = \sqrt{2} \max \left\{ \frac{4}{\underline{\mathcal{B}}}, \frac{1}{\kappa}\left(1 + \frac{4(\overline{\mathcal{B}} + R)}{\underline{\mathcal{B}}}\right), \frac{\overline{\mathcal{B}} + R}{\kappa^2} \right\} \tag{42}$$

and $\overline{\mathcal{B}}(R)$ and $\kappa(R)$ as given in Theorem 11 and Theorem 6, respectively.

Beweis. The proof builds on some prerequisites that have been established above and are briefly summarized.

i) In Theorem 6 it was proved that the *inf-sup* condition is satisfied:

$$\inf_{\delta\lambda \in L^2(]0,1[)} \sup_{\delta z \in \delta Z} \frac{\langle \delta\lambda, h'(z)[\delta z] \rangle}{\|\delta z\|_{Z^2} \|\delta\lambda\|_{L^2(]0,1[)}} \geq \kappa > 0.$$

ii) In Theorem 9 it was proved that $\mathcal{L}_{zz}$ is positive definite on the kernel of the constraints, i.e.,

$$\mathcal{L}_{zz}(\chi)[\delta z]^2 = T''(\xi)[\delta \xi]^2 + \langle h''(z)[\delta z]^2 \rangle \geq \frac{\underline{\mathcal{B}}}{4} \|\delta z\|_{Z^2}^2$$

for all $\delta z \in \delta Z$ such that $h'(z)[\delta z] = 0$.

iii) In Theorem 12 it was proved that $\mathcal{L}_{zz}$ is bounded from above as

$$|\mathcal{L}_{zz}(\chi)[\delta z]^2| = |T''(\xi)[\delta\xi]^2 + \langle h''(z)[\delta z]^2 \rangle| \leq (\overline{\mathcal{B}} + R)\|\delta z\|_{Z^2}^2.$$

Under these conditions, it follows from *Brezzi's Splitting Theorem* [9, Thm. 4.3] that $F'(x)$ is isomorphic. Further, it can be shown that for every right hand side $F(x)$ of the saddle point problem (24) there is exactly one solution $(\Delta z, \Delta\lambda)$ with

$$\begin{aligned}
\|\Delta z\|_{Z^2} \leq{}& \frac{4}{\underline{\mathcal{B}}} \|T'(\xi) + \langle \lambda, h'(z) \rangle\|_{Z^2} \\
&+ \frac{1}{\kappa}\left(1 + \frac{4(\overline{\mathcal{B}} + R)}{\underline{\mathcal{B}}}\right) \|h(z)\|_{L^2(]0,1[)}, \\
\|\Delta\lambda\|_{L^2(]0,1[)} \leq{}& \frac{1}{\kappa}\left(1 + \frac{4(\overline{\mathcal{B}} + R)}{\underline{\mathcal{B}}}\right) \|T'(\xi) + \langle \lambda, h'(z) \rangle\|_{Z^2} \\
&+ \frac{\overline{\mathcal{B}} + R}{\kappa^2}\left(1 + \frac{4(\overline{\mathcal{B}} + R)}{\underline{\mathcal{B}}}\right) \|h(z)\|_{L^2(]0,1[)}.
\end{aligned}$$

With $\|F(\chi)\| = \|T'(\xi) + \langle\lambda, h'(z)\rangle\|^2_{Z^2} + \|h(z)\|^2_{L^2(]0,1[)}$ follows that

$$\|\Delta z\|_{Z^2} \leq \sqrt{2}\max\left\{\frac{4}{\underline{\mathcal{B}}},\ \frac{1}{\kappa}\left(1+\frac{4(\overline{\mathcal{B}}+R)}{\underline{\mathcal{B}}}\right)\right\}\|F(\chi)\|,$$

$$\|\Delta\lambda\|_{L^2(]0,1[)} \leq \sqrt{2}\max\left\{\frac{1}{\kappa}\left(1+\frac{4(\overline{\mathcal{B}}+R)}{\underline{\mathcal{B}}}\right),\ \frac{\overline{\mathcal{B}}+R}{\kappa^2}\right\}\|F(\chi)\|,$$

which directly yields

$$\|\Delta\chi\|^2_{Y^2} \underset{(22b)}{=} \|\Delta z\|^2_{Z^2} + \|\Delta\lambda\|^2_{L^2(]0,1[)} \leq \omega_1^2\|F(\chi)\|$$

with $\omega_1 = \sqrt{2}\max\left\{\frac{4}{\underline{\mathcal{B}}}, \frac{1}{\kappa}\left(1+\frac{4(\overline{\mathcal{B}}+R)}{\underline{\mathcal{B}}}\right), \frac{\overline{\mathcal{B}}+R}{\kappa^2}\right\}$. This completes the proof, since

$$\|F'(\chi)^{-1}\|_{Y^2} = \sup_{\|F(\chi)\|_{Y^2}} \frac{\|\Delta\chi\|_{Y^2}}{\|F(\chi)\|_{Y^2}} \leq \omega_1.$$

□

4.5 Lipschitz Constant

We are almost ready to provide a Lipschitz constant for the free flight problem. One more bound is given in the following Lemma.

Lemma 14. *Let $\chi^{\star\star} = (z^{\star\star}, \lambda^{\star\star})$ be a global minimizer of (4) and the corresponding Lagrange multipliers. For any $\chi_{i\in\{1,2\}} \in \mathcal{B}(\chi^{\star\star}, R)$ there is a $\hat{\mathcal{B}}$ such that*

$$\|(F'(\chi_2) - F'(\chi_1))[\chi_2 - \chi_1]\|_{Y^2} \leq \omega_2\|\chi_2 - \chi_1\|_{Y^2} \tag{43}$$

with

$$\omega_2 = (8 + \hat{\mathcal{B}})R. \tag{44}$$

Beweis. From Lemma 4 and Lemma 4 it directly follows that

$$|L_2 - L_1| \leq 2R, \tag{45a}$$
$$\|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^\infty(]0,1[)} \leq 2R, \tag{45b}$$
$$\|\lambda_2 - \lambda_1\|_{L^\infty(]0,1[)} \leq R. \tag{45c}$$

Using these bounds as well as the Cauchy-Schwarz inequality and Young's inequality, we show that for any $\delta\chi \in \delta Z \times L^2(]0,1[)$ with $\|\delta\chi\|_{L^2(]0,1[)} \leq 1$ it holds that

$$
\begin{aligned}
&|\langle \lambda_2, h''(z_2)[z_2 - z_1, \delta z]\rangle - \langle \lambda_1, h''(z_1)[z_2 - z_1, \delta z]\rangle| \\
&\quad = |\int_0^1 \lambda_2(\delta\xi_\tau^T(\xi_{\tau,2} - \xi_{\tau,1}) - \delta L(L_2 - L_1)) \\
&\qquad - \lambda_1(\delta\xi_\tau^T(\xi_{\tau,2} - \xi_{\tau,1}) - \delta L(L_2 - L_1))d\tau| \\
&\quad = |\int_0^1 (\lambda_2 - \lambda_1)(\delta\xi_\tau^T(\xi_{\tau,2} - \xi_{\tau,1}) - \delta L(L_2 - L_1))d\tau| \\
&\quad \leq \int_0^1 |\lambda_2 - \lambda_1| \|\delta\xi_\tau\| \|\xi_{\tau,2} - \xi_{\tau,1}\| d\tau \\
&\qquad + |\delta L||L_2 - L_1| \int_0^1 |\delta\lambda| d\tau \\
&\quad \underset{\text{(CS)}}{\leq} \left[\int_0^1 \|\delta\xi_\tau\|^2 d\tau\right]^{1/2} \left[\int_0^1 (\lambda_2 - \lambda_1)^2 \|\xi_{\tau,2} - \xi_{\tau,1}\|^2 d\tau\right]^{1/2} \\
&\qquad + |\delta L|\ |L_2 - L_1| \|\lambda_2 - \lambda_1\|_{L^1(]0,1[)} \\
&\quad \underset{(45)}{\leq} \|\delta\xi_\tau\|_{L^2} \left[2R^2 \int_0^1 |\lambda_2 - \lambda_1| \|\xi_{\tau,2} - \xi_{\tau,1}\| d\tau\right]^{1/2} \\
&\qquad + R\,|\delta L|\ |L_2 - L_1| \\
&\quad \underset{\text{(CS)}}{\leq} \sqrt{2}R \|\delta\xi_\tau\|_{L^2} \|\lambda_2 - \lambda_1\|_{L^2}^{1/2} \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2}^{1/2} \\
&\qquad + R\,|\delta L|\ |L_2 - L_1| \\
&\quad \underset{\text{(Y)}}{\leq} \frac{\sqrt{2}}{2} R \|\delta\xi_\tau\|_{L^2} [\|\lambda_2 - \lambda_1\|_{L^2} + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2}] \\
&\qquad + R\,|\delta L|\ |L_2 - L_1| \\
&\quad \leq \frac{\sqrt{2}}{2} R\,[\|\lambda_2 - \lambda_1\|_{L^2} + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2}] \\
&\qquad + R\,|L_2 - L_1| \\
&\quad \leq R\left[\|\lambda_2 - \lambda_1\|_{L^2} + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2} + \|\xi_2 - \xi_1\|_{L^2} + |L_2 - L_1|\right] \\
&\quad \leq 2R\left[\|\lambda_2 - \lambda_1\|_{L^2}^2 + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2}^2 + \|\xi_2 - \xi_1\|_{L^2}^2 + |L_2 - L_1|^2\right]^{1/2} \\
&\quad \underset{\text{(22b)}}{=} 2R \|\chi_2 - \chi_1\|_{Y^2}
\end{aligned}
$$

as well as

$$
\begin{aligned}
&|\langle \lambda_2 - \lambda_1, (h'(z_2) - h'(z_1))[\delta z]\rangle| \\
&\qquad = |\int_0^1 (\lambda_2 - \lambda_1)\big((\xi_{\tau,2} - \xi_{\tau,1})^T \delta\xi_\tau - (L_2 - L_1)\delta L\big)\, d\tau| \\
&\qquad \le \int_0^1 |\lambda_2 - \lambda_1| \|\xi_{\tau,2} - \xi_{\tau,1}\| \|\delta\xi_\tau\| d\tau \\
&\qquad\quad + |L_2 - L_1||\delta L| \int_0^1 |\lambda_2 - \lambda_1| d\tau \\
&\qquad \underset{\text{(CS)}}{\le} \left[\int_0^1 \|\delta\xi_\tau\|^2 d\tau\right]^{1/2} \left[\int_0^1 (\lambda_2 - \lambda_1)^2 \|\xi_{\tau,2} - \xi_{\tau,1}\|^2 d\tau\right]^{1/2} \\
&\qquad\quad + |L_2 - L_1||\delta L| \|\lambda_2 - \lambda_1\|_{L^1(]0,1[)} \\
&\qquad \underset{(45)}{\le} \|\delta\xi_\tau\|_{L^2} \left[2R^2 \int_0^1 |\lambda_2 - \lambda_1|\; \|\xi_{\tau,2} - \xi_{\tau,1}\| d\tau\right]^{1/2} \\
&\qquad\quad + R|L_2 - L_1||\delta L| \\
&\qquad \le \sqrt{2}R \left[\int_0^1 (\lambda_2 - \lambda_1) \|\xi_{\tau,2} - \xi_{\tau,1}\| d\tau\right]^{1/2} \\
&\qquad\quad + R|L_2 - L_1| \\
&\qquad \underset{\text{(CS)}}{\le} \sqrt{2}R \|\lambda_2 - \lambda_1\|_{L^2}^{1/2} \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2}^{1/2} \\
&\qquad\quad + R|L_2 - L_1| \\
&\qquad \underset{\text{(Y)}}{\le} \frac{\sqrt{2}}{2} R \left[\|\lambda_2 - \lambda_1\|_{L^2} + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2}\right] \\
&\qquad\quad + R|L_2 - L_1| \\
&\qquad \le R\Big[\|\lambda_2 - \lambda_1\|_{L^2} + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2} + \|\xi_2 - \xi_1\|_{L^2} + |L_2 - L_1|\Big] \\
&\qquad \le 2R\Big[\|\lambda_2 - \lambda_1\|_{L^2}^2 + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2}^2 + \|\xi_2 - \xi_1\|_{L^2}^2 + |L_2 - L_1|^2\Big]^{1/2} \\
&\qquad \underset{\text{(22b)}}{=} 2R\|\chi_2 - \chi_1\|_{Y^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&|\langle \delta\lambda, (h'(z_2) - h'(z_1))[z_2 - z_1]\rangle| \\
&\quad = |\int_0^1 \delta\lambda((\xi_{\tau,2} - \xi_{\tau,1})^T(\xi_{\tau,2} - \xi_{\tau,1}) - (L_2 - L_1)^2)d\tau| \\
&\quad \leq \int_0^1 |\delta\lambda| \|\xi_{\tau,2} - \xi_{\tau,1}\|^2 d\tau + (L_2 - L_1)^2 \int_0^1 |\delta\lambda| d\tau \\
&\quad \underset{(45)}{\leq} 2R \int_0^1 |\delta\lambda| \|\xi_{\tau,2} - \xi_{\tau,1}\| d\tau + 2R|L_2 - L_1| \|\delta\lambda\|_{L^1(]0,1[)} \\
&\quad \underset{(\mathrm{CS})}{\leq} 2R \left[\int_0^1 \delta\lambda^2 d\tau\right]^{1/2} \left[\int_0^1 \|\xi_{\tau,2} - \xi_{\tau,1}\|^2 d\tau\right]^{1/2} \\
&\qquad + 2R|L_2 - L_1| \ \|\delta\lambda\|_{L^1(]0,1[)} \\
&\quad \leq 2R\|\delta\lambda\|_{L^2} \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2} \\
&\qquad + 2R|L_2 - L_1| \ \|\delta\lambda\|_{L^1(]0,1[)} \\
&\quad \leq 2R\|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2} + 2R|L_2 - L_1| \\
&\quad \leq 2R\Bigg[\|\lambda_2 - \lambda_1\|_{L^2} + \|\xi_{\tau,2} - \xi_{\tau,1}\|_{L^2} + \|\xi_2 - \xi_1\|_{L^2} + |L_2 - L_1|\Bigg] \\
&\quad \leq 4R\Bigg[\|\lambda_2 - \lambda_1\|^2_{L^2} + \|\xi_{\tau,2} - \xi_{\tau,1}\|^2_{L^2} + \|\xi_2 - \xi_1\|^2_{L^2} + |L_2 - L_1|^2\Bigg]^{1/2} \\
&\quad \underset{(22\mathrm{b})}{=} 4R\|\chi_2 - \chi_1\|_{Y^2}.
\end{aligned}
$$

As shown in Lemma 8 in the appendix, there is a $\hat{B} < \infty$ such that

$$
\begin{aligned}
&|\left(f''(\xi_2) - f''(\xi_1)\right)[\xi_2 - \xi_1, \delta\xi]| \\
&\qquad\qquad \leq \hat{\mathcal{B}}\sqrt{\|\xi_2 - \xi_1\|^2 + \|\xi_{\tau,2} - \xi_{\tau,1}\|^2}\sqrt{\|\delta\xi\|^2 + \|\delta\xi_\tau\|^2},
\end{aligned}
$$

which provides the following bound, as

$$
\begin{aligned}
&\ |\left(T''(\xi_2) - T''(\xi_1)\right)[\xi_2 - \xi_1, \delta\xi]| \\
&= |\int_0^1 \left(f''(\xi_2) - f''(\xi_1)\right)[\xi_2 - \xi_1, \delta\xi] d\tau| \\
&\leq \hat{\mathcal{B}} \int_0^1 \sqrt{\|\xi_2 - \xi_1\|^2 + \|\xi_{\tau,2} - \xi_{\tau,1}\|^2}\sqrt{\|\delta\xi\|^2 + \|\delta\xi_\tau\|^2} d\tau \\
&\underset{(\mathrm{CS})}{\leq} \hat{\mathcal{B}} \int_0^1 \|\xi_2 - \xi_1\|^2 + \|\xi_{\tau,2} - \xi_{\tau,1}\|^2 d\tau \int_0^1 \|\delta\xi\|^2 + \|\delta\xi_\tau\|^2 d\tau \\
&\leq \hat{\mathcal{B}} \left(\|\xi_2 - \xi_1\|^2_{L^2(]0,1[)} + \|\xi_{\tau,2} - \xi_{\tau,1}\|^2_{L^2(]0,1[)}\right)\left(\|\delta\xi\|^2_{L^2(]0,1[)} + \|\delta\xi_\tau\|^2_{L^2(]0,1[)}\right) \\
&\underset{(22\mathrm{b})}{\leq} \hat{B}\|\chi_2 - \chi_1\|_{Y^2} \|\delta\chi\|_{Y^2} \\
&\leq \hat{B}R\|\chi_2 - \chi_1\|_{Y^2}.
\end{aligned}
$$

Finally, we use the bounds derived above to show that for any δx with $\|\delta x\|_{Y^2} \leq$

1 it holds that

$$
\begin{aligned}
|(F'(\chi_2) - F'(\chi_1))[\chi_2 - \chi_1, \delta\chi]| = &\,|\,(T''(\xi_2) - T''(\xi_1))\,[\delta\xi, \xi_2 - \xi_1] \\
&+ \langle \lambda_2, h''(z_2)[\delta z, z_2 - z_1] \rangle \\
&\;\;- \langle \lambda_1, h''(z_1)[\delta z, z_2 - z_1] \rangle \\
&+ \langle \lambda_2 - \lambda_1, (h'(z_2) - h'(z_1))[\delta z] \rangle \\
&+ \langle \delta\lambda, (h'(z_2) - h'(z_1))[z_2 - z_1] \rangle| \\
\leq &\, \hat{\mathcal{B}} R \|\chi_2 - \chi_1\|_{Y^2} \\
&+ 2R\|\chi_2 - \chi_1\|_{Y^2} \\
&+ 2R\|\chi_2 - \chi_1\|_{Y^2} \\
&+ 4R\|\chi_2 - \chi_1\|_{Y^2} \\
\leq &\, (8 + \hat{\mathcal{B}})R\|\chi_2 - \chi_1\|_{Y^2} \\
\leq &\, \omega_2\|\chi_2 - \chi_1\|_{Y^2}
\end{aligned}
$$

with

$$\omega_2(R) = (8 + \hat{\mathcal{B}})R.$$

This directly yields the claim, as

$$
\begin{aligned}
\|(F'(\chi_2) - F'(\chi))[\chi_2 - \chi_1]\|_{L^2(]0,1[)} &= \sup_{\|\delta\chi\|_{Y^2}=1} |(F'(\chi_2) - F'(\chi_1))[\chi_2 - \chi_1, \delta\chi]| \\
&\leq \omega_2\|\chi_2 - \chi_1\|_{Y^2}. \qquad (46)
\end{aligned}
$$

□

4.6 Convergence of Newton's Method

We are now ready to connect the results outlined above to prove that the Newton-KKT method applied to the free flight optimization problem (4) converges under suitable conditions.

Theorem 15. *Let $\chi^{\star\star} = (z^{\star\star}, \lambda^{\star\star})$ be a global solution of* (4) *that satisfies the first and second order conditions for optimality with $\underline{\mathcal{B}} > 0$. Moreover let there be a $c > 0$ and a $u \in \mathbb{R}^2$ with $\|u\| = 1$ such that $u^T \xi_\tau^{\star\star} \geq c$ for almost all $\tau \in]0, 1[$. Finally, let $\omega := \omega_1\omega_2$, as given in Lemmata 13 und 14.*
Then there is a $R_C > 0$, such that the ordinary Newton iterates defined in Abschnitt 3.2 converge to $\chi^{\star\star}$ at an estimated rate

$$\|\chi^{k+1} - \chi^{\star\star}\|_{Y^2} \leq \frac{\omega}{2}\|\chi^k - \chi^{\star\star}\|_{Y^2}, \qquad (47)$$

if initialized with $\chi^0 \in \mathcal{B}(\chi^{\star\star}, R_C)$ and provided that the iterates χ^k remain in $\mathcal{B}(\chi^{\star\star}, R_C)$. Moreover, $\chi^{\star\star}$ is unique in $\mathcal{B}(\chi^{\star\star}, R_C)$.

Beweis. In Theoreme 6, 9 und 12 we showed that the *inf-sup* condition is satisfied, that, $\mathcal{L}_{zz}(\chi)$ is positive definite on the kernel of the constraint for all $x \in \mathcal{B}(\chi^{\star\star}, R_C)$, and that it is bounded from above. Consequently, $F'(\chi)$ is invertible with

$$\|F'(\chi)^{-1}\| \leq \omega_1 \qquad \forall\, \chi \in \mathcal{B}(\chi^{\star\star}, R_C),$$

as confirmed in Lemma 13. Further, it follows from Lemmata 13 und 14 that

$$\begin{aligned}
\|F'(\chi_1)^{-1}(F'(\chi_2) - F'(\chi))[\chi_2 - \chi_1]\|_{Y^2} \\
&\leq \|F'(\chi_1)^{-1}\|_{Y^2}\|(F'(\chi_2) - F'(\chi_1))[\chi_2 - \chi_1]\|_{Y^2} \\
&\leq \omega_1\omega_2\|\chi_2 - \chi_1\|_{Y^2} \\
&\leq \omega\|\chi_2 - \chi_1\|_{Y^2}
\end{aligned}$$

for $\chi_1, \chi_2 \in \mathcal{B}(\chi^{\star\star}, R_C)$. It is clear that since ω_1 is bounded and $\omega_2 = (8 + \hat{\mathcal{B}})R$, there is a $R_C > 0$ such that $\omega := \omega_1\omega_2 < 2$. We now define $e_k := \chi^k - \chi^{\star\star}$ and proceed for $\mu \in]0, 1[$ as follows:

$$\begin{aligned}
&\|\chi^k + \mu\Delta\chi^k - \chi^{\star\star}\|_{Y^2} \\
&\quad = \|e_k - \mu F'(\chi^k)^{-1}F(\chi^k)\|_{Y^2} \\
&\quad = \|e_k - \mu F'(\chi^k)^{-1}(F(\chi^k) - \underbrace{F(\chi^{\star\star})}_{=0})\|_{Y^2} \\
&\quad = \|(1-\mu)e_k - \mu F'(\chi^k)^{-1} \int_{s=0}^{1} \left(F'(\chi^k - se_k) - F'(\chi^k)\right) e_k \, ds \, \|_{Y^2} \\
&\quad \leq (1-\mu) \, \|e_k\|_{Y^2} + \frac{\mu}{2}\omega \, \|e_k\|_{Y^2},
\end{aligned}$$

which yields the claim with $\mu = 1$ as

$$\|e_{k+1}\|_{Y^2} \leq \frac{\omega}{2}\|e_k\|_{Y^2}.$$

In order to prove uniqueness in $\mathcal{B}(\chi^{\star\star}, R_C)$, assume there is a second solution $\chi^\star \neq \chi^{\star\star}$ with $F(\chi^\star) = 0$ and $\chi^\star \in \mathcal{B}(\chi^{\star\star}, R_C)$. Initialized with $\chi^0 := \chi^\star$ it certainly holds that $\chi^1 = \chi^\star$. However, from (47) we obtain

$$\|\chi^1 - \chi^{\star\star}\|_{Y^2} \leq \frac{\omega}{2}\|\chi^0 - \chi^{\star\star}\|_{Y^2} < \|\chi^0 - \chi^{\star\star}\|_{Y^2},$$

due to $\omega < 2$, which yields a contradiction. □

5 Conclusion

It has been demonstrated that the Newton-KKT method can be used to solve the free flight trajectory optimization problem under certain conditions. These conditions include i) the presence of certain constants that can be proven to exist for mild wind conditions and are likely to exist in most cases, ii) the requirement for the iterates to remain within a L^∞-neighborhood of the solution, and iii) a starting point that is sufficiently close to the solution. Such a suitable starting point can be found efficiently by calculating shortest paths on a specific graph [2]. Hence an important tool for efficient deterministic global optimization of the free flight problem has been established.

Declarations

Funding

This research was funded by the DFG Research Center of Excellence MATH$^+$ – Berlin Mathematics Research Center, Project TrU-4.

Competing interests

The authors have no relevant financial or non-financial interests to disclose.

Ethics approval

Not applicable.

Consent to participate

Not applicable.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Consent for publication

We confirm that all authors agree with the submission of this manuscript to *Public Transport Optimization: From Theory to Practice.*

Authors' contributions

Conceptualization, R.B and M.W.; methodology, F.D. and M.W.; validation, F.D.; formal analysis, F.D. and M.W.; investigation, F.D. and M.W.; resources, R.B., F.D. and M.W.; writing-original draft preparation, F.D. and M.W.; writing-review and editing, R.B.; supervision, R.B.; project administration, R.B. and M.W.; funding acquisition, R.B. and M.W.; All authors have read and agreed to the published version of the manuscript.

References

[1] R. Borndörfer, F. Danecker und M. Weiser. "A Discrete-Continuous Algorithm for Free Flight Planning". In: *Algorithms* 14.1 (2021), S. 4.

[2] R. Borndörfer, F. Danecker und M. Weiser. "A Discrete-Continuous Algorithm for Globally Optimal Free Flight Trajectory Optimization". In: *22nd Symposium on Algorithmic Approaches for Transportation Modelling, Optimization, and Systems (ATMOS 2022).* Hrsg. von M. D'Emidio und N. Lindner. Bd. 106. Open Access Series in Informatics (OASIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022, 2:1–2:13.

[3] R. Borndörfer, F. Danecker und M. Weiser. *Error Bounds for Discrete-Continuous Shortest Path Problems with Application to Free Flight Trajectory Optimization.* 2022.

[4] J. Y. Yen. "Finding the k Shortest Loopless Paths in a Network". In: *Management Science* 17.11 (1971), S. 712–716.

[5] J. Nocedal und S. Wright. *Numerical Optimization.* New York, NY: Springer, 2006.

[6] C. A. Wells, P. D. Williams, N. K. Nichols, D. Kalise und I. Poll. "Reducing Transatlantic Flight Emissions by Fuel-Optimised Routing". In: *Environmental Research Letters* 16.2 (2021), S. 025002.

[7] E. Zermelo. "Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung". In: *ZAMM* 11.2 (1931), S. 114–124.

[8] H. Maurer und J. Zowe. "First and Second-Order Necessary and Sufficient Optimality Conditions for Infinite-Dimensional Programming Problems". In: *Math. Programming* 16 (1979), S. 98–110.

[9] D. Braess. *Finite Elemente*. 5. Aufl. Heidelberg: Springer Spektrum Berlin, 2013.

[10] E. Casas und F. Tröltzsch. "Second Order Optimality Conditions and Their Role in PDE Control". In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 117.1 (März 2015), S. 3–44.

A Supplementary Material

A.1 Global Bounds

The derivative $f = t_\tau$ of parametrized time as defined in (2) consists of two terms, the tailwind term

$$f_1 = -\frac{\xi_\tau^T w}{g}, \tag{48}$$

$$g = \overline{v}^2 - w^T w, \tag{49}$$

and the length term

$$f_2 = g^{-1}\left((\xi_\tau^T w)^2 + g(\xi_\tau^T \xi_\tau)\right)^{1/2}. \tag{50}$$

At each time τ, we obtain

$$\underline{v}^2 := \overline{v}^2 - \overline{c}_0^2 \le g \le \overline{v}^2. \tag{51}$$

The directional derivatives of g in direction $\delta\xi$ and $\Delta\xi \in \delta X$ read

$$g'\delta\xi = -2w^T w_x \delta\xi \tag{52}$$

$$\Rightarrow \quad \|g'\| \le 2\overline{c}_0\overline{c}_1 \tag{53}$$

$$\delta\xi^T g''\delta\xi = -2\delta\xi^T w_x^T w_x \delta\xi - 2w_{xx}[w, \delta\xi, \delta\xi] \tag{54}$$

$$\Rightarrow \quad \|g''\| \le 2(\overline{c}_1^2 + \overline{c}_0\overline{c}_2). \tag{55}$$

$$g'''[\delta\xi, \delta\xi, \Delta\xi] = -6w_{xx}[w_x\delta\xi, \delta\xi, \Delta\xi] - 2w_{xxx}[w, \delta\xi, \delta\xi, \Delta\xi]) \tag{56}$$

$$\Rightarrow \quad \|g'''\| \le 2(3\overline{c}_1\overline{c}_2 + \overline{c}_0\overline{c}_3). \tag{57}$$

For the tailwind term, we consider

$$f_1'(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau] = g^{-2}(\xi_\tau^T w)(g'\delta\xi) - g^{-1}\xi_\tau^T w_x \delta\xi - g^{-1} w^T \delta\xi_\tau, \tag{58}$$

which is bounded by

$$|f_1'(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau]| \le \left(2\frac{\overline{c}_0^2\overline{c}_1}{\underline{v}^4} + \frac{\overline{c}_1}{\underline{v}^2}\right)\|\xi_\tau\|\|\delta\xi\| + \frac{\overline{c}_0}{\underline{v}^2}\|\delta\xi_\tau\|. \tag{59}$$

The second directional derivatives is

$$\begin{aligned} f_1''(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau][\tilde{\delta\xi}, \tilde{\delta\xi}_\tau] = {} & -2g^{-3}(g'\tilde{\delta\xi})(\xi_\tau^T w)(g'\delta\xi) && + g^{-2}(\tilde{\delta\xi}_\tau^T w)(g'\delta\xi) \\ & + g^{-2}(\xi_\tau^T w_x\tilde{\delta\xi})(g'\delta\xi) && + g^{-2}(\xi_\tau^T w)(\delta\xi^T g''\tilde{\delta\xi}) \\ & + g^{-2}(g'\tilde{\delta\xi})(\xi_\tau^T w_x\delta\xi) && - g^{-1}(\tilde{\delta\xi}_\tau w_x\delta\xi) \\ & - g^{-1}w_{xx}[\xi_\tau, \delta\xi, \tilde{\delta\xi}] && + g^{-2}(g'\tilde{\delta\xi})(w^T\delta\xi_\tau) \\ & - g^{-1}(\delta\xi_\tau^T w_x\tilde{\delta\xi}) \end{aligned} \tag{60}$$

and in particular

$$\begin{aligned} f_1''(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau]^2 = {} & -2g^{-3}(g'\delta\xi)^2(\xi_\tau^T w) && + 2g^{-2}(g'\delta\xi)(\xi_\tau^T w_x\delta\xi) \\ & + g^{-2}(\delta\xi^T g''\delta\xi)(\xi_\tau^T w) && - g^{-1}w_{xx}[\xi_\tau, \delta\xi, \delta\xi] \\ & - 2g^{-1}(\delta\xi_\tau^T w_x\delta\xi) && + 2g^{-2}(\delta\xi_\tau^T w)(g'\delta\xi), \end{aligned} \tag{61}$$

which yields

$$\begin{aligned}|f_1''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \leq{}& \left[8\frac{\overline{c}_0^3\overline{c}_1^2}{\underline{v}^6}+6\frac{\overline{c}_0\overline{c}_1^2}{\underline{v}^4}+2\frac{\overline{c}_0^2\overline{c}_2}{\underline{v}^4}+\frac{\overline{c}_2}{\underline{v}^2}\right]\|\xi_\tau\|\|\delta\xi\|\|\tilde{\delta\xi}\| \\ &+\left[2\frac{\overline{c}_0^2\overline{c}_1}{\underline{v}^4}+\frac{\overline{c}_1}{\underline{v}^2}\right]\|\delta\xi_\tau\|\|\tilde{\delta\xi}\| \\ &+\left[2\frac{\overline{c}_0^2\overline{c}_1}{\underline{v}^4}+\frac{\overline{c}_1}{\underline{v}^2}\right]\|\delta\xi\|\|\tilde{\delta\xi_\tau}\| \end{aligned} \tag{62}$$

and

$$\begin{aligned}|f_1''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2| \leq{}& \left[8\frac{\overline{c}_0^3\overline{c}_1^2}{\underline{v}^6}+6\frac{\overline{c}_0\overline{c}_1^2}{\underline{v}^4}+2\frac{\overline{c}_0^2\overline{c}_2}{\underline{v}^4}+\frac{\overline{c}_2}{\underline{v}^2}\right]\|\xi_\tau\|\|\delta\xi\|^2 \\ &+\left[4\frac{\overline{c}_0^2\overline{c}_1}{\underline{v}^4}+2\frac{\overline{c}_1}{\underline{v}^2}\right]\|\delta\xi\|\|\delta\xi_\tau\|, \end{aligned} \tag{63}$$

respectively. Finally, the third directional derivative is

$$\begin{aligned}
&f_1'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau] \\
&\quad= 6g^{-4}(g'\Delta\xi)(g'\delta\xi)^2(\xi_\tau^T w) && -4g^{-3}(g'\delta\xi)(\Delta\xi^T g''\delta\xi)(\xi_\tau^T w) \\
&\qquad -2g^{-3}(g'\Delta\xi)(\delta\xi^T g''\delta\xi)(\xi_\tau^T w) && +g^{-2}g'''[\delta\xi,\delta\xi,\Delta\xi](\xi_\tau^T w) \\
&\qquad -2g^{-3}(g'\delta\xi)^2(\xi_\tau^T w_x\Delta\xi) && +g^{-2}(\delta\xi^T g''\delta\xi)(\xi_\tau^T w_x\Delta\xi) \\
&\qquad -4g^{-3}(g'\Delta\xi)(g'\delta\xi)(\xi_\tau^T w_x\delta\xi) && +2g^{-2}(\Delta\xi^T g''\delta\xi)(\xi_\tau^T w_x\delta\xi) \\
&\qquad +2g^{-2}(g'\delta\xi)w_{xx}[\xi_\tau,\delta\xi,\Delta\xi] && +g^{-2}(g'\Delta\xi)w_{xx}[\xi_\tau,\delta\xi,\delta\xi] \\
&\qquad -g^{-1}w_{xxx}[\xi_\tau,\delta\xi,\delta\xi,\Delta\xi] && \\
&\qquad -2g^{-3}(g'\delta\xi)^2(\Delta\xi_\tau^T w) && +g^{-2}(\delta\xi^T g''\delta\xi)(\Delta\xi_\tau^T w) \\
&\qquad +2g^{-2}(g'\delta\xi)(\Delta\xi_\tau^T w_x\delta\xi) && -g^{-1}w_{xx}[\Delta\xi_\tau,\delta\xi,\delta\xi] \\
&\qquad -4g^{-3}(g'\Delta\xi)(g'\delta\xi)(\delta\xi_\tau^T w) && +2g^{-2}(\Delta\xi^T g''\delta\xi)(\delta\xi_\tau^T w) \\
&\qquad +2g^{-2}(g'\delta\xi)(\delta\xi_\tau^T w_x\Delta\xi) && +2g^{-2}(g'\Delta\xi)(\delta\xi_\tau^T w_x\delta\xi) \\
&\qquad -2g^{-1}w_{xx}[\delta\xi_\tau,\delta\xi,\Delta\xi],
\end{aligned} \tag{64}$$

which is bounded by

$$\begin{aligned}
&|f_1'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau]| \\
&\quad\leq \frac{\|\xi_\tau\|}{\underline{v}}\Bigg[\frac{\overline{c}_1^3}{\underline{v}^3}\left(48\frac{\overline{c}_0^4}{\underline{v}^4}+48\frac{\overline{c}_0^2}{\underline{v}^2}+6\right) \\
&\qquad\qquad +\frac{\overline{c}_1\overline{c}_2}{\underline{v}^2}\left(24\frac{\overline{c}_0^3}{\underline{v}^3}+18\frac{\overline{c}_0}{\underline{v}}\right) \\
&\qquad\qquad +\frac{\overline{c}_3}{\underline{v}}\left(2\frac{\overline{c}_0^2}{\underline{v}^2}+1\right)\Bigg]\|\delta\xi\|^2\|\Delta\xi\| \\
&\qquad +\left[\frac{\overline{c}_1^2}{\underline{v}^3}\left(8\frac{\overline{c}_0^3}{\underline{v}^3}+6\frac{\overline{c}_0}{\underline{v}}\right)+\frac{\overline{c}_2}{\underline{v}^2}\left(2\frac{\overline{c}_0^2}{\underline{v}^2}+1\right)\right]\|\delta\xi\|^2\|\Delta\xi_\tau\| \\
&\qquad +\left[\frac{\overline{c}_1^2}{\underline{v}^3}\left(16\frac{\overline{c}_0^3}{\underline{v}^3}+12\frac{\overline{c}_0}{\underline{v}}\right)+\frac{\overline{c}_2}{\underline{v}^2}\left(4\frac{\overline{c}_0^2}{\underline{v}^2}+2\right)\right]\|\delta\xi\|\|\delta\xi_\tau\|\|\Delta\xi\|.
\end{aligned} \tag{65}$$

Before we turn to the length term f_2, we first consider the term

$$F := (\xi_\tau^T w)^2 + g(\xi_\tau^T \xi_\tau) \tag{66}$$

with

$$\underline{v}^2 \|\xi_\tau\|^2 \leq F \leq \|\xi_\tau\|^2 \overline{\overline{v}}^2.$$

We also note that

$$\frac{g}{F} \leq \frac{1}{\|\xi_\tau\|^2}.$$

Then

$$\begin{aligned} F'(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau] =& 2(\xi_\tau^T w)((\delta\xi_\tau^T w) + (\xi_\tau^T w_x \delta\xi)) \\ &+ (g'\delta\xi)(\xi_\tau^T \xi_\tau) + 2g(\xi_\tau^T \delta\xi_\tau), \end{aligned} \tag{67}$$

which is bounded by

$$|F'(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau]| \leq 2\overline{\overline{v}}^2 \|\xi_\tau\| \|\delta\xi_\tau\| + 4\overline{c}_0 \overline{c}_1 \|\xi_\tau\|^2 \|\delta\xi\|, \tag{68}$$

The second derivative is

$$\begin{aligned} &F''(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau][\tilde{\delta\xi}, \tilde{\delta\xi_\tau}] \\ &\quad = 2(\xi_\tau^T w)(\delta\xi_\tau^T w_x \tilde{\delta\xi}) && + 2(\xi_\tau^T w_x \tilde{\delta\xi})(\delta\xi_\tau^T w) \\ &\qquad + 2(\tilde{\delta\xi}_\tau^T w)(\delta\xi_\tau^T w) && + 2(\xi_\tau^T w_x \tilde{\delta\xi})(\xi_\tau^T w_x \delta\xi) \\ &\qquad + 2(\xi_\tau^T w) w_{xx}[\xi_\tau, \delta\xi, \tilde{\delta\xi}] && + 2(\tilde{\delta\xi}_\tau^T w)(\xi_\tau^T w_x \delta\xi) \\ &\qquad + 2(\xi_\tau^T w)(\tilde{\delta\xi}_\tau^T w_x \delta\xi) && + (\tilde{\delta\xi}^T g'' \delta\xi)(\xi_\tau^T \xi_\tau) \\ &\qquad + 2(g'\delta\xi)(\tilde{\delta\xi}_\tau^T \xi_\tau) && + 2(g'\tilde{\delta\xi})(\xi_\tau^T \delta\xi_\tau) \\ &\qquad + 2g(\tilde{\delta\xi}_\tau^T \delta\xi_\tau) \end{aligned} \tag{69}$$

and in particular

$$\begin{aligned} F''(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau]^2 = & 4(\xi_\tau^T w)(\delta\xi_\tau^T w_x \delta\xi) && + 4(\delta\xi_\tau^T w)(\xi_\tau^T w_x \delta\xi) \\ & + 2(\delta\xi_\tau^T w)^2 && + 2(\xi_\tau^T w_x \delta\xi)^2 \\ & + 2(\xi_\tau^T w) w_{xx}[\xi_\tau, \delta\xi, \delta\xi] && + (\delta\xi^T g'' \delta\xi)(\xi_\tau^T \xi_\tau) \\ & + 4(g'\delta\xi)(\delta\xi_\tau^T \xi_\tau) && + 2g(\delta\xi_\tau^T \delta\xi_\tau), \end{aligned} \tag{70}$$

which yields

$$\begin{aligned} |F''(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau][\tilde{\delta\xi}, \tilde{\delta\xi_\tau}]| \leq & \left(4\overline{c}_1^2 + 4\overline{c}_0 \overline{c}_2\right) \|\xi_\tau\|^2 \|\delta\xi\| \|\tilde{\delta\xi}\| \\ & + 8\overline{c}_0 \overline{c}_1 \|\xi_\tau\| \|\delta\xi\| \|\tilde{\delta\xi}_\tau\| \\ & + 8\overline{c}_0 \overline{c}_1 \|\xi_\tau\| \|\delta\xi_\tau\| \|\tilde{\delta\xi}\| \\ & + 2\overline{\overline{v}}^2 \|\delta\xi_\tau\| \|\tilde{\delta\xi}_\tau\| \end{aligned} \tag{71}$$

and

$$\begin{aligned} |F''(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau]^2| \leq & \left(4\overline{c}_1^2 + 4\overline{c}_0 \overline{c}_2\right) \|\xi_\tau\|^2 \|\delta\xi\|^2 \\ & + 16\overline{c}_0 \overline{c}_1 \|\xi_\tau\| \|\delta\xi\| \|\delta\xi_\tau\| \\ & + 2\overline{\overline{v}}^2 \|\delta\xi_\tau\|^2, \end{aligned} \tag{72}$$

respectively. The third derivative is

$$
\begin{aligned}
& F'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&= 4(\tilde{\delta\xi}_\tau^T w)(\delta\xi_\tau^T w_x \delta\xi) && + 4(\xi_\tau^T w_x \tilde{\delta\xi})(\delta\xi_\tau^T w_x \delta\xi) \\
&\quad + 4(\xi_\tau^T w) w_{xx}[\delta\xi_\tau,\delta\xi,\tilde{\delta\xi}] && + 4(\delta\xi_\tau^T w_x \tilde{\delta\xi})(\xi_\tau^T w_x \delta\xi) \\
&\quad + 4(\delta\xi_\tau^T w)(\tilde{\delta\xi}_\tau^T w_x \delta\xi) && + 4(\delta\xi_\tau^T w) w_{xx}[\xi_\tau,\delta\xi,\tilde{\delta\xi}] \\
&\quad + 4(\delta\xi_\tau^T w)(\delta\xi_\tau^T w_x \tilde{\delta\xi}) && + 4(\xi_\tau^T w_x \delta\xi)(\tilde{\delta\xi}_\tau^T w_x \delta\xi) \\
&\quad + 4(\xi_\tau^T w_x \delta\xi) w_{xx}[\xi_\tau,\delta\xi,\tilde{\delta\xi}] && + 2(\tilde{\delta\xi}_\tau^T w) w_{xx}[\xi_\tau,\delta\xi,\delta\xi] \\
&\quad + 2(\xi_\tau^T w_x \tilde{\delta\xi}) w_{xx}[\xi_\tau,\delta\xi,\delta\xi] && + 2(\xi_\tau^T w) w_{xxx}[\xi_\tau,\delta\xi,\delta\xi,\tilde{\delta\xi}] \\
&\quad + 2(\xi_\tau^T w) w_{xx}[\tilde{\delta\xi}_\tau,\delta\xi,\delta\xi] && + g'''[\delta\xi,\delta\xi,\tilde{\delta\xi}](\xi_\tau^T\xi_\tau) \\
&\quad + 2(\delta\xi^T g'' \delta\xi)(\tilde{\delta\xi}_\tau^T \xi_\tau) && + 4(\tilde{\delta\xi}^T g'' \delta\xi)(\delta\xi_\tau^T \xi_\tau) \\
&\quad + 4(g'\delta\xi)(\delta\xi_\tau^T \tilde{\delta\xi}_\tau) && + 2(g'\tilde{\delta\xi})(\delta\xi_\tau^T \delta\xi_\tau),
\end{aligned}
\tag{73}
$$

which is bounded by

$$
\begin{aligned}
|F'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \le\ & 4\|\xi_\tau\|^2(\overline{c}_0\overline{c}_3 + 3\overline{c}_1\overline{c}_2)\|\delta\xi\|^2\|\tilde{\delta\xi}\| \\
& + 8\|\xi_\tau\|(\overline{c}_1^2 + \overline{c}_0\overline{c}_2)\|\delta\xi\|^2\|\tilde{\delta\xi}_\tau\| \\
& + 16\|\xi_\tau\|(\overline{c}_1^2 + \overline{c}_0\overline{c}_2)\|\delta\xi\|\|\delta\xi_\tau\|\|\tilde{\delta\xi}\| \\
& + 16\overline{c}_0\overline{c}_1\|\delta\xi\|\|\delta\xi_\tau\|\|\tilde{\delta\xi}_\tau\| \\
& + 8\overline{c}_0\overline{c}_1\|\delta\xi_\tau\|^2\|\tilde{\delta\xi}\|.
\end{aligned}
\tag{74}
$$

For the length term $f_2 = g^{-1}\sqrt{F}$, we thus obtain

$$
f_2'(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau] = -g^{-2}(g'\delta\xi)F^{1/2} + \frac{1}{2}g^{-1}F^{-1/2}F'[\delta\xi,\delta\xi_\tau], \tag{75}
$$

which is bounded by

$$
|f_2'(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]| \le \left(2\frac{\overline{c}_0\overline{c}_1\overline{\overline{v}}}{\underline{v}^4} + 4\frac{\overline{c}_0\overline{c}_1}{\underline{v}^3}\right)\|\xi_\tau\|\|\delta\xi\| + 2\underline{v}^{-3}\overline{\overline{v}}^2\|\delta\xi_\tau\|. \tag{76}
$$

The second derivative is

$$
\begin{aligned}
f_2''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi}_\tau] =\ & 2g^{-3}(g'\tilde{\delta\xi})(g'\delta\xi)F^{1/2} \\
& - g^{-2}(\delta\xi^T g'' \tilde{\delta\xi})F^{1/2} \\
& - \frac{1}{2}g^{-2}(g'\delta\xi)F^{-1/2}F'[\tilde{\delta\xi},\tilde{\delta\xi}_\tau] \\
& - \frac{1}{2}g^{-2}(g'\tilde{\delta\xi})F^{-1/2}F'[\delta\xi,\delta\xi_\tau] \\
& + \frac{1}{2}g^{-1}F^{-1/2}F''[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi}_\tau] \\
& - \frac{1}{4}g^{-1}F^{-3/2}F'[\delta\xi,\delta\xi_\tau]F'[\tilde{\delta\xi},\tilde{\delta\xi}_\tau]
\end{aligned}
\tag{77}
$$

and in particular

$$
\begin{aligned}
f_2''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2 = \quad & 2g^{-3}(g'\delta\xi)^2F^{1/2} \\
& - g^{-2}(\delta\xi^T g''\delta\xi)F^{1/2} \\
& - g^{-2}(g'\delta\xi)F^{-1/2}F'[\delta\xi,\delta\xi_\tau] \\
& + \frac{1}{2}g^{-1}F^{-1/2}F''[\delta\xi,\delta\xi_\tau]^2 \\
& - \frac{1}{4}g^{-1}F^{-3/2}(F'[\delta\xi,\delta\xi_\tau])^2,
\end{aligned}
\tag{78}
$$

which yields

$$
\begin{aligned}
& |f_2''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \\
& \quad \le \left[8\frac{\overline{c}_0^2\overline{c}_1^2\overline{\overline{v}}}{\underline{v}^6} + 12\frac{\overline{c}_0^2\overline{c}_1^2}{\underline{v}^5} + 2\frac{(\overline{c}_1^2+\overline{c}_0\overline{c}_2)\overline{\overline{v}}}{\underline{v}^4} + 2\frac{\overline{c}_1^2+\overline{c}_0\overline{c}_2}{\underline{v}^3}\right]\|\xi_\tau\|\|\delta\xi\|\|\tilde{\delta\xi}\| \\
& \quad + \left[4\frac{\overline{c}_0\overline{c}_1\overline{\overline{v}}^2}{\underline{v}^5} + 4\frac{\overline{c}_0\overline{c}_1}{\underline{v}^3}\right]\|\delta\xi\|\|\tilde{\delta\xi}_\tau\| \\
& \quad + \left[4\frac{\overline{c}_0\overline{c}_1\overline{\overline{v}}^2}{\underline{v}^5} + 4\frac{\overline{c}_0\overline{c}_1}{\underline{v}^3}\right]\|\delta\xi_\tau\|\|\tilde{\delta\xi}\| \\
& \quad + \left[\frac{\overline{\overline{v}}^4}{\underline{v}^5} + \frac{\overline{\overline{v}}^2}{\underline{v}^3}\right]\|\xi_\tau\|^{-1}\|\delta\xi_\tau\|\|\tilde{\delta\xi}_\tau\|
\end{aligned}
\tag{79}
$$

and

$$
\begin{aligned}
& |f_2''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2| \\
& \quad \le \left[8\frac{\overline{c}_0^2\overline{c}_1^2\overline{\overline{v}}}{\underline{v}^6} + 12\frac{\overline{c}_0^2\overline{c}_1^2}{\underline{v}^5} + 2\frac{(\overline{c}_1^2+\overline{c}_0\overline{c}_2)\overline{\overline{v}}}{\underline{v}^4} + 2\frac{\overline{c}_1^2+\overline{c}_0\overline{c}_2}{\underline{v}^3}\right]\|\xi_\tau\|\|\delta\xi\|^2 \\
& \quad + \left[8\frac{\overline{c}_0\overline{c}_1\overline{\overline{v}}^2}{\underline{v}^5} + 8\frac{\overline{c}_0\overline{c}_1}{\underline{v}^3}\right]\|\delta\xi\|\|\delta\xi_\tau\| \\
& \quad + \left[\frac{\overline{\overline{v}}^4}{\underline{v}^5} + \frac{\overline{\overline{v}}^2}{\underline{v}^3}\right]\|\xi_\tau\|^{-1}\|\delta\xi_\tau\|^2
\end{aligned}
\tag{80}
$$

The third derivative is

$$
\begin{aligned}
f_2'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] = &- 6g^{-4}(g'\tilde{\delta\xi})(g'\delta\xi)^2F^{1/2} \\
&+ 4g^{-3}(g'\delta\xi)(\tilde{\delta\xi}^T g''\delta\xi)F^{1/2} \\
&+ g^{-3}(g'\delta\xi)^2F^{-1/2}F'[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&+ 2g^{-3}(g'\tilde{\delta\xi})(\delta\xi^T g''\delta\xi)F^{1/2} \\
&- g^{-2}g'''[\delta\xi,\delta\xi,\tilde{\delta\xi}]F^{1/2} \\
&- \frac{1}{2}g^{-2}(\delta\xi^T g''\delta\xi)F^{-1/2}F'[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&+ g^{-3}(g'\tilde{\delta\xi})(g'\delta\xi)F^{-1/2}F'[\delta\xi,\delta\xi_\tau] \\
&- \frac{1}{2}g^{-2}(\tilde{\delta\xi}^T g''\delta\xi)F^{-1/2}F'[\delta\xi,\delta\xi_\tau] \\
&+ \frac{1}{4}g^{-2}(g'\delta\xi)F^{-3/2}F'[\delta\xi,\delta\xi_\tau]F'[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&- \frac{1}{2}g^{-2}(g'\delta\xi)F^{-1/2}F''[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&+ g^{-3}(g'\tilde{\delta\xi})(g'\delta\xi)F^{-1/2}F'[\delta\xi,\delta\xi_\tau] \\
&- \frac{1}{2}g^{-2}(\tilde{\delta\xi}g''\delta\xi)F^{-1/2}F'[\delta\xi,\delta\xi_\tau] \\
&+ \frac{1}{4}g^{-2}(g'\delta\xi)F^{-3/2}F'[\delta\xi,\delta\xi_\tau]F'[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&- \frac{1}{2}g^{-2}(g'\delta\xi)F^{-1/2}F''[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&+ \frac{1}{4}g^{-2}(g'\tilde{\delta\xi})F^{-3/2}(F'[\delta\xi,\delta\xi_\tau])^2 \\
&+ \frac{3}{8}g^{-1}F^{-5/2}(F'[\delta\xi,\delta\xi_\tau])^2F'[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&- \frac{1}{2}g^{-1}F^{-3/2}F'[\delta\xi,\delta\xi_\tau]F''[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&- \frac{1}{2}g^{-2}(g'\tilde{\delta\xi})F^{-1/2}F''[\delta\xi,\delta\xi_\tau]^2 \\
&- \frac{1}{4}g^{-1}F^{-3/2}F''[\delta\xi,\delta\xi_\tau]^2F'[\tilde{\delta\xi},\tilde{\delta\xi_\tau}] \\
&+ \frac{1}{2}g^{-1}F^{-1/2}F'''[\delta\xi,\delta\xi_\tau]^2[\tilde{\delta\xi},\tilde{\delta\xi_\tau}],
\end{aligned}
\tag{81}
$$

which is bounded by

$$
\begin{aligned}
&|f_2'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \\
&\qquad \le \frac{2\|\xi_\tau\|}{\underline{v}}\Bigg[\frac{\overline{c}_3}{\underline{v}}\left(\frac{\overline{c}_0}{\underline{v}}+\frac{\overline{\overline{v}}\,\overline{c}_0}{\underline{v}^2}\right) \\
&\qquad\qquad + \frac{3\overline{c}_1\overline{c}_2}{\underline{v}^2}\left(1+\frac{\overline{\overline{v}}}{\underline{v}}+6\frac{\overline{c}_0^2}{\underline{v}^2}+4\frac{\overline{\overline{v}}\,\overline{c}_0^2}{\underline{v}^3}\right) \\
&\qquad\qquad + \frac{6\overline{c}_1^3}{\underline{v}^3}\left(3\frac{\overline{c}_0}{\underline{v}}+2\frac{\overline{\overline{v}}\,\overline{c}_0}{\underline{v}^2}+8\frac{\overline{c}_0^3}{\underline{v}^3}+4\frac{\overline{\overline{v}}\,\overline{c}_0^3}{\underline{v}^4}\right)\Bigg]\|\delta\xi\|^2\|\tilde{\delta\xi}\| \\
&\qquad + \frac{4}{\underline{v}}\Bigg[\frac{\overline{c}_1^2}{\underline{v}^2}\left(1+\frac{\overline{v}^2}{\underline{v}^2}+9\frac{\overline{c}_0^2}{\underline{v}^2}+7\frac{\overline{c}_0^2\overline{\overline{v}}^2}{\underline{v}^4}\right) \\
&\qquad\qquad + \frac{\overline{c}_2}{\underline{v}}\left(\frac{\overline{c}_0}{\underline{v}}+\frac{\overline{v}^2\overline{c}_0}{\underline{v}^3}+\frac{\overline{c}_0^3}{\underline{v}^3}\right)\Bigg]\|\delta\xi\|^2\|\tilde{\delta\xi_\tau}\| \\
&\qquad + \frac{8}{\underline{v}}\Bigg[\frac{\overline{c}_1^2}{\underline{v}^2}\left(1+\frac{\overline{v}^2}{\underline{v}^2}+9\frac{\overline{c}_0^2}{\underline{v}^2}+7\frac{\overline{c}_0^2\overline{\overline{v}}^2}{\underline{v}^4}\right) \\
&\qquad\qquad + \frac{\overline{c}_2}{\underline{v}}\left(\frac{\overline{c}_0}{\underline{v}}+\frac{\overline{v}^2\overline{c}_0}{\underline{v}^3}+\frac{\overline{c}_0^3}{\underline{v}^3}\right)\Bigg]\|\delta\xi\|\|\delta\xi_\tau\|\|\tilde{\delta\xi}\| \\
&\qquad + \frac{8\overline{c}_0\overline{c}_1}{\|\xi_\tau\|\underline{v}^3}\left(1+3\frac{\overline{\overline{v}}^2}{\underline{v}^2}+2\frac{\overline{\overline{v}}^4}{\underline{v}^4}\right)\|\delta\xi\|\|\delta\xi_\tau\|\|\tilde{\delta\xi_\tau}\| \\
&\qquad + \frac{4\overline{c}_0\overline{c}_1}{\|\xi_\tau\|\underline{v}^3}\left(1+3\frac{\overline{\overline{v}}^2}{\underline{v}^2}+2\frac{\overline{\overline{v}}^4}{\underline{v}^4}\right)\|\delta\xi_\tau\|^2\|\tilde{\delta\xi}\| \\
&\qquad + \frac{3\overline{\overline{v}}^4}{\|\xi_\tau\|^2\underline{v}^5}\left(1+\frac{\overline{\overline{v}}^2}{\underline{v}^2}\right)\|\delta\xi_\tau\|^2\|\tilde{\delta\xi_\tau}\|.
\end{aligned}
\tag{82}
$$

Lemma 16. *Let $\|w\|_{L^\infty(\Omega)} \le \overline{c}_0 \le \overline{v}/\sqrt{5}$ and $\|w_x\|_{L^\infty(\Omega)} \le \overline{c}_1$ for every. Moreover, let $\underline{v}^2 := \overline{v}^2 - \overline{c}_0^2$ and $\overline{\overline{v}}^2 := \overline{v}^2 + \overline{c}_0^2$. Then, for any $\xi \in X$, the first directional derivative of f as given in* (2) *is bounded by*

$$|f'(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]| \le \overline{\alpha}_0\|\xi_\tau\|\|\delta\xi\| + \overline{\alpha}_1\|\delta\xi_\tau\| \tag{83}$$

with

$$\overline{\alpha}_0 = \frac{21\overline{c}_1}{4\underline{v}^2}, \qquad \overline{\alpha}_1 = \frac{7}{2\underline{v}}. \tag{84}$$

Beweis. We obtain f by adding f_1 and f_2. The first derivative of f can thus be bounded using (59), (76), and the triangle inequality.

$$
\begin{aligned}
|f'(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]| \le{}& \left(2\frac{\overline{c}_0^2\overline{c}_1}{\underline{v}^4}+\frac{\overline{c}_1}{\underline{v}^2}+2\frac{\overline{c}_0\overline{c}_1\overline{\overline{v}}}{\underline{v}^4}+4\frac{\overline{c}_0\overline{c}_1}{\underline{v}^3}\right)\|\xi_\tau\|\|\delta\xi\| \\
&+\left(\frac{\overline{c}_0}{\underline{v}^2}+2\frac{\overline{\overline{v}}^2}{\underline{v}^3}\right)\|\delta\xi_\tau\|
\end{aligned}
$$

With $\frac{\overline{c}_0}{\overline{v}} \le \frac{1}{\sqrt{5}}$, we note that

$$\frac{\overline{c}_0}{\underline{v}} \le \frac{1}{2}, \qquad \frac{\overline{\overline{v}}}{\underline{v}} \le \sqrt{\frac{3}{2}}, \quad \text{and} \quad \frac{\overline{v}}{\underline{v}} \le \frac{\sqrt{5}}{2}$$

and obtain

$$|f'(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]| \le \left(4+\sqrt{\frac{3}{2}}\right)\frac{\overline{c}_1}{\underline{v}^2}\|\xi_\tau\|\|\delta\xi\| + \frac{7}{2\underline{v}}\|\delta\xi_\tau\|.$$

Rounding up the values yields the bound

$$|f'(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]| \le \frac{21\overline{c}_1}{4\underline{v}^2}\|\xi_\tau\|\|\delta\xi\| + \frac{7}{2\underline{v}}\|\delta\xi_\tau\|.$$

□

Lemma 10. *Let* $\|w(p)\| \le \overline{c}_0 \le \overline{v}/\sqrt{5}$*,* $\|w_x(p)\| \le \overline{c}_1$*, and* $\|w_{xx}(p)\| \le \overline{c}_2$ *for every* $p \in \Omega$*. Moreover let* $\underline{v}^2 := \overline{v}^2 - \overline{c}_0^2$ *and* $\overline{\overline{v}}^2 := \overline{v}^2 + \overline{c}_0^2$*. Then, for any* $\xi \in X$*, the second directional derivative of* f *as given in* (2) *is bounded by*

$$\begin{aligned}|f''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \le{}& \overline{\beta}_0\|\xi_\tau\|\|\delta\xi\|\|\tilde{\delta\xi}\| + \overline{\beta}_1\|\delta\xi\|\|\tilde{\delta\xi_\tau}\| \\ &+ \overline{\beta}_1\|\delta\xi_\tau\|\|\tilde{\delta\xi}\| + \overline{\beta}_2\|\xi_\tau\|^{-1}\|\delta\xi_\tau\|\|\tilde{\delta\xi_\tau}\|\end{aligned}$$

with

$$\overline{\beta}_0 = 14\frac{\overline{c}_1^2}{\underline{v}^3} + 4\frac{\overline{c}_2}{\underline{v}^2}, \qquad \overline{\beta}_1 = 7\frac{\overline{c}_1}{\underline{v}^2}, \qquad \overline{\beta}_2 = \frac{4}{\underline{v}}.$$

Beweis. We obtain f by adding f_1 and f_2. The second derivative of f can thus be bounded using (62), (79), and the triangle inequality.

$$\begin{aligned}|f''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \le{}& \left[8\overline{c}_0^2\overline{c}_1^2\frac{\overline{c}_0+\overline{\overline{v}}}{\underline{v}^6} + 12\frac{\overline{c}_0^2\overline{c}_1^2}{\underline{v}^5}\right.\\ &+ 2\frac{\overline{c}_1^2\overline{\overline{v}} + \overline{c}_0\overline{c}_2\overline{\overline{v}} + 3\overline{c}_0\overline{c}_1^2 + \overline{c}_0^2\overline{c}_2}{\underline{v}^4}\\ &\left.+ 2\frac{\overline{c}_1^2 + \overline{c}_0\overline{c}_2}{\underline{v}^3} + \frac{\overline{c}_2}{\underline{v}^2}\right]\|\xi_\tau\|\|\delta\xi\|\|\tilde{\delta\xi}\|\\ &+ \left[4\frac{\overline{c}_0\overline{c}_1\overline{\overline{v}}^2}{\underline{v}^5} + 2\frac{\overline{c}_0^2\overline{c}_1}{\underline{v}^4} + 4\frac{\overline{c}_0\overline{c}_1}{\underline{v}^3} + \frac{\overline{c}_1}{\underline{v}^2}\right]\|\delta\xi_\tau\|\|\tilde{\delta\xi}\|\\ &+ \left[4\frac{\overline{c}_0\overline{c}_1\overline{\overline{v}}^2}{\underline{v}^5} + 2\frac{\overline{c}_0^2\overline{c}_1}{\underline{v}^4} + 4\frac{\overline{c}_0\overline{c}_1}{\underline{v}^3} + \frac{\overline{c}_1}{\underline{v}^2}\right]\|\delta\xi\|\|\tilde{\delta\xi_\tau}\|\\ &+ \left[\frac{\overline{\overline{v}}^4}{\underline{v}^5} + \frac{\overline{\overline{v}}^2}{\underline{v}^3}\right]\|\xi_\tau\|^{-1}\|\delta\xi_\tau\|\|\tilde{\delta\xi_\tau}\|.\end{aligned}$$

With $\frac{\overline{c}_0}{\overline{v}} \le \frac{1}{\sqrt{5}}$, we note that

$$\frac{\overline{c}_0}{\underline{v}} \le \frac{1}{2}, \qquad \frac{\overline{\overline{v}}}{\underline{v}} \le \sqrt{\frac{3}{2}}, \quad \text{and} \qquad \frac{\overline{v}}{\underline{v}} \le \frac{\sqrt{5}}{2}$$

and obtain

$$
\begin{aligned}
|f''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \leq & \left[\left(9+4\sqrt{\frac{3}{2}}\right)\frac{\overline{c}_1^2}{\underline{v}^3}+4\frac{\overline{c}_2}{\underline{v}^2}\right]\|\xi_\tau\|\|\delta\xi\|\|\tilde{\delta\xi}\| \\
&+\frac{13\overline{c}_1}{2\underline{v}^2}\|\delta\xi_\tau\|\|\tilde{\delta\xi}\| \\
&+\frac{13\overline{c}_1}{2\underline{v}^2}\|\delta\xi\|\|\tilde{\delta\xi}_\tau\| \\
&+\frac{15}{4\underline{v}}\|\xi_\tau\|^{-1}\|\delta\xi_\tau\|\|\tilde{\delta\xi}_\tau\|.
\end{aligned}
$$

Rounding up the values yields the bound

$$
\begin{aligned}
|f''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \leq & \left[14\frac{\overline{c}_1^2}{\underline{v}^3}+4\frac{\overline{c}_2}{\underline{v}^2}\right]\|\xi_\tau\|\|\delta\xi\|\|\tilde{\delta\xi}\| \\
&+7\frac{\overline{c}_1}{\underline{v}^2}\|\delta\xi_\tau\|\|\tilde{\delta\xi}\| \\
&+7\frac{\overline{c}_1}{\underline{v}^2}\|\delta\xi\|\|\tilde{\delta\xi}_\tau\| \\
&+\frac{4}{\underline{v}}\|\xi_\tau\|^{-1}\|\delta\xi_\tau\|\|\tilde{\delta\xi}_\tau\|.
\end{aligned}
$$

□

Lemma 7. *Let* $\|w(p)\| \leq \overline{c}_0 \leq \overline{v}/\sqrt{5}$, $\|w_x(p)\| \leq \overline{c}_1$, $\|w_{xx}(p)\| \leq \overline{c}_2$, *and* $\|w_{xxx}(p)\| \leq \overline{c}_3$ *for every* $p \in \Omega$. *Moreover let* $\underline{v}^2 := \overline{v}^2 - \overline{c}_0^2$ *and* $\overline{\overline{v}}^2 := \overline{v}^2 + \overline{c}_0^2$. *Then, for any* $\xi \in X$, *the third directional derivative of* f *as given in* (2) *is bounded by*

$$
\begin{aligned}
&|f'''(\xi,\xi_\tau)[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau]| \\
&\qquad \leq \left(\|\xi_\tau\|\overline{\gamma}_0\|\delta\xi\|^2+\overline{\gamma}_2\|\delta\xi\|\|\delta\xi_\tau\|+\frac{\overline{\gamma}_4}{\|\xi_\tau\|}\|\delta\xi_\tau\|^2\right)\|\Delta\xi\| \\
&\qquad +\left(\overline{\gamma}_1\|\delta\xi\|^2+\frac{\overline{\gamma}_3}{\|\xi_\tau\|}\|\delta\xi\|\|\delta\xi_\tau\|+\frac{\overline{\gamma}_5}{\|\xi_\tau\|^2}\|\delta\xi_\tau\|^2\right)\|\Delta\xi_\tau\|
\end{aligned}
$$

with

$$
\begin{aligned}
\overline{\gamma}_0 &= \frac{2}{\underline{v}^4}\left(37\overline{c}_1^3+21\overline{c}_1\overline{c}_2\underline{v}+2\overline{c}_3\underline{v}^2\right), & \overline{\gamma}_3 &= 40\frac{\overline{c}_1}{\underline{v}^2\|\xi_\tau\|}, \\
\overline{\gamma}_1 &= \frac{1}{\underline{v}^3}\left(29\overline{c}_1^2+7\underline{v}\overline{c}_2\right), & \overline{\gamma}_4 &= 20\frac{\overline{c}_1}{\underline{v}^2\|\xi_\tau\|}, \\
\overline{\gamma}_2 &= \frac{1}{\underline{v}^3}(57\overline{c}_1^2+13\underline{v}\overline{c}_2), & \overline{\gamma}_5 &= 18\frac{1}{\underline{v}\|\xi_\tau\|^2}.
\end{aligned}
$$

Beweis. We obtain f by adding f_1 and f_2. The third derivative of f can thus be bounded using (65), (82), and the triangle inequality.

$$
\begin{aligned}
&|f'''[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau]| \\
&\le \frac{\|\xi_\tau\|}{\underline{v}}\Bigg[\frac{\overline{c}_3}{\underline{v}}\left(1+2\frac{\overline{c}_0}{\underline{v}}+2\frac{\overline{\overline{v}}\overline{c}_0}{\underline{v}^2}+2\frac{\overline{c}_0^2}{\underline{v}^2}\right) \\
&\qquad\qquad +6\frac{\overline{c}_1\overline{c}_2}{\underline{v}^2}\left(1+1\frac{\overline{\overline{v}}}{\underline{v}}+3\frac{\overline{c}_0}{\underline{v}}+6\frac{\overline{c}_0^2}{\underline{v}^2}+4\frac{\overline{\overline{v}}\overline{c}_0^2}{\underline{v}^3}+4\frac{\overline{c}_0^3}{\underline{v}^3}\right) \\
&\qquad\qquad +6\frac{\overline{c}_1^3}{\underline{v}^3}\left(1+6\frac{\overline{c}_0}{\underline{v}}+4\frac{\overline{\overline{v}}\overline{c}_0}{\underline{v}^2}+8\frac{\overline{c}_0^2}{\underline{v}^2}+16\frac{\overline{c}_0^3}{\underline{v}^3}+8\frac{\overline{\overline{v}}\overline{c}_0^3}{\underline{v}^4}+8\frac{\overline{c}_0^4}{\underline{v}^4}\right)\Bigg]\|\delta\xi\|^2\|\Delta\xi\| \\
&\quad+\frac{1}{\underline{v}}\Bigg[2\frac{\overline{c}_1^2}{\underline{v}^2}\left(2+3\frac{\overline{c}_0}{\underline{v}}+2\frac{\overline{v}^2}{\underline{v}^2}+18\frac{\overline{c}_0^2}{\underline{v}^2}+4\frac{\overline{c}_0^3}{\underline{v}^3}+14\frac{\overline{c}_0^2\overline{\overline{v}}^2}{\underline{v}^4}\right) \\
&\qquad\qquad +\frac{\overline{c}_2}{\underline{v}}\left(1+4\frac{\overline{c}_0}{\underline{v}}+2\frac{\overline{c}_0^2}{\underline{v}^2}+4\frac{\overline{v}^2\overline{c}_0}{\underline{v}^3}+4\frac{\overline{c}_0^3}{\underline{v}^3}\right)\Bigg]\|\delta\xi\|^2\|\Delta\xi_\tau\| \\
&\quad+\frac{1}{\underline{v}}\Bigg[4\frac{\overline{c}_1^2}{\underline{v}^2}\left(2+4\frac{\overline{c}_0^3}{\underline{v}^3}+3\frac{\overline{c}_0}{\underline{v}}+2\frac{\overline{v}^2}{\underline{v}^2}+18\frac{\overline{c}_0^2}{\underline{v}^2}+14\frac{\overline{c}_0^2\overline{\overline{v}}^2}{\underline{v}^4}\right) \\
&\qquad\qquad +2\frac{\overline{c}_2}{\underline{v}}\left(1+4\frac{\overline{c}_0}{\underline{v}}+2\frac{\overline{c}_0^2}{\underline{v}^2}+4\frac{\overline{v}^2\overline{c}_0}{\underline{v}^3}+4^{\frac{\overline{c}_0^3}{\underline{v}^3}}\right)\Bigg]\|\delta\xi\|\|\delta\xi_\tau\|\|\Delta\xi\| \\
&\quad+\frac{8\overline{c}_0\overline{c}_1}{\|\xi_\tau\|\underline{v}^3}\left(1+3\frac{\overline{\overline{v}}^2}{\underline{v}^2}+2\frac{\overline{\overline{v}}^4}{\underline{v}^4}\right)\|\delta\xi\|\|\delta\xi_\tau\|\|\Delta\xi_\tau\| \\
&\quad+\frac{4\overline{c}_0\overline{c}_1}{\|\xi_\tau\|\underline{v}^3}\left(1+3\frac{\overline{\overline{v}}^2}{\underline{v}^2}+2\frac{\overline{\overline{v}}^4}{\underline{v}^4}\right)\|\delta\xi_\tau\|^2\|\Delta\xi\| \\
&\quad+\frac{3\overline{\overline{v}}^4}{\|\xi_\tau\|^2\underline{v}^5}\left(1+\frac{\overline{\overline{v}}^2}{\underline{v}^2}\right)\|\delta\xi_\tau\|^2\|\Delta\xi_\tau\|.
\end{aligned}
$$

With $\frac{\overline{c}_0}{\overline{\overline{v}}}\le\frac{1}{\sqrt{5}}$, we note that

$$
\frac{\overline{c}_0}{\underline{v}}\le\frac{1}{2},\qquad \frac{\overline{\overline{v}}}{\underline{v}}\le\sqrt{\frac{3}{2}},\quad\text{and}\qquad \frac{\overline{v}}{\underline{v}}\le\frac{\sqrt{5}}{2}
$$

and obtain

$$
\begin{aligned}
|f'''[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau]| \leq \frac{\|\xi_\tau\|}{\underline{v}} \Bigg[& \frac{\overline{c}_3}{\underline{v}} \left(\frac{5}{2} + \sqrt{\frac{3}{2}}\right) \\
& + 6\frac{\overline{c}_1\overline{c}_2}{\underline{v}^2}\left(\frac{9}{2} + 2\sqrt{\frac{3}{2}}\right) \\
& + 6\frac{\overline{c}_1^3}{\underline{v}^3}\left(\frac{17}{2} + 3\sqrt{\frac{3}{2}}\right)\Bigg] \|\delta\xi\|^2\|\Delta\xi\| \\
& + \frac{1}{\underline{v}}\left[\frac{57}{2}\frac{\overline{c}_1^2}{\underline{v}^2} + \frac{13}{2}\frac{\overline{c}_2}{\underline{v}}\right]\|\delta\xi\|^2\|\Delta\xi_\tau\| \\
& + \frac{1}{\underline{v}}\left[57\frac{\overline{c}_1^2}{\underline{v}^2} + 13\frac{\overline{c}_2}{\underline{v}}\right]\|\delta\xi\|\|\delta\xi_\tau\|\|\Delta\xi\| \\
& + 40\frac{\overline{c}_1}{\|\xi_\tau\|\underline{v}^2}\|\delta\xi\|\|\delta\xi_\tau\|\|\Delta\xi_\tau\| \\
& + 20\frac{\overline{c}_1}{\|\xi_\tau\|\underline{v}^2}\|\delta\xi_\tau\|^2\|\Delta\xi\| \\
& + \frac{135}{8\|\xi_\tau\|^2\underline{v}}\|\delta\xi_\tau\|^2\|\Delta\xi_\tau\|,
\end{aligned}
$$

Rounding up the values yields the bound

$$
\begin{aligned}
|f'''[\delta\xi,\delta\xi_\tau]^2[\Delta\xi,\Delta\xi_\tau]| \leq \frac{\|\xi_\tau\|}{\underline{v}} & \left[4\frac{\overline{c}_3}{\underline{v}} + 42\frac{\overline{c}_1\overline{c}_2}{\underline{v}^2} + 74\frac{\overline{c}_1^3}{\underline{v}^3}\right]\|\delta\xi\|^2\|\Delta\xi\| \\
& + \frac{1}{\underline{v}}\left[29\frac{\overline{c}_1^2}{\underline{v}^2} + 7\frac{\overline{c}_2}{\underline{v}}\right]\|\delta\xi\|^2\|\Delta\xi_\tau\| \\
& + \frac{1}{\underline{v}}\left[57\frac{\overline{c}_1^2}{\underline{v}^2} + 13\frac{\overline{c}_2}{\underline{v}}\right]\|\delta\xi\|\|\delta\xi_\tau\|\|\Delta\xi\| \\
& + 40\frac{\overline{c}_1}{\|\xi_\tau\|\underline{v}^2}\|\delta\xi\|\|\delta\xi_\tau\|\|\Delta\xi_\tau\| \\
& + 20\frac{\overline{c}_1}{\|\xi_\tau\|\underline{v}^2}\|\delta\xi_\tau\|^2\|\Delta\xi\| \\
& + 18\frac{1}{\|\xi_\tau\|^2\underline{v}}\|\delta\xi_\tau\|^2\|\Delta\xi_\tau\|.
\end{aligned}
$$

□

A.2 Bounds in a Neighborhood of a Minimizer

Below we derive bounds that hold in a L^∞-neighborhood of a global minimizer. Let $x^{\star\star} = (z^{\star\star}, \lambda^{\star\star})$ be a global minimizer of (4) and the corresponding Lagrange multipliers. Moreover, let $x_1, x_2 \in \mathcal{B}(x^{\star\star}, R)$ and define $\Delta x := x_2 - x_1$. Then it holds that $\|\Delta x\|_{Y^\infty} \leq 2R$ and consequently

$$\|\Delta\xi\|_{L^\infty(]0,1[)} \underset{(22a)}{\leq} 2R, \tag{85}$$

$$\|\Delta\xi_\tau\|_{L^\infty(]0,1[)} \underset{(22a)}{\leq} 2R. \tag{86}$$

Let $\|w\|_{L^\infty(\Omega)} \leq \overline{c}_0$, $\|w_x\|_{L^\infty(\Omega)} \leq \overline{c}_1$, $\|w_{xx}\|_{L^\infty(\Omega)} \leq \overline{c}_2$, and $\|w_{xxx}\|_{L^\infty(\Omega)} \leq \overline{c}_3$, then the following bounds hold,

$$|w(\xi_2) - w(\xi_1)| = |\int_0^1 w_x(\xi_1 + \mu\delta\xi)[\delta\xi]d\mu\,| \leq \overline{c}_1\|\delta\xi\| \leq 2R\overline{c}_1, \tag{87}$$

$$\|w_x(\xi_2) - w_{x,}(\xi_1)\| = |\int_0^1 w_{xx}(\xi_1 + \mu\delta\xi)d\mu\,| \quad \leq \overline{c}_2\|\delta\xi\| \leq 2R\overline{c}_2, \tag{88}$$

$$\|w_{xx}(\xi_2) - w_{xx}(\xi_1)\| = |\int_0^1 w_{xxx}(\xi_1 + \mu\delta\xi)d\mu\,| \quad \leq \overline{c}_3\|\delta\xi\| \leq 2R\overline{c}_3. \tag{89}$$

Moreover, we show that

$$\begin{aligned}
|g(\xi_2) - g(\xi_1)| &= |\overline{v}^2 - w(\xi_2)^T w(\xi_2) - \overline{v}^2 + w(\xi_1)^T w(\xi_1)| \\
&= |w(\xi_2)^T w(\xi_2) - w(\xi_1)^T w(\xi_1)| \\
&\leq 2\overline{c}_0\overline{c}_1\|\delta\xi\| \\
&\leq 4R\overline{c}_0\overline{c}_1
\end{aligned} \tag{90}$$

$$\begin{aligned}
|g(\xi_2)^2 - g(\xi_1)^2| &= |(g(\xi_2) - g(\xi_1))(g(\xi_2) + g(\xi_1))| \\
&\leq (2\overline{c}_0\overline{c}_1\|\delta\xi\|)(2\overline{v}^2) \\
&\leq 4\overline{c}_0\overline{c}_1\overline{v}^2\|\delta\xi\| \\
&\leq 8R\overline{c}_0\overline{c}_1\overline{v}^2
\end{aligned} \tag{91}$$

$$\begin{aligned}
|g(\xi_2)^3 - g(\xi_1)^3| &= |g(\xi_2) - g(\xi_1)|\;|g(\xi_1)^2 + 2g(\xi_1)g(\xi_2) + g(\xi_2)^2| \\
&\leq (2\overline{c}_0\overline{c}_1\|\delta\xi\|)(4\overline{v}^4) \\
&\leq 8\overline{c}_0\overline{c}_1\overline{v}^4\|\delta\xi\| \\
&\leq 16R\overline{c}_0\overline{c}_1\overline{v}^4
\end{aligned} \tag{92}$$

$$\begin{aligned}
|g'(\xi_2) - g'(\xi_1)| &= |\int_0^1 g''(\xi_1 + \mu\delta\xi)[\delta\xi]d\mu| \\
&\leq 2(\overline{c}_1^2 + \overline{c}_0\overline{c}_2)\|\delta\xi\| \\
&\leq 4R(\overline{c}_1^2 + \overline{c}_0\overline{c}_2)
\end{aligned} \tag{93}$$

$$\begin{aligned}
\|g''(\xi_2) - g''(\xi_1)\| &= |\int_0^1 g'''(\xi_1 + \mu\delta\xi)d\mu| \\
&\leq 2(3\overline{c}_1\overline{c}_2 + \overline{c}_0\overline{c}_3)\|\delta\xi\| \\
&\leq 2R(3\overline{c}_1\overline{c}_2 + \overline{c}_0\overline{c}_3)
\end{aligned} \tag{94}$$

Furthermore, with F as given in (66), and (28) we get

$$\underline{v}^2(L^{\star\star} - R)^2 \leq \underline{v}^2\|\xi_\tau\|^2 \leq F \leq \|\xi_\tau\|^2\overline{\overline{v}}^2 \leq (L^{\star\star} + R)^2\overline{\overline{v}}^2$$

and

$$\begin{aligned}
|F'(\xi, \xi_\tau)[\delta\xi, \delta\xi_\tau]| &\leq 2\overline{\overline{v}}^2\|\xi_\tau\|\|\delta\xi_\tau\| + 4\overline{c}_0\overline{c}_1\|\xi_\tau\|^2\|\delta\xi\| \\
&\leq 2\overline{\overline{v}}^2(L^{\star\star} + R)\|\delta\xi_\tau\| + 4\overline{c}_0\overline{c}_1(L^{\star\star} + R)^2\|\delta\xi\|.
\end{aligned} \tag{95}$$

This yields

$$
\begin{aligned}
&|F(\xi_2,\xi_{\tau,2})^{1/2} - F(\xi_1,\xi_{\tau,1})^{1/2}| \\
&\qquad\qquad \le \frac{1}{2}|\int_0^1 F(\xi_1+\mu\delta\xi)^{-1/2}F'(\xi_1+\mu\delta\xi)d\mu| \\
&\qquad\qquad \le \frac{\overline{\overline{v}}^2(L^{\star\star}+R)}{\underline{v}(L^{\star\star}-R)}\|\delta\xi_\tau\| + \frac{2\overline{c}_0\overline{c}_1(L^{\star\star}+R)^2}{\underline{v}(L^{\star\star}-R)}\|\delta\xi\| \qquad (96)
\end{aligned}
$$

$$
\begin{aligned}
&|F(\xi_2,\xi_{\tau,2})^{-1/2} - F(\xi_1,\xi_{\tau,1})^{-1/2}| \\
&\qquad\qquad \le \frac{1}{2}|\int_0^1 F(\xi_1+\mu\delta\xi)^{-3/2}F'(\xi_1+\mu\delta\xi)d\mu \\
&\qquad\qquad \le \frac{\overline{\overline{v}}^2(L^{\star\star}+R)}{\underline{v}^3(L^{\star\star}-R)^3}\|\delta\xi_\tau\| + \frac{2\overline{c}_0\overline{c}_1(L^{\star\star}+R)^2}{\underline{v}^3(L^{\star\star}-R)^3}\|\delta\xi\| \qquad (97)
\end{aligned}
$$

$$
\begin{aligned}
&|F(\xi_2,\xi_{\tau,2})^{-3/2} - F(\xi_1,\xi_{\tau,1})^{-3/2}| \\
&\qquad\qquad \le \frac{3}{2}|\int_0^1 F(\xi_1+\mu\delta\xi)^{-5/2}F'(\xi_1+\mu\delta\xi)d\mu| \\
&\qquad\qquad \le \frac{\overline{\overline{v}}^2(L^{\star\star}+R)}{\underline{v}^5(L^{\star\star}-R)^5}\|\delta\xi_\tau\| + \frac{2\overline{c}_0\overline{c}_1(L^{\star\star}+R)^2}{\underline{v}^5(L^{\star\star}-R)^5}\|\delta\xi\| \qquad (98)
\end{aligned}
$$

For f_1 as defined in (48), we obtain

$$
\begin{aligned}
&(f_1''(\xi_2,\xi_{\tau,2}) - f_1''(\xi_1,\xi_{\tau,1})))[\Delta\xi,\Delta\xi_\tau][\delta\xi,\delta\xi_\tau] \\
&\quad = g(\xi_1)^{-3} g(\xi_2)^{-3} \Big[- 2g(\xi_1)^3 (g'(\xi_2)\delta\xi)(\xi_{\tau,2}^T w(\xi_2))(g'(\xi_2)\Delta\xi) \\
&\qquad + 2g(\xi_2)^3 (g'(\xi_1)\delta\xi)(\xi_{\tau,1}^T w(\xi_1))(g'(\xi_1)\Delta\xi) \\
&\qquad + g(\xi_1)^3 g(\xi_2)(\delta\xi_\tau^T w(\xi_2))(g'(\xi_2)\Delta\xi) \\
&\qquad - g(\xi_2)^3 g(\xi_1)(\delta\xi_\tau^T w(\xi_1))(g'(\xi_1)\Delta\xi) \\
&\qquad + g(\xi_1)^3 g(\xi_2)(\xi_{\tau,2}^T w_x(\xi_2)\delta\xi)(g'(\xi_2)\Delta\xi) \\
&\qquad - g(\xi_2)^3 g(\xi_1)(\xi_{\tau,1}^T w_x(\xi_1)\delta\xi)(g'(\xi_1)\Delta\xi) \\
&\qquad + g(\xi_1)^3 g(\xi_2)(\xi_{\tau,2}^T w(\xi_2))(\Delta\xi^T g''(\xi_2)\delta\xi) \\
&\qquad - g(\xi_2)^3 g(\xi_1)(\xi_{\tau,1}^T w(\xi_1))(\Delta\xi^T g''(\xi_1)\delta\xi) \\
&\qquad + g(\xi_1)^3 g(\xi_2)(g'(\xi_2)\delta\xi)(\xi_{\tau,2}^T w_x(\xi_2)\Delta\xi) \\
&\qquad - g(\xi_2)^3 g(\xi_1)(g'(\xi_1)\delta\xi)(\xi_{\tau,1}^T w_x(\xi_1)\Delta\xi) \\
&\qquad - g(\xi_1)^3 g(\xi_2)^2 (\delta\xi_\tau w_x(\xi_2)\Delta\xi) \\
&\qquad + g(\xi_2)^3 g(\xi_1)^2 (\delta\xi_\tau w_x(\xi_1)\Delta\xi) \\
&\qquad - g(\xi_1)^3 g(\xi_2)^2 w_{xx}(\xi_2)[\xi_{\tau,2},\Delta\xi,\delta\xi] \\
&\qquad + g(\xi_2)^3 g(\xi_1)^2 w_{xx}(\xi_1)[\xi_{\tau,1},\Delta\xi,\delta\xi] \\
&\qquad + g(\xi_1)^3 g(\xi_2)(g'(\xi_2)\delta\xi)(w(\xi_2)^T\Delta\xi_\tau) \\
&\qquad - g(\xi_2)^3 g(\xi_1)(g'(\xi_1)\delta\xi)(w(\xi_1)^T\Delta\xi_\tau) \\
&\qquad - g(\xi_1)^3 g(\xi_2)^2 (\Delta\xi_\tau^T w_x(\xi_2)\delta\xi) \\
&\qquad + g(\xi_2)^3 g(\xi_1)^2 (\Delta\xi_\tau^T w_x(\xi_1)\delta\xi) \Big]
\end{aligned}
$$

Using the bounds from above we finally obtain

$$
\begin{aligned}
&| \left(f_1''(\xi_2,\xi_{\tau,2}) - f_1''(\xi_1,\xi_{\tau,1})\right) [\Delta\xi,\Delta\xi_\tau][\delta\xi,\delta\xi_\tau]| \\
&\qquad \le \hat\beta_1 R \sqrt{\|\Delta\xi\|^2 + \|\Delta\xi_\tau\|^2}\sqrt{\|\delta\xi\|^2 + \|\delta\xi_\tau\|^2} \qquad (99)
\end{aligned}
$$

with

$$
\begin{aligned}
\hat\beta_1 = \frac{4}{\underline{v}^{12}} \big(& 5 + 80\overline{c}_0\overline{c}_1\overline{v}^4 + 8\overline{c}_0\overline{c}_1\overline{v}^2 + 12\overline{c}_0\overline{c}_1 + 16\overline{c}_0\overline{c}_2 + 4\overline{c}_0\overline{c}_3 \\
&+ 16\overline{c}_1^2 + 12\overline{c}_1\overline{c}_2 + 4\overline{c}_1 + 4\overline{c}_2 + 2\overline{c}_3\big). \qquad (100)
\end{aligned}
$$

For f_2 as defined in (50) we obtain

$$
\begin{aligned}
&(f_2''(\xi_2,\xi_{\tau,2}) - f_2''(\xi_1,\xi_{\tau,1}))[\Delta\xi,\Delta\xi_\tau][\delta\xi,\delta\xi_\tau] \\
&= g(\xi_1)^{-3} g(\xi_2)^{-3} \Big[\; 2g(\xi_1)^3 (g'(\xi_2)\delta\xi)(g'(\xi_2)\Delta\xi) F(\xi_2)^{1/2} \\
&\qquad - 2g(\xi_2)^3 (g'(\xi_1)\delta\xi)(g'(\xi_1)\Delta\xi) F(\xi_1)^{1/2} \\
&\qquad - g(\xi_2) g(\xi_1)^3 (\Delta\xi^T g''(\xi_2)\delta\xi) F(\xi_2)^{1/2} \\
&\qquad + g(\xi_1) g(\xi_2)^3 (\Delta\xi^T g''(\xi_1)\delta\xi) F(\xi_1)^{1/2} \\
&\qquad - \frac{1}{2} g(\xi_2) g(\xi_1)^3 (g(\xi_2)'\Delta\xi) F(\xi_2)^{-1/2} F'(\xi_2)[\delta\xi,\delta\xi_\tau] \\
&\qquad + \frac{1}{2} g(\xi_1) g(\xi_2)^3 (g(\xi_1)'\Delta\xi) F(\xi_1)^{-1/2} F'(\xi_1)[\delta\xi,\delta\xi_\tau] \\
&\qquad - \frac{1}{2} g(\xi_2) g(\xi_1)^3 (g'(\xi_2)\delta\xi) F(\xi_2)^{-1/2} F'(\xi_2)[\Delta\xi,\Delta\xi_\tau] \\
&\qquad + \frac{1}{2} g(\xi_1) g(\xi_2)^3 (g'(\xi_1)\delta\xi) F(\xi_1)^{-1/2} F'(\xi_1)[\Delta\xi,\Delta\xi_\tau] \\
&\qquad + \frac{1}{2} g(\xi_2)^2 g(\xi_1)^3 F(\xi_2)^{-1/2} F''(\xi_2)[\Delta\xi,\Delta\xi_\tau][\delta\xi,\delta\xi_\tau] \\
&\qquad - \frac{1}{2} g(\xi_1)^2 g(\xi_2)^3 F(\xi_1)^{-1/2} F''(\xi_1)[\Delta\xi,\Delta\xi_\tau][\delta\xi,\delta\xi_\tau] \\
&\qquad - \frac{1}{4} g(\xi_2)^2 g(\xi_1)^3 F(\xi_2)^{-3/2} F'(\xi_2)[\Delta\xi,\Delta\xi_\tau] F'(\xi_2)[\delta\xi,\delta\xi_\tau] \\
&\qquad + \frac{1}{4} g(\xi_1)^2 g(\xi_2)^3 F(\xi_1)^{-3/2} F'(\xi_1)[\Delta\xi,\Delta\xi_\tau] F'(\xi_1)[\delta\xi,\delta\xi_\tau] \Big]
\end{aligned}
$$

Using the bounds from above, this yields

$$
\begin{aligned}
&|(f_2''(\xi_2,\xi_{\tau,2}) - f_2''(\xi_1,\xi_{\tau,1}))[\delta\xi,\delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \\
&\qquad\qquad \leq \hat{\beta}_2 R \sqrt{\|\Delta\xi\|^2 + \|\Delta\xi_\tau\|^2} \sqrt{\|\delta\xi\|^2 + \|\delta\xi_\tau\|^2}
\end{aligned}
\tag{101}
$$

with

$$
\begin{aligned}
\hat{\beta}_2 \leq \frac{4}{\underline{v}^{12}} \Bigg(& 20 + 10\overline{c}_1 + 7\overline{c}_2 + \overline{c}_3 + 10\overline{c}_0\overline{c}_1 + 36\overline{c}_0\overline{c}_1\overline{v}^2 + 88\overline{c}_0\overline{c}_1\overline{v}^4 \\
& + 20\overline{c}_0\overline{c}_2 + 8\overline{c}_0\overline{c}_3 + 20\overline{c}_1^2 + 24\overline{c}_1\overline{c}_2 \\
& + \left(\frac{3}{\underline{v}(L^{\star\star} - R)} + \frac{6}{\underline{v}^3(L^{\star\star} - R)^3} + \frac{6}{\underline{v}^5(L^{\star\star} - R)^5} \right) \\
& \quad \left(\overline{\overline{v}}^2 (L^{\star\star} + R) + 2\overline{c}_0\overline{c}_1 (L^{\star\star} + R)^2 \right) \Bigg).
\end{aligned}
\tag{102}
$$

Lemma 8. *Let $x^{\star\star} = (z^{\star\star}, \lambda^{\star\star})$ be a global minimizer of* (4) *and the corresponding Lagrange multipliers. Moreover, let $x_1, x_2 \in \mathcal{B}(x^{\star\star}, R)$ and define $\Delta x := x_2 - x_1$. Then there is a $\hat{B} < \infty$ such that*

$$
|(f''(\xi_2,\xi_{\tau,2}) - f''(\xi_1,\xi_{\tau,1}))[\Delta\xi,\Delta\xi_\tau][\tilde{\delta\xi},\tilde{\delta\xi_\tau}]| \leq \hat{B} R \, \|x_2 - x_1\| \, \|\delta x\|. \tag{103}
$$

Beweis. With (99) and (101) we obtain

$$
\begin{aligned}
&|(f''(\xi_2, \xi_{\tau,2}) - f''(\xi_1, \xi_{\tau,1}))[\Delta\xi, \Delta\xi_\tau][\tilde{\delta\xi}, \tilde{\delta\xi_\tau}]| \\
&\qquad \le |(f_1''(\xi_2, \xi_{\tau,2}) - f_1''(\xi_1, \xi_{\tau,1}))[\Delta\xi, \Delta\xi_\tau][\tilde{\delta\xi}, \tilde{\delta\xi_\tau}]| \\
&\qquad\quad + |(f_2''(\xi_2, \xi_{\tau,2}) - f_2''(\xi_1, \xi_{\tau,1}))[\Delta\xi, \Delta\xi_\tau][\tilde{\delta\xi}, \tilde{\delta\xi_\tau}]| \\
&\qquad \le \hat{B}R\sqrt{\|\Delta\xi\|^2 + \|\Delta\xi_\tau\|^2}\sqrt{\|\delta\xi\|^2 + \|\delta\xi_\tau\|^2} \\
&\qquad \le \hat{B}R\,\|x_2 - x_1\|\,\|\delta x\|
\end{aligned}
$$

with $\hat{B} = \max\{\hat{\beta}_1, \hat{\beta}_2\}$. □