Numerical Riemann solutions in multi-pieces for 2-D gas dynamics systems I. Contact discontinuities

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Abstract: The numerical solutions of Riemann problems in three, four, five and six pieces, which only contain contact discontinuities, are presented by using Taylor FVM MmB schemes on regular triangular meshes for 2-D gas dynamics systems. The 2-D Riemann initial data are as defined in [1], under the assumption that each jump in initial data outside of the origin projects exactly one planar wave of shocks, centered rarefaction waves , or contact discontinuities. The main ends of the paper are that spirals will be shown for some configurations and the relations of the solutions between different distibutions of Riemann initial data are explained by the numerical solutions of modified Riemann problems.

Key words and phrases: Riemann problem, gas dynamics systems, spiral, MmB schemes.

1. Introduction

Two dimensional Riemann problems in four-pieces, which are constants in each quadrant, have theoretically been studied for 2-d scalar conservation law in [2][3], 2-D 2×2 conservation laws in [4] and 2-D gas dynamics systems in [1].

In [1], some conjectures on the solution structure of the Riemann problem in four pieces, that initial data are constant in each quadrant, have been given for two dimensional flow of polytropic gas. In that paper, initial data were selected under the assumption that each jump in initial data outside of the origin projects exactly one planar wave of shocks, rarefaction waves, or contact discontinuities. The main and interesting result is that there is a spiral for some case.

Correspondingly, numerical solutions of the Riemann problems have been presented in [5][6] for 2-D scalar conservation law, 2-D 2×2 conservation laws [7] and 2-D isentropic and adiabatic flows [8][9][10]. For Riemann problems in three pieces, which only contain contact discontinuites , numerical solutions have been calculated by MmB schemes on regular triangular meshes for 2-D gas dynamics systems[11].

In this paper, we are mainly interested in the numerical discription on the solution structure of Riemann problems in multi-pieces, which only contain contact discontinuities, for the two dimensional gas dynamics systems—adiabatic flow

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0 \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y = 0 \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y = 0 \\ (\rho(e + \frac{u^2 + v^2}{2}))_t + (\rho u(h + \frac{u^2 + v^2}{2}))_x + (\rho v(h + \frac{u^2 + v^2}{2}))_y = 0 \end{cases} \tag{1.1}$$

$$e = \frac{p}{(\gamma - 1)\rho}, \qquad h = e + \frac{p}{\rho}$$

where ρ, (u,v) and p denote density, velocity and pressure, respectively. and Riemann data in multi-pieces are discribed as follows,

$$(\rho, p, u, v)|_{t=0} = ST_i, \qquad i = 1, ..., m \tag{1.2}$$

where ST_i (i=1,...,m) are constant states. See Fig. 1.1.

Fig. 1.1 Distribution of Riemann Problem

Due to the analyses in [1], the characteristics in a direction θ= (μ, ν) for (1.1) are writtten as

$$\lambda_0 = u_\theta, \qquad \textit{flow} \quad \textit{characteristic}$$

$$\lambda_\pm = u_\theta \pm c, \qquad \textit{wave} \quad \textit{characteristics}$$

where $u_\theta = \mu u + \nu v$ is velocity in θ direction and c is sound speed, c=$\sqrt{\gamma p/\rho}$.

By Rankine-Hugoniot condition and [1], the conditions of initial data are for a discontinuous line in a direction.

Fig. 1.2

where u_μ and u_ν is tangential velocity and normal velocity, repectively,

(i) rarefaction wave (R)

$$\begin{cases} u_{\mu,1} = u_{\mu,2} \\ u_{\nu,2} = u_{\nu,1} + \int_{\rho_1}^{\rho_2} \dfrac{\sqrt{p'}}{\rho} d\rho \\ \rho_1 \neq \rho_2 \\ p_1 \rho_1^{-\gamma} = p_2 \rho_2^{-\gamma} \end{cases}$$

There are two classes of rarefaction waves according to wave characteristics: forward rarefaction wave—$\overrightarrow{R}$ $(\rho_1 > \rho_2)$ and backward rarefaction wave — $\overleftarrow{R}$ $(\rho_1 < \rho_2)$.

(ii) shock wave (S)

$$\begin{cases} u_{\mu,1} = u_{\mu,2} \\ u_{\nu,2} + \sqrt{\frac{\rho_1}{\rho_2} p'_{12}} = u_{\nu,1} + \sqrt{\frac{\rho_2}{\rho_1} p'_{21}} \\ \rho_1 \neq \rho_2 \\ \frac{p_2}{p_1} = \frac{(\gamma+1)\rho_2 - (\gamma-1)\rho_1}{(\gamma+1)\rho_1 - (\gamma-1)\rho_2} \end{cases}$$

wherer $p'_{12} = p'_{21} = \dfrac{p_2 - p_1}{\rho_2 - \rho_1}$. forward shock wave—$\overrightarrow{S}$ $(\rho_1 < \rho_2)$ and backward shock wave — $\overleftarrow{S}$ $(\rho_1 > \rho_2)$.

(iii) contact discontinuity (J)

$$\begin{cases} u_{\mu,1} \neq u_{\mu,2} \\ u_{\nu,1} = u_{\nu,2} \\ \rho_1 \neq \rho_2 \\ p_1 = p_2 \end{cases} \tag{1.3}$$

Due to the signals of Curl(u,v)=$v_x - u_y$ defined in [2], the contact discontinuities are dvided into two classes which can be found in [9],

$$J^+, \quad if \quad Curl(u,v) = +\infty; \qquad J^-, \quad if \quad Curl(u,v) = -\infty$$

It is very difficult to give the exact theoretical solution since the structure of the solution of 2-D Riemann problem for 2-D gas dynamics systems is very complicated. Therefore it is necessary to present numerical Riemann solutions for 2-D gas dynamics systems. In section 2 of the paper, a simple discription on numerical methods (see [11]) , MmB schemes on regular triangular meshes, is presented for 2-D conservation laws; then classifications of Riemann problems in three pieces and more than three pieces and corresponding numerical solutions, which only contain contact discontinuities, are given in section 3 and section 4, respectively; In section 5, the relations of Riemann problems between three pieces and more than three pieces are discussed for (1.1) (1.2). We know that Riemann problems in three pieces are simplest.

2. Numerical methods

Here we recall some discriptions of constructing Taylor FVM MmB schemes on regular triangular meshes from [11]. First we consider the initial value problem for 2-D scalar conservation law,

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0, \\ u(x, y, t)|_{t=0} = u_0(x, y) \end{cases} \tag{2.1}$$

where $u_0(x, y)$ is a piecewise smooth function.

We figure the local regular triangular meshes at point (x_i, y_i) as follows,

Fig. 2.1 Local Regular Triangular Meshes

We first discretize (2.1) by Taylor expansion in time direction,

$$\begin{aligned} u^{n+1} &= u^n + \Delta t u_t^n + \tfrac{1}{2}\Delta t^2 u_{tt}^n \\ &= u^n - \Delta t(f_x + g_y) + \tfrac{1}{2}\Delta t^2[(f_u(f_x+g_y))_x + (g_u(f_x+g_y))_y] \end{aligned} \tag{2.2}$$

and then integrate (2.2) on C_i, we have

$$\int\int_{C_i} u^{n+1} ds = \int\int_{C_i} u^n ds - \Delta t \int_l (f\nu^x + g\nu^y) dl + \frac{1}{2}\Delta t^2 \int_l (f_u\nu^x + g_u\nu^y)(f_x + g_y) dl \tag{2.3}$$

In (2.3), take u_i as the integral average u on C_i, that is

$$u_i = \frac{1}{Ar(c_i)} \int\int_{C_i} u ds \tag{2.4}$$

To discretize $\int_l (f_u\nu^x + g_u\nu^y)(f_x + g_y)dl$, by transformation of the coordinates(see Fig.2.2), we have

Fig. 2.2 Transformation of Coordinates

$$\begin{cases} x' = \nu^x x + \nu^y y \\ y' = \nu^x y - \nu^y x \end{cases}$$

then

$$\begin{cases} f_x = f_{x'}\nu^x - f_{y'}\nu^y \\ g_y = g_{x'}\nu^y + g_{y'}\nu^x \end{cases}$$

and

$$f_x + g_y = f'_{x'} + g'_{y'}$$

where

$$f' = f\nu^x + g\nu^y, \qquad g' = g\nu^x - f\nu^y$$

(2.4) becomes

$$\begin{aligned} u_i^{n+1} = u_i^n - & \frac{\Delta t}{Ar(C_i)} \sum_{j=1}^{6} (f_{I_j}\nu_{i_j}^x + g_{I_j}\nu_{i_j}^y)|g_{i_j}g_{i_{j+1}}| \\ & + \frac{1}{2}\frac{\Delta t^2}{Ar(C_i)} \sum_{j=1}^{6} [(f_u\nu_{i_j}^x + g_u\nu_{i_j}^y)(f'_{x'} + g'_{y'})]_{I_j}|g_{i_j}g_{i_{j+1}}| \end{aligned} \tag{2.5}$$

According to (2.4), let

$$\begin{aligned} & (g'_{y'})_{I_j} = 0 \\ & A_{i_j} = (f_u\nu_{i_j}^x + g_u\nu_{i_j}^y)_{I_j} \\ & \quad = \begin{cases} \dfrac{(f_{i_j} - f_i)\nu_{i_j}^x + (g_{i_j} - g_i)\nu_{i_j}^y}{u_{i_j} - u_i} & if \quad u_{i_j} \neq u_i \\ (f_u\nu_{i_j}^x + g_u\nu_{i_j})|_{u_i}, & if \quad u_{i_j} = u_i \end{cases} \\ & (f'_{x'})_{I_j} = \frac{1}{h}(f'_{i_j} - f'_i) \end{aligned}$$

then we get a second order accurate scheme which is like Lax-Wendroff scheme .

$$\begin{aligned} u_i^{n+1} = u_i^n & - \frac{1}{2}\frac{\Delta t|g_{i_1}g_{i_2}|}{Ar(C_i)} \sum_{j=1}^{3} [(f_{i_j} - f_{i_{j+3}})\nu_{i_j}^x + (g_{i_j} - g_{i_{j+3}})\nu_{i_j}^y)] \\ & + \frac{1}{2}\frac{\Delta t^2|g_{i_1}g_{i_2}|}{Ar(C_i)h} \sum_{j=1}^{3} [a_{i_j}^2(u_{i_j} - u_i) - a_{i_{j+3}}^2(u_i - u_{i_{j+3}})] \end{aligned} \tag{2.6}$$

By the experience of constructing MmB schemes on rectangular meshes [5], modified schemes of (2.6) are given in the following conservative form,

$$u_i^{n+1} = u_i^n - S_i \sum_{j=1}^{3} (\widetilde{fg}_{i_j} - \widetilde{fg}_{i_{j+3}}) \tag{2.7}$$

where

$$\widetilde{fg}_{i_j} = \frac{1}{2}(\overline{fg}_i + \overline{fg}_{i_j}) - \frac{1}{2}(|a_{i_j}| + a_{i_j}^-(1 + a_{i_j}^-\lambda)Q_{i_j}^- - a_{i_j}^+(1 - a_{i_j}^+\lambda)Q_{i_j}^+)(u_{i_j} - u_i)$$

$$\begin{aligned} \widetilde{fg}_{i_{j+3}} = \tfrac{1}{2}(\overline{fg}_i + \overline{fg}_{i_{j+3}}) - \frac{1}{2}(|a_{i_{j+3}}| & + a_{i_{j+3}}^-(1 + a_{i_{j+3}}^-\lambda)Q_{i_{j+3}}^- \\ & - a_{i_{j+3}}^+(1 - a_{i_{j+3}}^+\lambda)Q_{i_{j+3}}^+)(u_i - u_{i_{j+3}}) \end{aligned}$$

$$\overline{fg}_k = f_k\nu_k^x + g_k\nu_k^y, \qquad k = i, i_1, ..., i_6$$

and

$$
\begin{aligned}
&S_i = |g_{i_j} g_{i_{j+1}}|/Ar(C_i), \qquad |g_{i_j} g_{i_{j+1}}| = const. \\
&Q^{\pm}_{i_j} = Q(r^{\pm}_{i_j}), \qquad Q^{\pm}_{i_{j+3}} = Q(r^{\pm}_{i_{j+3}}) \\
&r^{+}_{i_j} = \frac{a^{+}_{i_{j+3}}(u_i - u_{i_{j+3}})}{a^{+}_{i_j}(u_{i_j} - u_i)} \qquad r^{-}_{i_j} = \frac{a^{-}_{i_{j_j}}(u_{i_{j_j}} - u_{i_j})}{a^{-}_{i_j}(u_{i_j} - u_i)} \\
&r^{+}_{i_{j+3}} = \frac{a^{+}_{i_{j+3_{j+3}}}(u_{i_{j+3}} - u_{i_{j+3_{j+3}}})}{a^{+}_{i_{j+3}}(u_i - u_{i_{j+3}})} \qquad r^{-}_{i_{j+3}} = \frac{a^{-}_{i_j}(u_{i_j} - u_i)}{a^{-}_{i_{j+3}}(u_i - u_{i_{j+3}})}
\end{aligned}
$$

$$
a_{i_j} = \begin{cases} [(f_{i_j} - f_i)\nu^x_{i_j} + (g_{i_j} - g_i)\nu^y_{i_j}]/(u_{i_j} - u_i), & u_{i_j} \neq u_i \\ \dfrac{\partial f}{\partial u}\nu^x_{i_j} + \dfrac{\partial g}{\partial u}\nu^y_{i_j}|_{u_i}, & u_{i_j} = u_i \end{cases}
$$

$$
a_{i_{j+3}} = \begin{cases} [(f_i - f_{i_{j+3}})\nu^x_{i_{j+3}} + (g_i - g_{i_{j+3}})\nu^y_{i_{j+3}}]/(u_i - u_{i_{j+3}}), & u_{i_{j+3}} \neq u_i \\ \dfrac{\partial f}{\partial u}\nu^x_{i_{j+3}} + \dfrac{\partial g}{\partial u}\nu^y_{i_{j+3}}|_{u_i}, & u_{i_{j+3}} = u_i \end{cases}
$$

Under some restriction on $\lambda = \Delta t/h$, $Q^{\pm}_{i_j}$ and $Q^{\pm}_{i_{j+3}}$, we can obtain :

u^{n+1}_i **is a convex combination of** $u^n_{i_1}$**, ...,** $u^n_{i_6}$

that is, MmB, see the detail discriptions in [11].

Consider the initial value problem for 2-D systems in conservation laws,

$$
\begin{cases} U_t + F(U)_x + G(U)_y = 0 \\ U(x,y,t)|_{t=0} = U_0(x,y) \end{cases} \tag{2.8}
$$

where $U_0(x,y)$ is a piecewise smooth vector function and $U = (u_1, ..., u_n)^T$,$F(U) = (f_1(U), ..., f_n(U))^T$ and $G(U) = (g_1(U), ..., g_n(U))^T$.

Here the genaralized schemes of (2.7) for (2.8) are in the following forms,

$$
U^{n+1}_i = U^n_i - S_i \sum_{j=1}^{3} (\widetilde{FG}_{i_j} - \widetilde{FG}_{i_{j+3}})
$$

where

$$
\widetilde{FG}_{i_j} = \frac{1}{2}(\overline{FG}_i + \overline{FG}_{i_j}) - \frac{1}{2} R_{i_j}(|\Lambda_{i_j}| + \Lambda^{-}_{i_j}(I + \Lambda^{-}_{i_j}\lambda)Q^{-}_{i_j} - \Lambda^{+}_{i_j}(I - \Lambda^{+}_{i_j}\lambda)Q^{+}_{i_j})R^{-1}_{i_j}(U_{i_j} - U_i)
$$

$$\widetilde{FG}_{i_{j+3}} = \tfrac{1}{2}(\overline{FG}_i + \overline{Fg}_{i_{j+3}}) \quad -\frac{1}{2}R_{i_{j+3}}(|\Lambda_{i_{j+3}}| + \Lambda^-_{i_{j+3}}(I + \Lambda^-_{i_{j+3}}\lambda)Q^-_{i_{j+3}}$$
$$-\Lambda^+_{i_{j+3}}(I - \Lambda^+_{i_{j+3}}\lambda)Q^+_{i_{j+3}})R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}})$$
$$\overline{FG}_k = F_k\nu^x_k + G_k\nu^y_k, \qquad k = i, i_1, ..., i_6$$
$$Q^{\pm} = diag(Q^{1,\pm}, ..., Q^{n,\pm}), \qquad Q^{k,\pm}_{i_j} = Q(r^{k,\pm}_{i_j}), \qquad Q^{k,\pm}_{i_{j+3}} = Q(r^{k,\pm}_{i_{j+3}})$$
$$r^{k,+}_{i_j} = \frac{(R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}}))^k}{(R^{-1}_{i_j}(U_{i_j} - U_i))^k} \qquad r^{k,-}_{i_j} = \frac{(R^{-1}_{i_{j_j}}(U_{i_{j_j}} - U_{i_j}))^k}{(R^{-1}_{i_j}(U_{i_j} - U_i))^k}$$
$$r^{k,+}_{i_{j+3}} = \frac{(R^{-1}_{i_{j+3_{j+3}}}(U_{i_{j+3}} - U_{i_{j+3_{j+3}}}))^k}{(R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}}))^k} \qquad r^{k,-}_{i_{j+3}} = \frac{(R^{-1}_{i_j}(U_{i_j} - U_i))^k}{(R^{-1}_{i_{j+3}}(U_i - U_{i_{j+3}}))^k}$$
$$k = 1, .., n$$
$$A^{\pm} = R\Lambda^{\pm}R^{-1}, \qquad \Lambda^{\pm} = \tfrac{1}{2}(\Lambda \pm |\Lambda|)$$
$$A_{i_j} = R_{i_j}\Lambda_{i_j}R^{-1}_{i_j} \qquad A_{i_{j+3}} = R_{i_{j+3}}\Lambda_{i_{j+3}}R^{-1}_{i_{j+3}}$$

and as defined in Roe [12], we let

$$(F(U_{i_j}) - F(U_i))\nu^x_{i_j} + (G(U_{i_j}) - G(U_i))\nu^y_{i_j} = A_{i_j}(U_{i_j} - U_i)$$

and

$$(F(U_i) - F(U_{i_{j+3}}))\nu^x_{i_{j+3}} + (G(U_i) - G(U_{i_{j+3}}))\nu^y_{i_{j+3}} = A_{i_{j+3}}(U_i - U_{i_{j+3}})$$

3. Riemann problems in three pieces

Firstly, we must clarify that there is no 2-D Riemann problem in two-pieces which only contain contact discontinuities unless the problem is a one- dimensional Riemann problem. For example, assume two states (u_1, v_1) and (u_2, v_2), and the directions of contact discontinuity lines are (ν^x_1, ν^y_1) and (ν^x_2, ν^y_2). then from (1.3) they satisfy

$$\begin{aligned} \nu^x_1 u_1 + \nu^y_1 v_1 &= \nu^x_1 u_2 + \nu^y_1 v_2 \\ \nu^x_2 u_1 + \nu^y_2 v_1 &= \nu^x_2 u_2 + \nu^y_2 v_2 \end{aligned} \tag{3.1}$$

see Fig. 3.1

Fig. 3.1

Obviously, if $(\nu_1^x, \nu_1^y) \neq (\nu_2^x, \nu_2^y)$, then (u_1, v_1) $=(u_2, v_2)$, it is a constant state; Otherwise, if $(u_1, v_1) \neq (u_2, v_2)$, then (ν_1^x, ν_1^y) and (ν_2^x, ν_2^y) are same. So it is a one dimensional Riemann problem.

Consider three distributions of Riemann problems in three pieces as follows

Fig. 3.2 Distribution of Riemann Problem in Three Pieces

For Fig.3.2-a, if we don't consider the signals of Curl(u,v), there is only one case, the distribution of velocities for initial data is counterclockwise. In this case there is a piral. See Fig.3.3-a.

The Riemann data: Take $\rho_1 = 0.5$, $\rho_2 = 1$, $\rho_3 = 1.5$, $p_1 = p_2 = p_3 = 2$, $u_1 = u_2 = \sqrt{3}/2$, $u_3 = -\sqrt{3}/2$, $v_1 = -1$, $v_2 = 1$, $v_3 = 0$.

a. Distribution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines

d. Pressure Contour Lines

Fig. 3.3-I Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

Where pseudo-velocity (U,V) is expressed (U,V)=(u-x/t,v-y/t). From Fig. 3.3-I, the interactions of the three contact discontinuities produce a spiral in the pseudo-subsonic region and wave characteristic lines are tangent to pesudo-sonic lines, see [1].

For Fig.3.2-b, there is only one case, the distribution of velocities for initial data is clockwise. See Fig.3.3-b

The Riemann data: Take $\rho_1 = 0.5$, $\rho_2 = 1.5$, $\rho_3 = 1$, $p_1 = p_2 = p_3 = 1$, $u_1 = u_2 = -\sqrt{3}/2$, $u_3 = \sqrt{3}/2$, $v_1 = -1$, $v_2 = 1$, $v_3 = 0$.

a. Distribution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines

d. Pressure Contour Lines

Fig. 3.3-II Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

In this case, the interactions of the three contact discontinuities produce two symmetric shock waves which bound the pseudo-subsonic region and a contact discontinuity in the subsonic region.

For Fig.3.2-c, there is no configuration which only contain contact discontinuities in this situation. It is easy to prove this conclusion. If there are three states (u_1, v_1), (u_2, v_2) and (u_3, v_3), and corresponding directions of contact discontinuity lines are (ν_1^x, ν_1^y), (ν_2^x, ν_2^y) and (ν_3^x, ν_3^y). Then we have

$$\begin{aligned}
\nu_1^x u_1 + \nu_1^y v_1 &= \nu_1^x u_2 + \nu_1^y v_2 \\
\nu_2^x u_2 + \nu_2^y v_2 &= \nu_2^x u_3 + \nu_2^y v_3 \\
\nu_3^x u_3 + \nu_3^y v_3 &= \nu_3^x u_1 + \nu_3^y v_1
\end{aligned}$$

From Fig.3.1-c, $(\nu_2^x, \nu_2^y) = -(\nu_3^x, \nu_3^y)$, then we obtain

$$\nu_2^x u_1 + \nu_2^y v_1 = \nu_2^x u_2 + \nu_2^y v_2$$

We know that $(\nu_1^x, \nu_1^y) \neq (\nu_2^x, \nu_2^y)$, then due to the above discussion $(u_1, v_1) = (u_2, v_2)$, it makes a contradiction to the distribution of Riemann problem.

From the above analyses, we know that Riemann problems in three pieces are simplest for 2-D gas dynamics systems. The structure of Riemann solutions in three pices is a basic one for the solutions of Riemann problems in more than three pieces, see the following sections.

4. Riemann problems in more than three pieces

In this section we present the classifications and numerical solutions of Riemann problems in four, five and six pieces which only contain contact discontinuities.

4.1 Riemann problems in four pieces

Consider the following distribution of Riemann problems in four pieces

Fig. 4.1 Distribution of Riemann Problem in Four Pieces

There are two cases which only contain contact discontinuities. The first one is called counterclockwise see Fig.4.2-a. There is a spiral in the case.

The Riemann data: Take $\rho_1 = \rho_3 = 0.5$, $\rho_2 = \rho_4 = 1.5$, $p_1 = p_2 = p_3 = p_4 = 2.0$, $u_1 = u_2 = 1$, $u_3 = u_4 = -1$, $v_1 = v_4 = -1$, $v_2 = v_3 = 1$.

a. Distibution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines

d. Pressure Contour Lines

Fig.4.2-I Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

As Fig. 3.3-I in section 3, in this case there is a spiral in pseudo- subsonic region.

The second one is called clockwise see Fig.4.2-b. In this case there are two shock waves.

The Riemann data: Take $\rho_1 = \rho_3 = 0.5$, $\rho_2 = \rho_4 = 1.5$, $p_1 = p_2 = p_3 = p_4 = 1$, $u_1 = u_2 = -1$, $u_3 = u_4 = 1$, $v_1 = v_4 = -1$, $v_2 = v_3 = 1$.

a. Distribution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines d. Pressure Contour Lines

Fig.4.2-II Mesh Points 201×201, $\lambda = 0.15$, Time Steps n=200

In this case, the interactions of the four contact discontinuities also produce two symmetric shock waves, but there are no contact discontinuity in pseudo-subsonic region as we choose the initial data of velocities in a symmetric form.

4.2 Riemann problems in five-pieces

The distribution of Riemann problem in five pieces:

Fig.4.3 Distribution of Riemann Problem in Five Pieces

There are 5 cases for this distribution which only contain contact discontinuities. The first one is counterclockwise see Fig.4.4-a, there is a spiral in this case.

The Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = 2$, $u_1 = u_2 = 1.289$, $u_3 = u_5 = -0.711$, $u_4 = -1.289$, $v_1 = v_5 = -1$, $v_2 = v_3 = 1$, $v_4 = 0$.

a. Distribution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines

d. Pressure Contour Lines

Fig.4.4-I Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

From Fig. 4.4-I, we can find that the local structure of the solution is like the structure of solution in Fig. 3.3-I.

The second one is clockwise see Fig.4.4-b.

The Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = 1$, $u_1 = u_2 = -1$, $u_3 = u_5 = 1$, $u_4 = 0.402$, $v_1 = v_5 = -1$, $v_2 = v_3 = 1$, $v_4 = 0$.

a. Distribution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines

d. Pressure Contour Lines

Fig.4.4-II Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=200

In this case, we find that in the interaction region the structure of solution about ST_1, ST_2 and ST_3 is like Fig.3.3-II, that is, the interaction of the contact discontinuities produce two shock waves and one contact discontinuity in the pseudo-subsonic region. Also for the structure of the solution of ST_3, ST_4 and ST_5.

The classifications of the others by velocities are figured as follows:

Fig.4.5

Corresponding numerical solutions for Fig.4.5 are presented:

For Fig.4.5-a, the Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = 1$, $u_1 = u_2 = u_4 = -0.5$, $u_3 = -1.5$, $u_5 = 1.5$, $v_1 = v_5 = -\sqrt{3}$, $v_2 = v_3 = 0$, $v_4 = \sqrt{3}$

a. Density Contour Lines　　　　b. Pressure Contour Lines

Fig.4.6-I　Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

From Fig. 4.6-I, we know that the structure of solution of ST_2, ST_3 and ST_4, ST_4, ST_5 and ST_1 is like the structure of the solution in Fig. 3.3-II.

For Fig.4.5-b, the Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = 1.5$, $u_1 = u_2 = 1.5$, $u_3 = 0$, $u_4 = -1$, $u_5 = -1.5$, $v_1 = v_5 = 0$, $v_2 = v_3 = \sqrt{3}/2$, $v_4 = -\sqrt{3}/2$.

a. Density Contour Lines b. Pressure Contour Lines

Fig.4.6-II Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

For Fig.4.5-c, the Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = 1.5$, $u_1 = u_2 = 1$, $u_3 = 1.5$, $u_4 = -0.5$, $u_5 = -1.5$, $v_1 = v_5 = 0$, $v_2 = v_3 = \sqrt{3}$, $v_4 = -\sqrt{3}$.

a. Density Contour Lines　　　　b. Pressure Contour Lines

Fig.4.6-III Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

In the above two cases, the structures of the solutions are like the structures of solutions both Fig.3.3-I and Fig.3.3-II.

4.3 Riemann problems in six-pieces

The distribution of Riemann problem in six pieces:

Fig.4.7 Distribution of Riemann Problem in Six Pieces

There are 6 cases which only contain contact discontinuities. The first one is counterclockwise see Fig.4.8, there is a spiral in the case.

The Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = \rho_6 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 2$, $u_1 = u_2 = \sqrt{3}$, $u_3 = u_6 = 0$, $u_3 = u_4 = -\sqrt{3}$, $v_1 = v_5 = -1$, $v_2 = v_4 = 1$, $v_3 = 2$, $v_6 = -2$.

a. Distribution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines

d. Pressure Contour Lines

Fig.4.8 Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=200

The structure of the solution in Fig. 4.8 is like the structure of the solution in Fig.3.3-I locally.

The second one is clockwise see Fig.4.9.

The Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = \rho_6 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1.5$, $u_1 = u_2 = -\sqrt{3}$, $u_3 = u_6 = 0$, $u_4 = u_5 = \sqrt{3}$, $v_1 = v_5 = -1$, $v_2 = v_4 = 1$, $v_3 = v_6 = 0$.

a. Distribution of Velocity

b. Pseudo-Stream Field

c. Density Contour Lines

d. Pressure Contour Lines

Fig.4.9 Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

From Fig.4.9, in the interaction region, the structure of the solution in ST_1, ST_2 and ST_3, and ST_4, ST_5 and ST_6 is like the solution structure in Fig.3.3-II, respectively.

The classifications of the others are discribed as follows:

Fig.4.10

For Fig.4.10-a, the Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = \rho_6 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1$, $u_1 = u_2 = u_4 = u_5 = \sqrt{3}/2$, $u_3 = u_6 = -\sqrt{3}/2$, $v_1 = v_4 = 1$, $v_2 = v_5 = -1$, $v_3 = v_6 = 0$.

a. Density Contour Lines b. Pressure Contour Lines

Fig.4.11-I Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

For Fig.4.10-b, the Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = \rho_6 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1$, $u_1 = u_2 = 0$, $u_4 = u_5 = \sqrt{3}/2$, $u_3 = u_6 = -\sqrt{3}/2$, $v_1 = -0.5$, $v_2 = 0.5$, $v_3 = 1$, $v_4 = 2$, $v_5 = -2$, $v_6 = -1$.

a. Density Contour Lines

b. Pressure Contour Lines

Fig.4.11-II Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

For Fig.4.10-c, the Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = \rho_6 = 1$, $p_1 = p_2 = p_3 = p_4 = p_4 = p_6 = 1$, $u_1 = u_2 = u_4 = u_5 = \sqrt{3}/2$, $u_3 = \sqrt{3}$, $u_6 = -\sqrt{3}$, $v_1 = -1.5$, $v_2 = -0.5$, $v_3 = 0$, $v_4 = 0.5$, $v_5 = 1.5$, $v_6 = 0$.

a. Density Contour Lines

b. Pressure Contour Lines

Fig.4.11-III Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=300

For Fig.4.10-d, the Riemann data: Take $\rho_1 = \rho_4 = 0.5$, $\rho_2 = \rho_5 = 1.5$, $\rho_3 = \rho_6 = 1$, $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1$, $u_1 = u_2 = -\sqrt{3}/2$, $u_4 = u_5 = \sqrt{3}$, $u_3 = u_6 = -\sqrt{3}$, $v_1 = 2$, $v_2 = -2$, $v_3 = -1.5$, $v_4 = 0.5$, $v_5 = -0.5$, $v_6 = 1.5$.

a. Density Contour Lines b. Pressure Contour Lines

Fig.4.11-IV Mesh Points 201×201, $\lambda = 0.1$, Time Steps n=250

As discussed above, the structure of the solutions in Fig. 4.11-I, II, III and IV is locally like the structure of the solutions in Fig. 3.3-I and II, respectively.

From the above numerical results, the spirals are clearly shown in the counterclockwise cases and in the other cases there are shock waves when the initial Riemann data just contain contact discontonuities.

5. Relations of Riemann solutions between in three pieces and more than three pieces

As shown in section 3 and 4, the solutions of Riemann peoblems for 2-D gas dynamics are much more complicated than in one dimension. It is very difficult to give exact solutions for any distribution of the 2-D Riemann problem. Obviously the first thing that we can do is to find the simplest distribution of Riemann problem for 2-D gas dynamics systems. In section 3, we show that there is no 2-D Riemann problem in two-pieces and exsits Riemann problem in three pieces which only contain contact discontinuities. In this section, we explain some relations between Riemann problem in three pieces and Riemann problem in more than three pieces.

In order to find some structure of the solution of Riemann problem in more than three pieces which is like the structure of the solution of Riemann problem in three pieces, We consider the following modified Riemann problem in multi-pieces.

a. Original Riemann Problem   b. Modified Riemann Problem

Fig.5.1 Distribution of Riemann Problem in Multi-Pieces

It means that we insert a mid-state(MS=$\sum_{k=1}^{m}(md - st_k)$) near the origin.

Here we only choose one case for each distribution of Riemann problem among four, five and six pieces and present the density contour lines of the numerical solutios of the modified Riemann problems.

(i) Riemann problem in four pieces.

Here we choose the case of clockwise by taking more mesh points and one mesh point in the mid-state, see Fig.5.2.

For mor mesh points in mid-state, we choose two configurations, the first one is that the mid-state is taken as an average, that is,

$$U_{ms} = [\sum_{k=1}^{m} \int\int_{md-st_k} U_k dxdy]/Ar(MS)$$

Time steps n=150, see Fig.5.2-a. Due to last section, the data of mid-state are

$$\rho_{ms} = 1.0, \qquad p_{ms} = 1.0, \qquad u_{ms} = 0.0, \qquad v_{ms} = 0.0$$

In the second one, the mid-state is taken as two symmetric shock waves between ST_1 and ST_3, the data of mid-state are

$$\rho_{ms} = 1.0, \qquad p_{ms} = 2.44, \qquad u_{ms} = 0.0, \qquad v_{ms} = 0.0$$

see Fig.5.2-b.

a. Mid-State as Average b. Mid-State as Two Symmetric Shocks

Fig.5.2 Density Contour Lines, Mesh Points 201×201

$\lambda = 0.15$, Time Steps n=150

and one mesh point in mid-state is as follows

Fig.5.2-c Density Contour Lines, Mesh Points 201×201

$\lambda = 0.15$, Time Steps n=250

(ii) Riemann problem in five pieces.

Consider the second case (Fig.4.4-II) by taking more mesh points and one mesh point in the mid-state, the data of mid-state are chosen as the above average, see Fig.5.3.

a. More Points, Time Steps n=150 b. One Point, Time Steps n=200

Fig.5.3 Density Contour Lines, Mesh Points 201×201, $\lambda = 0.1$

(iii) Riemann problem in six pieces

Consider the counterclockwise case of Riemann problem in six pieces in section 4 by taking more mesh ponts and one mesh point in mid-state, the data of mid-state are taken as an average as the above discription, see Fig.5.4.

a. More Points, Time Steps n=150　　　　b. One Point, Time Steps n=200

Fig.5.4 Density Contour Lines, Mesh Points 201×201, $\lambda = 0.1$

We may also present the following mofified Riemann problems for the Riemann problem in four pieces.

a. Distribution of velocity,　　　　b. Density contour lines

Fig.5.5 Mesh Points 201×201, $\lambda = 0.15$, Time Steps n=200

In the above figures, we know that numerical solution in the case of one mesh point is the same as the original problem and the structure of the solution in the case of more mesh points is like the structure of the solution of original Riemann problem near the border of the interaction region. For the detail discriptions of the solution structure of Riemann problem in multi-pieces, we should study for the mixed Riemann problem in three pieces which contain shock waves, rarefaction waves and contact discontinuities.

In addition, due to the structure of triangular meshes, Riemann problem in three pieces should also be a basic one for us to construct numerical methods which are produced depending on Riemann solver. So it is necessary to analyse Riemann problem in three pieces first.

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    I. Interaction of contact discontinuities.

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