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# Enumeration and Random Realization of Triangulated Surfaces

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## Abstract

We discuss different approaches for the enumeration of triangulated surfaces. In particular, we enumerate all triangulated surfaces with 9 and 10 vertices. We also show how geometric realizations of orientable surfaces with few vertices can be obtained by choosing coordinates randomly.

## 1 Introduction

The enumeration of triangulations of the 2-dimensional sphere  $S^2$  was started by Brückner [17] at the end of the 19th century. It took Brückner several years and nine thick manuscript books to compose a list of triangulated 2-spheres with up to 13 vertices; cf. [18]. His enumeration was complete and correct up to 10 vertices. On 11, 12, and 13 vertices his census comprised 1251, 7616, and 49451 triangulations, respectively. These numbers were off only slightly; they were later corrected by Grace [26] (for 11 vertices), by Bowen and Fisk [13] (for 12 vertices), and by Royle [42] (for 13 vertices). In fact, Royle made use of the program `plantri` by Brinkmann and McKay [16] to enumerate triangulations of the 2-sphere with up to 23 vertices (see the manual of `plantri` and also Royle [42]). Table 1 and Table 2 list the respective numbers. Precise formulas for rooted triangulations of the 2-sphere with  $n$  vertices were determined by Tutte [50].

It follows from work of Steinitz [45, §46] that every triangulated 2-sphere can be reduced to the boundary of the tetrahedron by a sequence of *edge contractions*. In other words, the boundary of the tetrahedron is the only *irreducible triangulation* of the 2-sphere from which every  $n$ -vertex triangulation can be obtained by a suitable sequence of *vertex splits*. The program `plantri` implements this procedure and allows for a fast enumeration of triangulations of the 2-sphere  $S^2$ .

Barnette and Edelson [6] have shown that every 2-manifold has only finitely many irreducible triangulations. The respective numbers were determined for the torus (21 examples) by Grünbaum and Lavrenchenko [35], for the projective plane (2 examples) by Barnette [5], for the Klein bottle (29 examples) by Lawrencenko and Negami [36] and Sulanke [46], and, recently, for the orientable surface of genus 2 (396784 examples) and the non-orientable surfaces of genus 3 (9708 examples) and genus 4 (6297982 examples) by Sulanke [47] [48]. With his program `surftri` [49] (based on `plantri`), Sulanke generated all triangulations with up to (at least) 14 vertices for these surfaces by vertex-splitting; see [49] for respective counts of triangulations.

Table 1: Triangulated 2-spheres with  $11 \leq n \leq 23$  vertices.

$n$	Types
11	1249
12	7595
13	49566
14	339722
15	2406841
16	17490241
17	129664753
18	977526957
19	7475907149
20	57896349553
21	453382272049
22	3585853662949
23	28615703421545

Here, we will survey further enumeration approaches. In particular, we give a complete enumeration of all triangulated surfaces with up to 10 vertices.

For an arbitrary orientable or non-orientable surface  $M$  the Euler characteristic  $\chi(M)$  of  $M$  equals, by Euler's equation, the alternating sum of the number of vertices  $n = f_0$ , the number of edges  $f_1$ , and the number of triangles  $f_2$ , i.e.,

$$n - f_1 + f_2 = \chi(M). \tag{1}$$

By double counting of incidences between edges and triangles of a triangulation, it follows that  $2f_1 = 3f_2$ . Thus, the number of vertices  $n$  determines  $f_1$  and  $f_2$ , that is, a triangulated surface  $M$  of Euler characteristic  $\chi(M)$  on  $n$  vertices has  $f$ -vector

$$f = (f_0, f_1, f_2) = (n, 3n - 3\chi(M), 2n - 2\chi(M)). \tag{2}$$

An *orientable surface*  $M(g, +)$  of genus  $g$  has homology  $H_*(M(g, +)) = (\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z})$  and Euler characteristic  $\chi(M(g, +)) = 2 - 2g$ , whereas a *non-orientable surface*  $M(g, -)$  of genus  $g$  has homology  $H_*(M(g, -)) = (\mathbb{Z}, \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, 0)$  and Euler characteristic  $\chi(M(g, -)) = 2 - g$ . The smallest possible  $n$  for a triangulation of a 2-manifold  $M$  (with the exception of the orientable surface of genus 2, the Klein bottle, and the non-orientable surface of genus 3, where an extra vertex has to be added, respectively) is determined by Heawood's bound [28]

$$n \geq \left\lceil \frac{1}{2}(7 + \sqrt{49 - 24\chi(M)}) \right\rceil. \tag{3}$$

Corresponding minimal and combinatorially unique triangulations of the real projective plane  $\mathbb{RP}^2$  with 6 vertices ( $\mathbb{RP}_6^2$ ) and of the 2-torus with 7 vertices (Möbius'

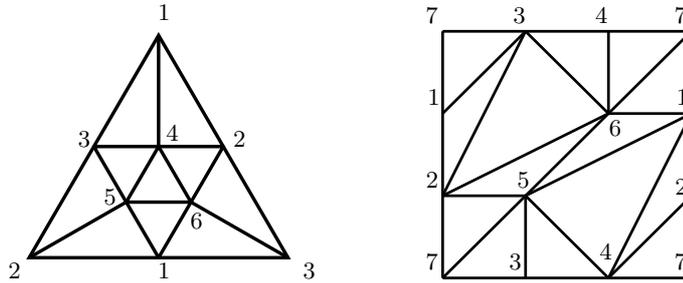


Figure 1: The real projective plane  $\mathbb{RP}_6^2$  and Möbius' torus.

torus [40]) were already known in the 19th century; see Figure 1. However, it took until 1955 to complete the construction of series of examples of minimal triangulations for all non-orientable surfaces (Ringel [41]) and until 1980 for all orientable surfaces (Jungerman and Ringel [32]).

A complete classification of triangulated surfaces with up to 8 vertices was obtained by Datta [20] and Datta and Nilakantan [21].

In Section 2, we discuss algorithms for the enumeration of triangulated surfaces, and we give the numbers (up to combinatorial equivalence) of triangulated surfaces with 9 and 10 vertices.

For triangulated orientable surfaces of genus  $g \geq 1$  it is, in general, a difficult problem to decide realizability. Surprisingly, for triangulations with few vertices, 3-dimensional geometric realizations (with straight edges, flat triangles, and without self-intersections) can be obtained by choosing coordinates randomly; see Section 3.

## 2 Enumeration of Triangulated Surfaces

At present, there are three essentially different enumeration schemes available to algorithmically generate triangulated surfaces with a fixed number  $n$  of vertices:

### 1. Generation from irreducible triangulations

An edge of a triangulated surface is *contractible* if the vertices of the edge can be identified without changing the topological type of the triangulation. A triangulation of a surface is *irreducible* if it has no contractible edge.

The triangulations of a given 2-manifold  $M$  with  $n$  vertices can therefore be obtained in two steps: First, determine all irreducible triangulations of  $M$  with  $\leq n$  vertices, and, second, generate additional reducible triangulations of  $M$  from the irreducible ones with  $< n$  vertices by *vertex-splitting*.

*Comments:* Although every 2-manifold  $M$  has only finitely many irreducible triangulations [6], it is non-trivial to classify or enumerate these. In the case of  $S^2$ ,

however, the boundary of the tetrahedron is the only irreducible triangulation. Resulting reducible triangulations of  $S^2$  with up to 23 vertices were generated by Royle [42] (with the program `plantri` by Brinkmann and McKay [16]). For further results, in particular by Sulanke, see the overview in the introduction.

## 2. *Strongly connected enumeration*

The basic idea here is to start with a single triangle, which has three edges as its boundary. Then select the lexicographically smallest edge of the boundary. This edge has to lie in a second triangle (since in a triangulated surface every edge is contained in exactly two triangles). In order to complement our pivot edge to a triangle we can choose as a third vertex either a vertex of the current boundary or a new vertex that has not yet been used. For every such choice we compose a new complex by adding the respective triangle to the previous complex. We continue with adding triangles until, eventually, we obtain a closed surface.

*Comments:* At every step of the procedure, the complexes that we produce are strongly connected. By the lexicographic choice of the pivot edges, the vertex-stars are closed in lexicographic order (which helps to sort out pseudomanifolds at an early stage of the generation; see the discussion below). Nevertheless, the generation is not very systematic: Albeit we choose the pivot edges in lexicographic order the resulting triangulations do not have to be lexicographically minimal.

Strongly connected enumeration was used by Altshuler and Steinberg [1], [4] to determine all combinatorial 3-manifolds with up to 9 vertices, by Bokowski [2] (cf. also [7]) to enumerate all 59 neighborly triangulations with 12 vertices of the orientable surface of genus 6, and by Lutz and Sullivan to enumerate all 4787 triangulated 3-manifolds of edge degree  $\leq 5$ .

## 3. *Lexicographic enumeration*

It is often very useful to list a collection of objects, e.g., triangulated surfaces, in *lexicographic order*: Every listed triangulated surface with  $n$  vertices is the lexicographically smallest set of triangles combinatorially equivalent to this triangulation and is lexicographically smaller than the next surface in the list.

### 3'. *Mixed lexicographic enumeration*

A variant of 3.; see below.

We present in the following an algorithm for the lexicographic and the mixed lexicographic enumeration of triangulated surfaces. (Similar enumeration schemes for vertex-transitive triangulations have been described earlier by Kühnel and Lassmann [34] and Köhler and Lutz [33].)

Let  $\{1, 2, \dots, n\}$  be the ground set of  $n$  vertices. Then a *triangulation of a surface/a triangulated surface* with  $n$  vertices is

- a connected 2-dimensional simplicial complex  $M \subseteq 2^{\{1, \dots, n\}}$
- such that the link of every vertex of  $M$  is a circle.

In particular,

- $M$  is *pure*, that is, every maximal face of  $M$  is 2-dimensional,
- and every edge of  $M$  is contained in exactly two triangles.

As an example, we consider the case  $n = 5$ . On the ground set  $\{1, 2, \dots, 5\}$  there are  $\binom{5}{3} = 10$  triangles,  $\binom{5}{2} = 10$  edges, and  $2^{10}$  different sets of triangles, of which only few compose a triangulated surface with 5 vertices. Our aim will be to find those sets of triangles that indeed form a triangulated surface. One way to proceed is by *backtracking*:

Start with some triangle and add further triangles as long as no edge is contained in more than two triangles. If this condition is violated, then backtrack. A set of triangles is *closed* if every of its edges is contained in exactly two triangles. If the link of every vertex of a closed set of triangles is a circle, then this set of triangles gives a triangulated surface: OUTPUT surface.

We are interested in enumerating triangulated surfaces *up to combinatorial equivalence*, i.e., up to relabeling the vertices. Thus we can, without loss of generality, assume that the triangle 123 should be present in the triangulation and therefore can be chosen as the starting triangle. More than that, we can assume that the triangulated surfaces that we are going to enumerate should come in lexicographic order.

In a lexicographically minimal triangulation, the collection  $B_{\deg(1)}$  of triangles containing the vertex 1 is of the form

$$123, 124, 135, 146, 157, 168, \dots, 1(\deg(1) - 1)(\deg(1) + 1), 1\deg(1)(\deg(1) + 1),$$

where  $\deg(1)$  is the *degree* of the vertex 1, i.e., the number of neighbors of the vertex 1. Obviously,  $3 \leq \deg(1) \leq n - 1$ .

On 5 vertices, the vertex 1 has the possible beginning segments  $B_3$  and  $B_4$ ; see Figure 2.

In a lexicographically sorted list of (lexicographically smallest) triangulated surfaces with  $n$  vertices, those with beginning segment  $B_k$  are listed before those with beginning segment  $B_{k+1}$ , etc. Thus, we start the backtracking with beginning segment  $B_3$  and enumerate all corresponding triangulated surfaces, then restart the backtracking with beginning segment  $B_4$ , and so on. In triangulations with beginning segment

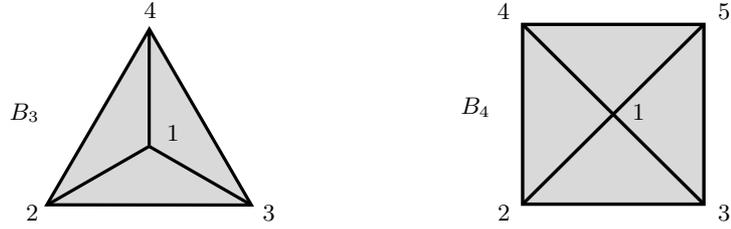


Figure 2: Beginning segments

$$\begin{array}{l}
 123 \\
 124 \\
 125 \\
 134 \\
 135 \\
 145 \\
 234 \\
 235 \\
 245 \\
 345
 \end{array}
 \begin{pmatrix}
 12 & 13 & 14 & 15 & 23 & 24 & 25 & 34 & 35 & 45 \\
 1 & 1 & & & 1 & & & & & \\
 1 & & 1 & & & 1 & & & & \\
 1 & & & 1 & & & 1 & & & \\
 & 1 & 1 & & & & & 1 & & \\
 & & 1 & & 1 & & & & 1 & \\
 & & & 1 & 1 & & & & & 1 \\
 & & & & 1 & 1 & & 1 & & \\
 & & & & 1 & & 1 & & 1 & \\
 & & & & & 1 & 1 & & & 1 \\
 & & & & & & & 1 & 1 & 1
 \end{pmatrix}$$

Figure 3: The triangle-edge-incidence matrix for  $n = 5$  vertices.

$B_k$  all vertices have degree *at least*  $k$ . Otherwise such a triangulation has a vertex of degree  $j < k$ . However, by relabeling the vertices, it can be achieved that the relabeled triangulation has beginning segment  $B_j$  (and thus would have appeared earlier in the lexicographically sorted list). Contradiction.

We store the triangles on the ground set as rows of a (sparse) triangle-edge-incidence matrix; see Figure 3 for the triangle-edge-incidence matrix in the case of  $n = 5$  vertices. The backtracking in terms of the triangle-edge-incidence matrix then can be formulated as follows: Start with the zero row vector and add to it all rows corresponding to the triangles of the beginning segment  $B_3$ . The resulting vector has entries

- 0 (the corresponding edge does not appear in  $B_3$ ),
- 1 (the corresponding edge is a boundary edge of  $B_3$ ),
- 2 (the corresponding edge appears twice in the triangles of  $B_3$ ).

We next add (the corresponding rows of) further triangles to (the sum vector of) our beginning segment. As soon as a resulting entry is larger than two, we backtrack, since in such a combination of triangles the edge corresponding to the entry is contained in at least three triangles, which is forbidden. If a resulting vector has entries 0 and 2 only, then the corresponding set of triangles is closed and thus is a candidate

for a triangulated surface. In case of  $n = 5$  vertices, the backtracking (in short) is as follows:

	12	13	14	15	23	24	25	34	35	45		
- :	( 0	0	0	0	0	0	0	0	0	0	)	
$B_3$ :	( 2	2	2	0	1	1	0	1	0	0	)	
$B_3 + 234$ :	( 2	2	2	0	2	2	0	2	0	0	)	<i>Candidate!</i>
$B_3$ :	( 2	2	2	0	1	1	0	1	0	0	)	
$B_3 + 235$ :	( 2	2	2	0	2	1	1	1	1	0	)	
$B_3 + 235 + 245$ :	( 2	2	2	0	2	2	2	1	1	1	)	
$B_3 + 235 + 245 + 345$ :	( 2	2	2	0	2	2	2	2	2	2	)	<i>Candidate!</i>
$B_3 + 235 + 245$ :	( 2	2	2	0	2	2	2	1	1	1	)	
$B_3 + 235$ :	( 2	2	2	0	2	1	1	1	1	0	)	
$B_3$ :	( 2	2	2	0	1	1	0	1	0	0	)	
- :	( 0	0	0	0	0	0	0	0	0	0	)	
$B_4$ :	( 2	2	2	2	1	1	0	0	1	1	)	
$B_4 + 234$ :	( 2	2	2	2	2	2	0	1	1	1	)	
$B_4 + 234 + 345$ :	( 2	2	2	2	2	2	0	2	2	2	)	<i>Candidate!</i>
$B_4 + 234$ :	( 2	2	2	2	2	2	0	1	1	1	)	
$B_4$ :	( 2	2	2	2	1	1	0	0	1	1	)	
$B_4 + 235$ :	( 2	2	2	2	2	1	1	0	2	1	)	
$B_4 + 235 + 245$ :	( 2	2	2	2	2	2	2	0	2	2	)	<i>Candidate!</i>
$B_4 + 235$ :	( 2	2	2	2	2	1	1	0	2	1	)	
$B_4$ :	( 2	2	2	2	1	1	0	0	1	1	)	
- :	( 0	0	0	0	0	0	0	0	0	0	)	

It remains to check whether every candidate is indeed a triangulated surface. For this we have to verify that the link of every vertex is a circle, which is a purely combinatorial condition that can easily be tested. Moreover, we have to check for every candidate, whether it is combinatorially isomorphic to a triangulation that has appeared in the backtracking before. For example, the triangulations  $B_4 + 234 + 345$  and  $B_4 + 235 + 245$  are isomorphic to  $B_3 + 235 + 245 + 345$ . In fact, we should have stopped to add triangles to  $B_4 + 234$ : The link of vertex 2 in this case is a closed circle of length 3, which should not occur, since the beginning segment  $B_4$  has degree 4. Similarly, we should have backtracked at  $B_4 + 235$ . Finally, we neglect triangulations with less than  $n$  vertices (such as  $B_3 + 234$ ). As result in the case  $n = 5$ , we obtain that there is a unique triangulation of the 2-sphere with 5 vertices.

The basic property that every edge is contained in exactly two triangles is often called *the pseudomanifold property*: Every closed set of triangles forms a *2-dimensional pseudomanifold*.

It is rather easy to construct 2-dimensional pseudomanifolds that are not surfaces: The left pseudomanifold in Figure 4 has the middle vertex as an isolated singularity with its vertex-link consisting of two disjoint triangles. The right pseudomanifold in Figure 4 has no singularities and thus is a surface, but it is not connected.

Since there are way more 2-dimensional pseudomanifolds than there are (connected) surfaces, it is necessary to sort out those pseudomanifolds that are not sur-

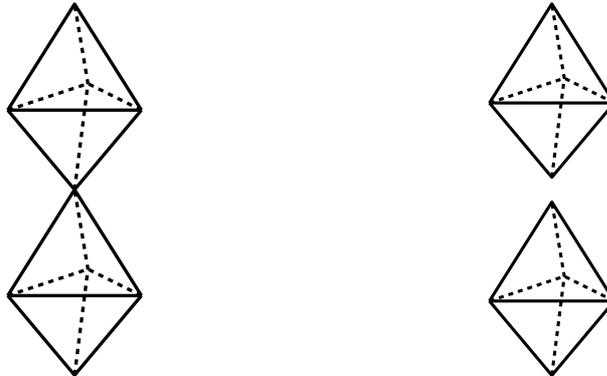


Figure 4: Examples of pseudomanifolds

faces as early as possible in our backtracking. In a 2-dimensional pseudomanifold the link of every vertex is a 1-dimensional pseudomanifold (i.e., every vertex lies in exactly two edges). Thus, every vertex-link is a union of disjoint circles whereas the link of a vertex in a proper surface is one single circle:

- Backtrack if the link of a vertex of a current sum of triangles consists of a closed circle plus at least one extra edge.

Since at isolated singularities there are more edges in the link than we want to have, we can try to avoid this as follows:

*Mixed lexicographic enumeration:* Start the backtracking with the beginning segment  $B_{n-1}$  (instead of  $B_3$ ), then proceed in reversed order with  $B_{n-2}, \dots$ , until  $B_3$  is processed.

In the case that the beginning segment is  $B_k$  we have the additional criterion:

- (Lexicographic enumeration:) Backtrack if the closed link of a vertex has less than  $k$  edges.
- (Mixed lexicographic enumeration:) Backtrack if the number of edges in the link of a vertex or if the degree of a vertex is larger than  $k$ .

In the mixed lexicographic approach, the sub-collections of triangulations that contain a vertex of the same maximal degree are sorted lexicographically. In particular, every triangulation in such a sub-collection begins with the same beginning segment (which is of type  $B_k$  for some  $k$ ). However, the complete list of triangulations is not produced in lexicographic order anymore.

By symmetry, we can, in addition, exclude the following cases (for the lexicographic as well as for the mixed lexicographic approach):

Table 2: Triangulated surfaces with up to 10 vertices.

$n$	Surface	Types	$n$	Surface	Types	$n$	Surface	Types
4	$S^2$	1	8	$S^2$	14	10	$S^2$	233
				$T^2$	7		$T^2$	2109
5	$S^2$	1		$\mathbb{RP}^2$	16		$M(2, +)$	865
6	$S^2$	2		$K^2$	6		$M(3, +)$	20
	$\mathbb{RP}^2$	1	9	$S^2$	50		$\mathbb{RP}^2$	1210
				$T^2$	112		$K^2$	4462
7	$S^2$	5		$M(3, -)$	134		$M(3, -)$	11784
	$T^2$	1		$K^2$	187		$M(4, -)$	13657
	$\mathbb{RP}^2$	3		$M(4, -)$	37		$M(5, -)$	7050
				$M(5, -)$	2		$M(6, -)$	1022
							$M(7, -)$	14

- Do not use triangles of the form  $23j$  with odd  $5 \leq j \leq k$ . (Since a resulting surface would, by relabeling, be isomorphic to a triangulation with beginning segment  $B_k$  plus triangle  $23(j-1)$ , which is lexicographically smaller). If the triangle  $23i$  with even  $6 \leq i \leq k$  is used, then do not use the triangles  $24j$  with odd  $3 \leq j \leq i-3$ .

Finally, we test for every resulting (connected) surface whether it has, up to combinatorial equivalence, appeared previously in the enumeration. For this, we first compute as combinatorial invariants the  $f$ -vector, the sequence of vertex degrees, and the *Altshuler-Steinberg determinant* [3] (i.e., the determinant  $\det(AA^T)$  of the vertex-triangle incidence matrix  $A$ ) of an example. If these invariants are equal for two resulting surfaces, then we take one triangle of the first complex and test for all possible ways it can be mapped to the triangles of the second complex whether this map can be extended to a simplicial isomorphism of the two complexes. (Alternatively, one can use McKay's fast graph isomorphism testing program `nauty` [39] to determine whether the vertex-facet incidence graphs of the two complexes are isomorphic or not.)

**Theorem 1** *There are exactly 655 triangulated surfaces with 9 vertices and 42426 triangulated surfaces with 10 vertices.*

The respective triangulations (in mixed lexicographic order) can be found online at [38]. Table 2 gives the detailed numbers of orientable and non-orientable surfaces that appear with up to 10 vertices. The corresponding combinatorial symmetry groups  $G$  of the examples are listed in Table 3.

Table 3: Symmetry groups of triangulated surfaces with up to 10 vertices.

$n$	Surface	$ G $	$G$	Types	$n$	Surface	$ G $	$G$	Types
4	$S^2$	24	$T^* = S_4,$	1	9	$S^2$	1	trivial	16
			3-transitive				2	$\mathbb{Z}_2$	25
5	$S^2$	12	$S_3 \times \mathbb{Z}_2$	1			4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	5
6	$S^2$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	1			6	$S_3$	1
		48	$O^* = \mathbb{Z}_2 \wr S_3,$ vertex-trans.	1			12	$S_3 \times \mathbb{Z}_2$	2
	$\mathbb{RP}^2$	60	$A_5,$	1			28	$D_7 \times \mathbb{Z}_2$	1
			2-transitive				1	trivial	52
7	$S^2$	2	$\mathbb{Z}_2$	1			2	$\mathbb{Z}_2$	46
		4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	1			3	$\mathbb{Z}_3$	1
	$T^2$	6	$S_3$	2			4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	7
		20	$D_{10}$	1			6	$\mathbb{Z}_6$	1
	$\mathbb{RP}^2$	42	$AGL(1, 7),$	1			12	$S_3 \times \mathbb{Z}_2$	2
			2-transitive				18	$D_9,$ vertex-trans.	1
	$\mathbb{RP}^2$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	1			108	$\mathbb{Z}_3^2 : D_6,$ vertex-trans.	1
			6		$S_3$	1			1
	$\mathbb{RP}^2$	24	$T^* = S_4$	1			2	$\mathbb{Z}_2$	52
			6		$S_3$	1			3
8	$S^2$	1	trivial	2			4	$\mathbb{Z}_4$	1
		2	$\mathbb{Z}_2$	5				$\mathbb{Z}_2 \times \mathbb{Z}_2$	11
	$T^2$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	3			6	$S_3$	5
		8	$D_4$	1			24	$T^* = S_4$	1
	$\mathbb{RP}^2$	12	$S_3 \times \mathbb{Z}_2$	1			1	trivial	131
			24		$T^* = S_4$ $D_6 \times \mathbb{Z}_2$	1			2
	$\mathbb{RP}^2$	2	$\mathbb{Z}_2$	2			4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	6
			3		$\mathbb{Z}_3$	1			6
	$\mathbb{RP}^2$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	2			12	$S_3 \times \mathbb{Z}_2$	2
			6		$S_3$	1			1
	$\mathbb{RP}^2$	32	$[32, 43],$	1			2	$\mathbb{Z}_2$	24
			vertex-trans.				3	$\mathbb{Z}_3$	1
	$\mathbb{RP}^2$	1	trivial	2			6	$\mathbb{Z}_6$	1
			2		$\mathbb{Z}_2$	8			1
	$\mathbb{RP}^2$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	5			1	trivial	19
			14		$D_7$	1			2
	$K^2$	1	trivial	1			3	$\mathbb{Z}_3$	1
			2		$\mathbb{Z}_2$	4			4
	$K^2$	2	$\mathbb{Z}_2$	4			6	$S_3$	1
			8		$D_4$	1			6
	$K^2$	8	$D_4$	1			18	$S_3 \times \mathbb{Z}_3,$ vertex-trans.	1

Table 3: Symmetry groups of triangulated surfaces (continued).

$n$	Surface	$ G $	$G$	Types	$n$	Surface	$ G $	$G$	Types	
10	$S^2$	1	trivial	137	$K^2$	1	trivial	4057		
		2	$\mathbb{Z}_2$	69		2	$\mathbb{Z}_2$	367		
		3	$\mathbb{Z}_3$	1		4	$\mathbb{Z}_4$	1		
		4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	13		8	$\mathbb{Z}_2^3$	2		
		6	$S_3$	6			$D_4$	3		
		8	$\mathbb{Z}_2^3$	2		10	$D_5$	1		
			$D_4$	2		20	$D_{10}$ ,			
		16	$D_8$	1			vertex-trans.	1		
		24	$T^* = S_4$	1		$M(3, -)$	1	trivial	11308	
		32	$D_8 \times \mathbb{Z}_2$	1			2	$\mathbb{Z}_2$	448	
			$T^2$	1		trivial	1763	3	$\mathbb{Z}_3$	12
				2		$\mathbb{Z}_2$	292	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	11
				3		$\mathbb{Z}_3$	10	6	$S_3$	5
4	$\mathbb{Z}_4$			2	$M(4, -)$	1	trivial	13037		
	$\mathbb{Z}_2 \times \mathbb{Z}_2$			33		2	$\mathbb{Z}_2$	556		
6	$S_3$			3	3	$\mathbb{Z}_3$	6			
8	$\mathbb{Z}_2^3$			2	4	$\mathbb{Z}_4$	1			
	$D_4$			1		$\mathbb{Z}_2 \times \mathbb{Z}_2$	45			
12	$S_3 \times \mathbb{Z}_2$			1	6	$\mathbb{Z}_6$	1			
20	$D_{10}$ ,					$S_3$	5			
	vertex-trans.			1	8	$\mathbb{Z}_2^3$	1			
	$AGL(1, 5)$			1		$D_4$	4			
$M(2, +)$	1			trivial	789	48	$O^* = \mathbb{Z}_2 \wr S_3$	1		
	2	$\mathbb{Z}_2$	61	$M(5, -)$	1	trivial	6792			
	3	$\mathbb{Z}_3$	7		2	$\mathbb{Z}_2$	214			
	4	$\mathbb{Z}_4$	2		3	$\mathbb{Z}_3$	22			
		$\mathbb{Z}_2 \times \mathbb{Z}_2$	4		4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	16			
	8	$\mathbb{Z}_8$	1		6	$\mathbb{Z}_6$	2			
	16	$\langle 2, 2   2 \rangle$	1			$S_3$	4			
	$M(3, +)$	1	trivial		8	$M(6, -)$	1	trivial	926	
2		$\mathbb{Z}_2$	3		2		$\mathbb{Z}_2$	71		
3		$\mathbb{Z}_3$	4	3	$\mathbb{Z}_3$		18			
4		$\mathbb{Z}_4$	2	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$		6			
		$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	12	$A_4$		1			
12		$A_4$	1	$M(7, -)$	1		trivial	4		
21		$\mathbb{Z}_7 \times \mathbb{Z}_3$	1		2		$\mathbb{Z}_2$	4		
$\mathbb{RP}^2$		1	trivial		923		3	$\mathbb{Z}_3$	1	
	2	$\mathbb{Z}_2$	242		5	$\mathbb{Z}_5$	1			
	3	$\mathbb{Z}_3$	2		6	$S_3$	1			
	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	29		9	$\mathbb{Z}_9$	1			
	6	$S_3$	10		12	$A_4$	1			
	12	$S_3 \times \mathbb{Z}_2$	2		60	$A_5$ ,				
		$A_4$	1		vertex-trans.	1				
	18	$D_9$	1							

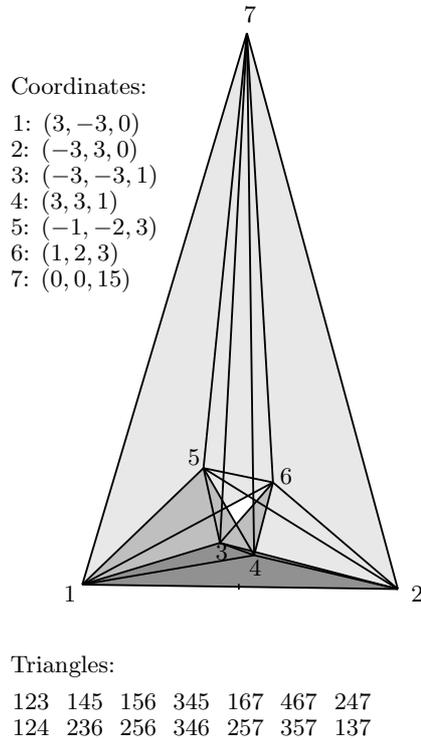


Figure 5: Császár's torus.

### 3 Random Realization

It was asked by Grünbaum [27, Ch. 13.2], whether every triangulated orientable surface can be embedded geometrically in  $\mathbb{R}^3$ , i.e., whether it can be realized with straight edges, flat triangles, and without self intersections? By Steinitz' theorem (cf. [44], [45], [51]), every triangulated 2-sphere is realizable as the boundary complex of a convex 3-dimensional polytope. For the 2-torus of genus 1 the realizability problem is still open.

**Conjecture 2** (Duke [22]) *Every triangulated torus is realizable in  $\mathbb{R}^3$ .*

A first explicit geometric realization (see Figure 5) of Möbius' minimal 7-vertex triangulation [40] of the 2-torus was given by Császár [19] (cf. also [24] and [37]). Bokowski and Eggert [10] showed that there are altogether 72 different "types" of realizations of the Möbius torus, and Fendrich [23] verified that triangulated tori with up to 11 vertices are realizable.

Brehm and Bokowski [8], [9], [14], [15] constructed geometric realizations for several triangulated orientable surfaces of genus  $g = 2, 3, 4$  with minimal numbers of vertices  $n = 10, 10, 11$ , respectively.

Neighborly triangulations (i.e., triangulations that have as its 1-skeleton the complete graph  $K_n$ ) of higher genus were considered as candidates for counter-examples to the Grünbaum realization problem for a while (cf. [19] and [12, p. 137]). Neighborly orientable surfaces have genus  $g = (n-3)(n-4)/12$  and therefore  $n \equiv 0, 3, 4, 7 \pmod{12}$  vertices, with  $g = 6$  and  $n = 12$  as the first case beyond the tetrahedron and the 7-vertex torus.

**Theorem 3** (Bokowski and Guedes de Oliveira [11]) *The triangulated orientable surface  $N_{54}^{12}$  of genus 6 with 12 vertices of Alshuler's list [2] is not geometrically embeddable in  $\mathbb{R}^3$ .*

In fact, Bokowski and Guedes de Oliveira showed that there is no oriented matroid compatible with the triangulation  $N_{54}^{12}$ , from which the non-realizability follows. Recently, Schewe reimplemented the approach of Bokowski and Guedes de Oliveira:

**Theorem 4** (Schewe [43]) *Every orientable surface of genus  $g \geq 5$  has a triangulation which is not geometrically embeddable in  $\mathbb{R}^3$ .*

From an algorithmic point of view the realizability problem for triangulated surfaces is decidable (cf. [12, p. 50]), but there is *no* algorithm known that would solve the realization problem for instances with, say, 10 vertices in reasonable time.

Surprisingly, the following simple heuristic can be used to realize tori and surfaces of genus 2 with up to 10 vertices.

**Random Realization** *For a given orientable surface with  $n$  vertices pick  $n$  integer points in a cube of size  $k^3$  (for some fixed  $k \in \mathbb{N}$ ) uniformly at random. Test whether this (labeled) set of  $n$  points in  $\mathbb{R}^3$  yields a geometric realization of the surface. If not, try again.*

Suppose, we are given  $n$  (integer) points  $\vec{x}_1, \dots, \vec{x}_n$  (in general position) in  $\mathbb{R}^3$  together with a triangulation of an orientable surface. It then is an elementary linear algebra exercise to check whether these  $n$  points provide a geometric realization of the surface: For every pair of a triangle  $i_1i_2i_3$  together with a combinatorially disjoint edge  $i_4i_5$  of the triangulation we have to test whether the geometric triangle  $\vec{x}_{i_1}\vec{x}_{i_2}\vec{x}_{i_3}$  and the edge  $\vec{x}_{i_4}\vec{x}_{i_5}$  have empty intersection.

Since the surfaces were enumerated in (mixed) lexicographic order with different triangulations in the list sometimes differing only slightly, it is promising to try:

**Recycling of Coordinates** *As soon as a realization has been found for a triangulated orientable surface with  $n$  vertices, test whether the corresponding set of coordinates yields realizations for other triangulations with  $n$  vertices as well. Moreover, perturb the coordinates slightly and try again.*

Random realization and recycling of coordinates was used to obtain realizations for 864 of the 865 triangulations of the orientable surface of genus 2 with 10 vertices, leaving one case open. This last example (`manifold_2_10_41348` in the catalog [38])

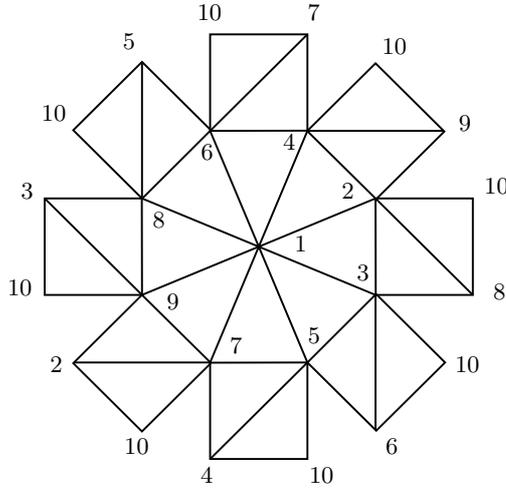


Figure 6: The double torus `manifold_2_10_41348`.

has the highest symmetry group of order 16 among the 865 examples; see Figure 6. It was realized geometrically by Jürgen Bokowski by construction of an explicit rubber-band model.

**Theorem 5** (Bokowski and Lutz; cf. [7]) *All 865 vertex-minimal 10-vertex triangulations of the orientable surface of genus 2 can be realized geometrically in  $\mathbb{R}^3$ .*

**Conjecture 6** *Every triangulation of the orientable surface of genus 2 is realizable in  $\mathbb{R}^3$ .*

A priori, nothing is known on the expected number of tries that are necessary to obtain a geometric realization for a given triangulated orientable surface by random realization: For triangulations of a surface of genus  $g \geq 1$  any proof showing the finiteness of the expected number of tries would immediately imply the realizability of the given triangulation.

*Computational Experiments:*

For the random realization of triangulated orientable surfaces with 10 vertices we chose  $k = 2^{15} = 32768$  as the side length of the cube.

- It took an average of about 700 tries to realize one of the 233 triangulations of  $S^2$  with 10 vertices (in non-convex position).
- It took an average of about 418000 tries to realize one of the 2109 triangulated tori with 10 vertices.

- For the 865 vertex-minimal 10-vertex triangulations of the orientable surface of genus 2 we initially set a limit of 200 million tries for every example. Random realizations were found for about  $\frac{1}{5}$  of the 864 triangulations. All other of the 864 realizations were found by recycling of coordinates. The computations were run for 3 months on ten Pentium R 2.8 GHz processors.
- It happened a few times that for a given triangulation realizations with identical coordinates were found by different processors after different numbers of tries. (Thus the length of cycles of the random generator plays a role.)

By successively rounding the coordinates of one of the randomly realized surfaces it is in most cases relatively easy to obtain realizations of the respective surface with much smaller coordinates. (For triangulations of  $S^2$  it is an open problem whether there are *convex* realizations with small coordinates; cf. [51, Ex. 4.16].) An enumerative search for realizations with small coordinates was carried out in [29] (this time, the computations were run for eight days on ten Pentium R 2.8 GHz processors) and [30]; see these two references and [38] for explicit coordinates and visualizations.

**Theorem 7** (Hougardy, Lutz, and Zelke [29]) *All the 865 vertex-minimal 10-vertex triangulations of the orientable surface of genus 2 have realizations in the  $(4 \times 4 \times 4)$ -cube, but cannot be realized (in general position) in the  $(3 \times 3 \times 3)$ -cube.*

A posteriori, the existence of small triangulations explains the successfulness of the described random search for realizations.

With a more advanced simulated annealing approach, realizations of surfaces can be obtained much easier:

**Theorem 8** (Hougardy, Lutz, and Zelke [31]) *All 20 vertex-minimal 10-vertex triangulations of the orientable surface of genus 3 can be realized geometrically in  $\mathbb{R}^3$ .*

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