Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

Frank H. Lutz[1]

Combinatorial 3-Manifolds with 10 Vertices

[1] *Main address:* Technische Universität Berlin, Fakultät II - Mathematik und Naturwissenschaften, Institut für Mathematik, Sekr. MA 3-2, Straße des 17. Juni 136, D-10623 Berlin, Germany, lutz@math.tu-berlin.de. *Additional address:* Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB), Takustraße 7, D-14195 Berlin, Germany, lutz@zib.de.

ZIB-Report 06-14 (März 2006)

Combinatorial 3-Manifolds with 10 Vertices

Frank H. Lutz

Abstract

We give a complete enumeration of combinatorial 3-manifolds with 10 vertices: There are precisely 247882 triangulated 3-spheres with 10 vertices as well as 518 vertex-minimal triangulations of the sphere product $S^2 \times S^1$ and 615 triangulations of the twisted sphere product $S^2 \underline{\times} S^1$.

All the 3-spheres with up to 10 vertices are shellable, but there are 29 vertex-minimal non-shellable 3-balls with 9 vertices.

1 Introduction

Let M be a triangulated 3-manifold with n vertices and face vector $f = (n, f_1, f_2, f_3)$. By Euler's equation, $n - f_1 + f_2 - f_3 = 0$, and by double counting the edges of the ridge-facet incidence graph, $2f_2 = 4f_3$, it follows that

$$f = (n, f_1, 2f_1 - 2n, f_1 - n). \tag{1}$$

A complete characterization of the f-vectors of the 3-sphere S^3, the sphere product $S^2 \times S^1$, the twisted sphere product (or 3-dimensional Klein bottle) $S^2 \underline{\times} S^1$, and of the real projective 3-space $\mathbb{R}\mathbf{P}^3$ was given by Walkup.

Theorem 1 (Walkup [30]) *For every* 3*-manifold* M *there is an integer* $\gamma(M)$ *such that*

$$f_1 \geq 4n + \gamma(M) \tag{2}$$

for every triangulation of M *with* n *vertices and* f_1 *edges. Moreover there is an integer* $\gamma^*(M) \geq \gamma(M)$ *such that for every pair* (n, f_1) *with* $n \geq 0$ *and*

$$\binom{n}{2} \geq f_1 \geq 4n + \gamma^*(M) \tag{3}$$

there is a triangulation of M *with* n *vertices and* f_1 *edges. In particular,*

(a) $\gamma^* = \gamma = -10$ *for* S^3*,*

(b) $\gamma^* = \gamma = 0$ *for* $S^2 \underline{\times} S^1$*,*

(c) $\gamma^* = 1$ *and* $\gamma = 0$ *for* $S^2 \times S^1$*, where, with the exception* $(9, 36)$*, all pairs* (n, f_1) *with* $n \geq 0$ *and* $4n + \gamma(M) \leq f_1 \leq \binom{n}{2}$ *occur,*

(d) $\gamma^* = \gamma = 7$ *for* $\mathbb{R}\mathbf{P}^3$*, and*

(e) $\gamma^*(M) \geq \gamma(M) \geq 8$ *for all other* 3*-manifolds* M*.*

Table 1: Combinatorial 3-manifolds with up to 10 vertices.

Vertices\Types	All	S^3	$S^2 \times S^1$	$S^2 \underline{\times} S^1$
5	1	1	–	–
6	2	2	–	–
7	5	5	–	–
8	39	39	–	–
9	1297	1296	–	1
10	249015	247882	518	615

By Walkup's theorem, vertex-minimal triangulations of $S^2 \underline{\times} S^1$, $S^2 \times S^1$, and $\mathbb{R}\mathbf{P}^3$ have 9, 10, and 11 vertices, respectively. The 3-sphere can be triangulated vertex-minimally as the boundary of the 4-simplex with 5 vertices. But otherwise, rather little is known on vertex-minimal triangulations of 3- and higher-dimensional manifolds. See [24], [25], and [29] for a discussion, further references, and for various examples of small triangulations of 3-manifolds.

The exact numbers of different combinatorial types of triangulations of S^3, $S^2 \underline{\times} S^1$, and $S^2 \times S^1$ with up to 9 vertices and of neighborly triangulations (i.e., triangulations with complete 1-skeleton) with 10 vertices were obtained by

Grünbaum and Sreedharan [15]	(simplicial 4-polytopes with 8 vertices),
Barnette [8]	(combinatorial 3-spheres with 8 vertices),
Altshuler [2]	(combinatorial 3-manifolds with up to 8 vertices),
Altshuler and Steinberg [5]	(neighborly 4-polytopes with 9 vertices),
Altshuler and Steinberg [6]	(neighborly 3-manifolds with 9 vertices),
Altshuler and Steinberg [7]	(combinatorial 3-manifolds with 9 vertices),
Altshuler [3]	(neighborly 3-manifolds with 10 vertices).

In this paper, the enumeration of 3-manifolds is continued: We completely classify triangulated 3-manifolds with 10 vertices. Moreover, we determine the combinatorial automorphism groups of all triangulations with up to 10 vertices, and we test for all 3-spheres (and all 3-balls) with up to 10 vertices (with up to 9 vertices) whether they are constructible, shellable, or vertex-decomposable.

2 Enumeration

We used a backtracking approach, described as mixed lexicographic enumeration in [26], to determine all triangulated 3-manifolds with 10 vertices: The vertex-links of a triangulated 3-manifold with 10 vertices are triangulated 2-spheres with up to 9 vertices. Altogether, there are 73 such 2-spheres, which are processed in decreasing size. As a first vertex-star of a 3-manifold that we are going to build we take the cone over one of the respective 2-spheres and then add further tetrahedra (in lexicographic order) as long as this is possible. If, for example, a triangle of a partial complex that we have built is contained in three tetrahedra, then this violates the *pseudo-manifold property* that in a triangulated 3-manifold every triangle is contained in *exactly two* tetrahedra. We backtrack, remove the last tetrahedron that we have added, and try to add to our partial complex the next tetrahedron (with respect to the lexicographic order). See [26] for further details on the enumeration.

Table 2: Combinatorial 3-manifolds with 10 vertices.

f-vector\Types	All	S^3	$S^2\times S^1$	$S^2 \underline{\times} S^1$
(10,30,40,20)	30	30	–	–
(10,31,42,21)	124	124	–	–
(10,32,44,22)	385	385	–	–
(10,33,46,23)	952	952	–	–
(10,34,48,24)	2142	2142	–	–
(10,35,50,25)	4340	4340	–	–
(10,36,52,26)	8106	8106	–	–
(10,37,54,27)	13853	13853	–	–
(10,38,56,28)	21702	21702	–	–
(10,39,58,29)	30526	30526	–	–
(10,40,60,30)	38575	38553	10	12
(10,41,62,31)	42581	42498	37	46
(10,42,64,32)	39526	39299	110	117
(10,43,66,33)	28439	28087	162	190
(10,44,68,34)	14057	13745	145	167
(10,45,70,35)	3677	3540	54	83
Total:	249015	247882	518	615

Theorem 2 *There are precisely* 249015 *triangulated* 3*-manifolds with* 10 *vertices:* 247882 *of these are triangulated* 3*-spheres,* 518 *are vertex-minimal triangulations of the sphere product* $S^2\times S^1$*, and* 615 *are triangulations of the twisted sphere product* $S^2 \underline{\times} S^1$*.*

Table 1 gives the total numbers of all triangulations with up to 10 vertices. The numbers of 10-vertex triangulations are listed in detail in Table 2. All triangulations can be found online at [22]. The topological types were determined with the bistellar flip program BISTELLAR [23]; see [10] for a description.

For a given triangulation, it is a purely combinatorial task to determine its combinatorial symmetry group. We computed the respective groups with a program written in GAP [14].

Corollary 3 *There are exactly* 1*,* 1*,* 5*,* 36*,* 408*, and* 7443 *triangulated* 3*-manifolds with* 5*,* 6*,* 7*,* 8*,* 9*, and* 10 *vertices, respectively, that have a non-trivial combinatorial symmetry group.*

The symmetry groups along with the numbers of combinatorial types of triangulations that correspond to a particular group are listed in Table 3. Altogether, there are 14 examples that have a vertex-transitive symmetry group; see [18].

All simplicial 3-spheres with up to 7 vertices are polytopal. However, there are two non-polytopal 3-spheres with 8 vertices, the Grünbaum and Sreedharan sphere [15] and the Barnette sphere [8]. The classification of triangulated 3-spheres with 9 vertices into polytopal and non-polytopal spheres was started by Altshuler and Steinberg [5], [6], [7] and completed by Altshuler, Bokowski, and Steinberg [4] and Engel [13]. For neighborly simplicial 3-spheres with 10 vertices the numbers of polytopal and non-polytopal spheres were determined by Altshuler [3], Bokowski and Garms [11], and Bokowski and Sturmfels [12].

Problem 4 *Classify all simplicial* 3*-spheres with* 10 *vertices into polytopal and non-polytopal spheres.*

Table 3: Symmetry groups of triangulated 3-manifolds with up to 10 vertices.

n	Manifold	$\lvert G\rvert$	G	Types
5	S^3	120	S_5, transitive	1
6	S^3	48	$O^* = \mathbb{Z}_2 \wr S_3$	1
		72	$S_3 \wr \mathbb{Z}_2$, transitive	1
7	S^3	8	D_4	2
		12	$S_3 \times \mathbb{Z}_2$	1
		14	D_7, transitive	1
		48	$D_4 \times D_3$	1
8	S^3	1	trivial	3
		2	$\mathbb{Z}_2$	13
		4	$\mathbb{Z}_4$	1
			$\mathbb{Z}_2 \times \mathbb{Z}_2$	9
		6	S_3	1
		8	$\mathbb{Z}_2^3$	1
			D_4	3
		12	$S_3 \times \mathbb{Z}_2$	4
		16	$D_4 \times \mathbb{Z}_2$	1
			D_8, transitive	1
		60	$D_5 \times D_3$	1
		384	$\mathbb{Z}_2 \wr S_4$, transitive	1
9	S^3	1	trivial	889
		2	$\mathbb{Z}_2$	319
		3	$\mathbb{Z}_3$	3
		4	$\mathbb{Z}_4$	6
			$\mathbb{Z}_2 \times \mathbb{Z}_2$	46
		6	$\mathbb{Z}_6$	1
			S_3	8
		8	$\mathbb{Z}_2^3$	3
			D_4	5
		12	$S_3 \times \mathbb{Z}_2$	10
		18	D_9, transitive	1
		24	$T^* = S_4$	3
		72	$D_6 \times D_3$	1
		80	$D_5 \times D_4$	1
	$S^2 \underline{\times} S^1$	18	D_9, transitive	1
10	S^3	1	trivial	240683
		2	$\mathbb{Z}_2$	6675
		3	$\mathbb{Z}_3$	10
		4	$\mathbb{Z}_4$	53
			$\mathbb{Z}_2 \times \mathbb{Z}_2$	358
		5	$\mathbb{Z}_5$	1
		6	$\mathbb{Z}_6$	1
			S_3	19
		8	$\mathbb{Z}_2^3$	15
			D_4	31
		10	$\mathbb{Z}_{10}$, transitive	1
			D_5	4
		12	$S_3 \times \mathbb{Z}_2$	15
		16	$D_4 \times \mathbb{Z}_2$	3
		20	D_{10}, transitive	1
			$AGL(1,5)$, transitive	2
		24	$T^* = S_4$	1
			$D_6 \times \mathbb{Z}_2$	2
		48	$O^* = \mathbb{Z}_2 \wr S_3$	2
		84	$D_7 \times D_3$	1
		96	$D_6 \times D_4$	1
		120	S_5	1
		200	$D_5 \wr \mathbb{Z}_2$, transitive	1
		240	$S_5 \times \mathbb{Z}_2$, transitive	1
	$S^2 \times S^1$	1	trivial	420
		2	$\mathbb{Z}_2$	95
		10	$\mathbb{Z}_{10}$, transitive	1
		16	$\langle 2,2,2 \rangle_2$	1
		20	D_{10}, transitive	1
	$S^2 \underline{\times} S^1$	1	trivial	469
		2	$\mathbb{Z}_2$	127
		4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	14
		8	D_4	2
		10	D_5	1
		20	D_{10}, transitive	2

Table 4: Combinatorial 3-balls with up to 9 vertices.

Vertices\Types	All	Non-Shellable	Not Vertex-Decomposable
4	1	–	–
5	3	–	–
6	12	–	–
7	167	–	2
8	10211	–	628
9	2451305	29	623819

3 3-Balls

Along with the enumeration of triangulated 3-spheres with up to 10 vertices we implicitly have enumerated all triangulated 3-balls with up to 9 vertices: Let B^3_{n-1} be a triangulated 3-ball with $n-1$ vertices and let v_n be a new vertex. Then the union $B^3_{n-1} \cup (v_n * \partial B^3_{n-1})$ of B^3_{n-1} with the cone $v_n * \partial B^3_{n-1}$ over the boundary ∂B^3_{n-1} with respect to v_n is a triangulated 3-sphere. Thus there are at most as many combinatorially distinct 3-spheres with n vertices as there are combinatorially distinct 3-balls with $n-1$ vertices. If, on the contrary, we delete the star of a vertex from a triangulated 3-sphere S^3_n with n vertices, then, obviously, we obtain a 3-ball with $n-1$ vertices. If we delete the star of a different vertex from S^3_n then we might or might not obtain a combinatorially different ball. Let $\#B^3(n-1)$ and $\#S^3(n)$ be the numbers of combinatorially distinct 3-balls and 3-spheres with $n-1$ and n vertices, respectively. Then

$$\#S^3(n) \leq \#B^3(n-1) \leq n \cdot \#S^3(n).$$

For the explicit numbers of simplicial 3-balls with up to 9 vertices see Table 4.

4 Vertex-Decomposability, Shellability, and Constructibility

The concepts of *vertex-decomposability*, *shellability*, and *constructibility* describe three particular ways to assemble a simplicial complex from the collection of its facets (cf. Björner [9] and see the surveys [16], [19], and [31]). The following implications are strict for (pure) simplicial complexes:

$$\text{vertex decomposable} \Longrightarrow \text{shellable} \Longrightarrow \text{constructible.}$$

It follows from Newman's and Alexander's fundamental works on the foundations of combinatorial and PL topology from 1926 [27] and 1930 [1] that a constructible d-dimensional simplicial complex in which every $(d-1)$-face is contained in exactly two or at most two d-dimensional facets is a PL d-sphere or a PL d-ball, respectively.

A *shelling* of a triangulated d-ball or d-sphere is a linear ordering of its f_d facets $F_1, ..., F_{f_d}$ such that if we remove the facets from the ball or sphere in this order, then at every intermediate step the remaining simplicial complex is a simplicial ball. A simplicial ball or sphere is *shellable* if it has a shelling; it is *extendably shellable* if any partial shelling $F_1, ..., F_i$, $i < f_d$, can be extended to a shelling; and it is *strongly non-shellable* if it has no *free* facet that can be removed from the triangulation without loosing ballness.

A triangulated d-ball or d-sphere is *constructible* if it can be decomposed into two constructible balls of smaller size and if, in addition, the intersection of the two balls is a constructible ball of dimension $d-1$; it is *vertex-decomposable* if we can remove the star of a vertex v and the remaining complex and the link of v are again vertex-decomposable balls.

We tested vertex-decomposability and shellability with a straightforward backtracking implementation.

Corollary 5 *All triangulated* 3*-spheres with* $n \leq 10$ *vertices are shellable and therefore constructible.*

An example of a non-constructible and thus non-shellable 3-sphere with 13 vertices has been constructed in [19]. It remains open whether there are non-shellable respectively non-constructible 3-spheres with 11 and 12 vertices.

Corollary 6 *All triangulated* 3*-balls with* $n \leq 8$ *vertices are extendably shellable.*

Examples of non-shellable 3-balls can be found at various places in the literature (cf. the references in [19], [20], and [31]) with the smallest previously known non-shellable 3-ball by Ziegler [31] with 10 vertices.

Corollary 7 *There are precisely twenty-nine vertex-minimal non-shellable simplicial* 3*-balls with* 9 *vertices, ten of which are strongly non-shellable. The twenty-nine balls have between* 18 *and* 22 *facets, with one unique ball* $B\_3\_9\_18$ *having* 18 *facets and* f*-vector* $(9, 33, 43, 18)$.

A list of facets and visualization of the ball $B\_3\_9\_18$ is given in [20].

The cone over a simplicial d-ball with respect to a new vertex is a $(d+1)$-dimensional ball. It is shellable respectively vertex-decomposable if and only if the original ball is shellable respectively vertex-decomposable (cf. [28]).

Corollary 8 *There are non-shellable* 3*-balls with* $d+6$ *vertices and* 18 *facets for* $d \geq 3$.

Each of the 29 non-shellable 3-balls with 9 vertices can be split into a pair of shellable balls.

Corollary 9 *All triangulated* 3*-balls with* $n \leq 9$ *vertices are constructible.*

Klee and Kleinschmidt [17] showed that all simplicial d-balls with up to $d+3$ vertices are vertex-decomposable.

Corollary 10 *There are not vertex-decomposable* 3*-balls with* $d+4$ *vertices and* 10 *facets for* $d \geq 3$.

In fact, there are exactly two not vertex-decomposable 3-balls with 7 vertices; see [21] for a visualization of these two balls. One of the examples has 10 tetrahedra, the other has 11 tetrahedra.

For the numbers of not vertex-decomposable 3-balls with up to 9 vertices see Table 4.

Corollary 11 *All triangulated* 3*-spheres with* $n \leq 8$ *vertices are vertex-decomposable.*

Klee and Kleinschmidt [17] constructed an example of a not vertex-decomposable polytopal 3-sphere with 10 vertices.

Corollary 12 *There are precisely* 7 *not vertex-decomposable* 3*-spheres with* 9 *vertices, which are all non-polytopal. Moreover, there are* 14468 *not vertex-decomposable* 3*-spheres with* 10 *vertices.*

Four of the seven examples with 9 vertices are neighborly with 27 tetrahedra, the other three have 25, 26, and 26 tetrahedra, respectively. The 25 tetrahedra of the smallest example are:

1234	1235	1246	1257	1268
1278	1345	1456	1567	1679
1689	1789	2348	2359	2378
2379	2468	2579	3458	3568
3569	3689	3789	4568	5679.

References

[1] J. W. Alexander. The combinatorial theory of complexes. *Ann. Math.* **31**, 292–320 (1930).

[2] A. Altshuler. Combinatorial 3-manifolds with few vertices. *J. Comb. Theory, Ser. A* **16**, 165–173 (1974).

[3] A. Altshuler. Neighborly 4-polytopes and neighborly combinatorial 3-manifolds with ten vertices. *Can. J. Math.* **29**, 400–420 (1977).

[4] A. Altshuler, J. Bokowski, and L. Steinberg. The classification of simplicial 3-spheres with nine vertices into polytopes and nonpolytopes. *Discrete Math.* **31**, 115–124 (1980).

[5] A. Altshuler and L. Steinberg. Neighborly 4-polytopes with 9 vertices. *J. Comb. Theory, Ser. A* **15**, 270–287 (1973).

[6] A. Altshuler and L. Steinberg. Neighborly combinatorial 3-manifolds with 9 vertices. *Discrete Math.* **8**, 113–137 (1974).

[7] A. Altshuler and L. Steinberg. An enumeration of combinatorial 3-manifolds with nine vertices. *Discrete Math.* **16**, 91–108 (1976).

[8] D. Barnette. The triangulations of the 3-sphere with up to 8 vertices. *J. Comb. Theory, Ser. A* **14**, 37–52 (1973).

[9] A. Björner. Topological methods. *Handbook of Combinatorics* (R. Graham, M. Grötschel, and L. Lovász, eds.), Chapter 34, 1819–1872. Elsevier, Amsterdam, 1995.

[10] A. Björner and F. H. Lutz. Simplicial manifolds, bistellar flips and a 16-vertex triangulation of the Poincaré homology 3-sphere. *Exp. Math.* **9**, 275–289 (2000).

[11] J. Bokowski and K. Garms. Altshuler's sphere M^{10}_{425} is not polytopal. *Eur. J. Comb.* **8**, 227–229 (1987).

[12] J. Bokowski and B. Sturmfels. Polytopal and nonpolytopal spheres. An algorithmic approach. *Isr. J. Math.* **57**, 257–271 (1987).

[13] P. Engel. The enumeration of four-dimensional polytopes. *Discrete Math.* **91**, 9–31 (1991).

[14] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.4. `http://www.gap-system.org`, 2004.

[15] B. Grünbaum and V. P. Sreedharan. An enumeration of simplicial 4-polytopes with 8 vertices. *J. Comb. Theory* **2**, 437–465 (1967).

[16] M. Hachimori and G. M. Ziegler. Decompositions of simplicial balls and spheres with knots consisting of few edges. *Math. Z.* **235**, 159–171 (2000).

[17] V. Klee and P. Kleinschmidt. The d-step conjecture and its relatives. *Math. Oper. Res.* **12**, 718–755 (1987).

[18] E. G. Köhler and F. H. Lutz. Triangulated Manifolds with Few Vertices: Vertex-Transitive Triangulations I. `arXiv:math.GT/0506520`, 2005, 74 pages.

[19] F. H. Lutz. Small examples of nonconstructible simplicial balls and spheres. *SIAM J. Discrete Math.* **18**, 103–109 (2004).

[20] F. H. Lutz. A vertex-minimal non-shellable simplicial 3-ball with 9 vertices and 18 facets. *Electronic Geometry Models* No. 2003.05.004 (2004). `http://www.eg-models.de/2003.05.004`.

[21] F. H. Lutz. Vertex-minimal not vertex-decomposable balls. *Electronic Geometry Models* No. 2003.06.001 (2004). `http://www.eg-models.de/2003.06.001`.

[22] F. H. Lutz. The Manifold Page, 1999–2006. `http://www.math.tu-berlin.de/diskregeom/stellar/`.

[23] F. H. Lutz. `BISTELLAR`, Version Nov/2003. `http://www.math.tu-berlin.de/diskregeom/stellar/BISTELLAR`, 2003.

[24] F. H. Lutz. Triangulated Manifolds with Few Vertices: Geometric 3-Manifolds. `arXiv:math.GT/0311116`, 2003, 48 pages.

[25] F. H. Lutz. Triangulated Manifolds with Few Vertices: Combinatorial Manifolds. `arXiv:math.CO/0506372`, 2005, 37 pages.

[26] F. H. Lutz. Enumeration and random realization of triangulated surfaces. `arXiv:math.CO/0506316`, 2005, 18 pages; to appear in *Discrete Differential Geometry* (A. I. Bobenko, J. M. Sullivan, P. Schröder, and G.M. Ziegler, eds.), Oberwolfach Seminars, Birkhäuser, Basel.

[27] M. H. A. Newman. On the foundations of combinatory analysis situs. I, II. *Proc. Royal Acad. Amsterdam* **29**, 611–626, 627–641 (1926).

[28] J. S. Provan and L. J. Billera. Decompositions of simplicial complexes related to diameters of convex polyhedra. *Math. Oper. Res.* **5**, 576–594 (1980).

[29] E. Swartz. From spheres to manifolds. Preprint, 2005, 42 pages.

[30] D. W. Walkup. The lower bound conjecture for 3- and 4-manifolds. *Acta Math.* **125**, 75–107 (1970).

[31] G. M. Ziegler. Shelling polyhedral 3-balls and 4-polytopes. *Discrete Comput. Geom.* **19**, 159–174 (1998).