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Nash Equilibria in Online Sequential Routing Games

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NASH EQUILIBRIA IN ONLINE SEQUENTIAL ROUTING GAMES

TOBIAS HARKS*

Abstract. In this paper, we study the efficiency of Nash equilibria for a sequence of routing games. In a routing game players route demand from source to destination in a network. Their strategy is to select routes in order to minimize their individual travel time. We assume that the games are played consecutively in time in an online fashion: by the time of playing game i, future games $i+1, \dots, n$ are not known, and, once players of game i are in equilibrium, their corresponding strategies remain fixed. The cost function is given by the total routing cost, when all games have been played. We analyze the efficiency of a sequence of Nash equilibria in terms of competitive analysis arising in the online optimization field. Our main results are summarized in the following: (i) for nonatomic players the online algorithm SeqNash that produces a sequence of Nash equilibria is $\frac{4n}{2+n}$-competitive for affine linear latency functions and $\frac{4n^2}{(1+n)^2}$-competitive for linear latency functions; (ii) for atomic players SeqNash is $\min\{\frac{2(3\mathcal{K}+1)n}{n\mathcal{K}+3n+3\mathcal{K}+1}, \frac{5\mathcal{K}+1}{\mathcal{K}+5}\}$-competitive for affine linear latency functions; (iii) a lower bound of $\frac{3n-2}{n}$ in both cases (iv) for general polynomial latency functions, we prove lower and upper bounds on the competitive ratio of SeqNash that grow exponentially in the degree of the considered polynomials for the nonatomic and atomic case. For $n = 1$, these results include the first known bounds on the price of anarchy for games with atomic players for general polynomial latency functions.

1. Introduction

In this paper, we introduce the concept of *sequential routing games*. In this concept, we assume a sequence of routing games $\sigma = 1, \dots, n$ that are played consecutively in time in an online fashion. By the time of playing game i, future games $i+1, \dots, n$ are not known. We further assume that once players of game i are in equilibrium, their corresponding strategies remain fixed, that is, the strategies are irrevocable. We analyze the efficiency of an online algorithm, called SeqNash that produces a flow consisting of the sequence of Nash equilibria. Our measure of efficiency is defined in terms of competitive analysis coming from the online optimization field. An online algorithm Alg is called *c-competitive* if its cost of Alg is never larger than c times the cost of an optimal offline solution. The optimal offline solution in our model is derived by minimizing the total routing cost when all games have been played. Note that for deriving the optimal offline solution, the sequence σ is known a priori.

Our work is motivated by the application of selfish routing to the source routing concept in telecommunication networks, see Qiu, Yang, Zhang, and Shenker [1] and Friedman [2] for an engineering perspective and Roughgarden [3] and Altman, Basar, Jimenez, and Shimkin [4] for a theoretical perspective on this topic. In the source routing model, sources are responsible for selecting paths to route data to the corresponding sink. The links in the network advertise their current status (price) that is based on the current congestion situation. If the link prices correspond to the expected delay on that link, minimum cost routing is a natural goal for time critical real-time applications. The main focus of this line of research is to quantify the efficiency loss of a Nash equilibrium compared to the system optimum. Here, one assumption is crucial: if the traffic matrix changes, all sources may possibly change their routes and form a new equilibrium.

From a practical point of view, the main drawback of source routing in the Internet is the communication overhead in *continuously* maintaining the state of all available routes. Furthermore, frequent rerouting attempts during data transmission may interfere with the widely used

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congestion control protocol TCP that controls the data rate, as reported by La, Walrand, and Anantharam in [5]. Thus, rerouting may result in severe performance degradation for time critical applications, such as Internet Telephony or video.

The purpose of this paper is to introduce a new model where sources starting at the same time select their routes *only* during connection setup phase. Once these flows are at equilibrium their routing decisions remain fixed. Thus, continuously gathering information about the network state is dispensable after this initial routing game.

We study in this paper the sequence $\sigma = 1, \dots, n$ of games, where players of game i choose strategies without taking future games $j = i + 1, \dots, n$ into account. It turns out, that a combination of the online optimization field with algorithmic game theory provides a fruitful way to analyze the efficiency of routing strategies in this new model. Surprisingly, the inefficiency of the sequence of Nash equilibria, where nonatomic and atomic players are allowed, can be bounded by a constant factor for a wide class of latency functions.

1.1. Related Work

In the last years there has been an exciting development in algorithmic game theory trying to quantify the efficiency loss of Nash equilibria (user equilibria) in non-cooperative games. The fact that there exists an efficiency loss of the user equilibrium compared to a system optimum is well known in the transportation literature, see Braess [6] and Dubey [7]. A first attempt to exactly quantify this so called "price of anarchy" is given by Papadimitriou and Koutsoupias [8] in the context of a load balancing game in communication networks. Roughgarden and Tardos [9] applied this approach to quantify the price of anarchy in nonatomic selfish routing games. In nonatomic games, a large number of players is assumed, each consuming an infinitesimal part of the resources. In particular, Roughgarden and Tardos [9] proved for a set of separable affine cost functions a bound of $\frac{4}{3}$ on the price of anarchy. A series of several other papers analyzed the price of anarchy for more general cost functions and model features; see for example Czumaj and Vöcking [10], Correa Schulz, and Stier-Moses [11, 12], Perakis [13], and Roughgarden [3].

For atomic routing games, that is, some players may control a significant part of the entire demand, Roughgarden and Tardos examined the price of anarchy in for an unsplittable variant [9]. Subsequently, Awerbuch, Azar, and Epstein [14] and Christodoulou and Koutsoupias [15] studied the price of anarchy for linear latency functions. Cominetti, Correa and Stier-Moses [16] provided new bounds on the price of anarchy for general atomic routing games that revised previous work of Roughgarden [17] and Correa, Schulz, and Stier-Moses [12]. Subsequently, Hayrapetyan, Tardos and, Wexler [18] improved these bounds for special network topologies.

In the online routing field, mainly call admission control problems have been considered. An overview article about these problems is given by Leonardi in [19]. In the paper by Awerbuch, Azar, and Plotkin [20], online routing algorithms are presented to maximize throughput under the assumption that routings are irrevocable. They, however, restrict the analysis to single path routing and present competitive bounds that depend on the number of nodes in the network. Our work is motivated by the paper by Harks, Heinz, and Pfetsch [21], where online multicommodity routing problems are considered. They present a greedy online algorithm for a different convex cost function that is $\frac{4K}{2+K}$ competitive, where K is the number of commodities. In their framework, only single demands are released consecutively.

1.2. Our Results

Our main result states that the online algorithm SeqNash that produces a flow consisting of a sequence of Nash equilibria is $\frac{4n}{2+n}$-competitive for affine linear latency functions and nonatomic routing. This result contains the bound on the price of anarchy of $\frac{4}{3}$ for affine linear latency functions of Roughgarden and Tardos [9] as a special case of our model, where $n = 1$. We present lower bounds for affine latency functions of $\frac{4}{3}$ and $\frac{3n-2}{n}$ for any deterministic online algorithm and SeqNash, respectively.

For games with atomic atomic players we show that SeqNash is $\min\{\frac{2(3\mathcal{K}+1)n}{n\mathcal{K}+3n+3\mathcal{K}+1}, \frac{5\mathcal{K}+1}{\mathcal{K}+5}\}$-competitive for affine linear latency functions. This result contains the bound on the price of anarchy of $\frac{3\mathcal{K}+1}{2\mathcal{K}+2}$ for atomic routing games with affine linear latency functions that has previously been established by Cominetti, Correa and Stier-Moses [16] as a special case of our model, where $n = 1$. For purely linear latency functions we present an improved upper bound of $\frac{1}{8}\,(2+\sqrt{2})\,(1+\sqrt{2})\,\sqrt{2}$ on the price of anarchy.

Finally, for polynomial latency functions, we present upper and lower bounds on the competitive ratio of SeqNash that grow exponentially in the maximum degree of the allowed polynomials. These results include improvements of known bounds of the price of anarchy for gemes with atomic players involving polynomial latency functions with degree $2 \leq d \leq 4$ and presents the first bounds for arbitrary degree d.

Among many works in the algorithmic game theory field, this is the first paper that combines techniques arising in the online optimization field with techniques coming from algorithmic game theory.

2. Online Sequential Routing Games

An instance of the *Online Sequential Routing Game* (OnlineSRG) consists of a directed network $D = (V, A)$ and nondecreasing continuous price or latency functions $\ell_a : \mathbb{R}_+ \to \mathbb{R}_+$ for each link $a \in A$. Furthermore, a sequence $\sigma = 1, \dots, n$ of routing games are given. We denote for each game i the set of different types of players by $[K_i] = \{(i,1), \dots, (i,n_i)\}$ with $|[K_i]| = \mathcal{K}_i$ denoting the number of players of game i. Let $[\mathcal{K}] = \bigcup_{i=1}^{n} [K_i]$ denote the union of the sets $[K_1], \dots, [K_n]$. The total number of players is given by $\mathcal{K} = \sum_{i=1}^{n} \mathcal{K}_i$. For each $ij \in [K_i]$, a flow of rate $d_{ij} > 0$ must be routed from the origin s_{ij} to the destination t_{ij}. We allow for nonatomic and atomic players. In contrast to nonatomic routing games where infinitely many agents are carrying the flow, in the atomic variant, each player controls the entire flow for his demand.

For ease of notation, we express in the following the strategy of player ij in terms of a routing assignment. A routing assignment, or *flow*, for player $ij \in [K_i]$ is a nonnegative vector $\boldsymbol{f}^{ij} \in \mathbb{R}_+^A$. This flow is *feasible* if for all $v \in V$

$$\sum_{a \in \delta^+(v)} f_a^{ij} - \sum_{a \in \delta^-(v)} f_a^{ij} = \gamma(v), \tag{1}$$

where $\delta^+(v)$ and $\delta^-(v)$ are the arcs leaving and entering v, respectively; furthermore, $\gamma(v) = d_{ij}$ if $v = s_{ij}$, $\gamma(v) = -d_{ij}$ if $v = t_{ij}$, and $\gamma(v) = 0$ otherwise.

Alternatively, one can consider a *path flow* for a player $ij \in [K_i]$. Let $\mathcal{P}_{ij}$ be the set of all paths from s_{ij} to t_{ij} in D. A path flow is a nonnegative vector $(f_P^{ij})_{P \in \mathcal{P}_{ij}}$. The corresponding flow on link $a \in A$ for player $ij \in [K_i]$ is then

$$f_a^{ij} := \sum_{P \ni a} f_P^{ij}.$$

We define $\mathcal{F}_i$ with $0 \leq i \leq n$ to be the set of vectors $(\boldsymbol{f}^1, \dots, \boldsymbol{f}^i)$ such that $\boldsymbol{f}^j$ is a feasible flow for games $j,\ j = 1, \dots, i$. The entire flow for a sequence of games $\sigma = (1, \dots, n)$ is denoted by $\boldsymbol{f} = (\boldsymbol{f}^1, \dots, \boldsymbol{f}^n)$. We define $\boldsymbol{f}_a = (f_a^{ij})_{ij \in [\mathcal{K}]}$ as the vector of flow values of the players ij on arc a.

We denote by f_a^i the aggregate flow of game i on link a, i.e.,

$$f_a^i := \sum_{ij \in [K_i]} f_a^{ij},$$

and define by

$$f_a := \sum_{i=1}^{n} f_a^i$$

the total flow on link a. The *current* cost of a feasible flow $\boldsymbol{f}$ on link $a \in A$ of game i is defined by

$$C_a^i(f_a^i; f_a^1, \dots, f_a^{i-1}) = \ell_a\Big(\sum_{j=1}^{i} f_a^j\Big) f_a^i \tag{2}$$

This expression can be obtained as the routing cost on arc a for a feasible flow for game i, *given* the flows $(\boldsymbol{f}^1, \dots, \boldsymbol{f}^{i-1})$ of previous games $1, \dots, i-1$. The current cost for game i is given by the sum of arc costs

$$C^i(\boldsymbol{f}^i; \boldsymbol{f}^1, \dots, \boldsymbol{f}^{i-1}) = \sum_{a \in A} C_a^i(f_a^i; f_a^1, \dots, f_a^{i-1}).$$

In the sequel of this paper we assume that $\ell_a(x)\, x$ is a convex function. Hence $C_a^i(\cdot)$ is also a convex function.

The individual current cost for player $ij \in [K_i]$ on arc a is given by:

$$C_a^{ij}(f_a^i; f_a^1, \dots, f_a^{i-1}) = \ell_a\Big(\sum_{j=1}^{i} f_a^j\Big) f_a^{ij} \tag{3}$$

The total individual current cost for player $ij \in [K_i]$ is given by:

$$C^{ij}(\boldsymbol{f}^i; \boldsymbol{f}^1, \dots, \boldsymbol{f}^{i-1}) = \sum_{a \in A} C_a^{ij}(f_a^i; f_a^1, \dots, f_a^{i-1}).$$

The aggregate cost of a flow on link $a \in A$ is defined by

$$C_a(f_a) = \ell_a(f_a)\, f_a. \tag{4}$$

The total cost of all sequentially played games is given by:

$$C(\boldsymbol{f}) = \sum_{a \in A} C_a(f_a) = \sum_{a \in A} \ell_a(f_a)\, f_a = \sum_{a \in A} \ell_a\left(\sum_{i=1}^{n} f_a^i\right)\left(\sum_{i=1}^{n} f_a^i\right). \tag{5}$$

This cost function reflects the congestion cost provided the entire sequence of games has been played. Note that players of game i are only aware of their current cost. If players of later games select overlapping strategies compared to previous players, the previous players may experience higher costs compared to their initial costs. For a sequence of games $\sigma = 1, \dots, n$, we investigate in this paper the online algorithm SeqNash, that consists of the sequence of Nash equilibria for the corresponding games $1, \dots, n$. We focus on the *efficiency* of SeqNash compared to the offline optimum Opt. For $n = 1$ our model reduces to the standard setting of a routing game with nonatomic or atomic players.

2.1. Characterizing Nash Equilibria for Nonatomic Players.

A strategy distribution or flow for game i is at Nash equilibrium when no player has an incentive to unilaterally change his strategy. We assume that players of game i decide on their strategies without taking future games $j = i+1, \dots, n$ into account. It is straight-forward to check that a Nash flow $\boldsymbol{f}^i$ for nonatomic players is the optimum of the following convex optimization problem, see for example Roughgarden and Tardos [9].

$$\begin{aligned}
\min \quad & \sum_{a \in A} \int_0^{f_a^i} \ell_a\Big(\sum_{k=1}^{i-1} f_a^k + z\Big)\, dz & & (6)\\
\text{s.t.} \quad & \sum_{a \in \delta^+(v)} f_a^{ij} - \sum_{a \in \delta^-(v)} f_a^{ij} = \gamma_{ij}(v) & & \forall\, v \in V,\ ij \in [K_i]\\
& f_a^{ij} \geq 0 & & \forall\, a \in A,\ ij \in [K_i],
\end{aligned}$$

where $\gamma_{ij}(v)$ is defined as in (1). The following conditions are necessary and sufficient to characterize a Nash equilibrium for game i.

Lemma 2.1. *A feasible flow $\boldsymbol{f}^i$ for the nonatomic game i is at Nash equilibrium if and only if it satisfies:*

$$\sum_{a\in A} \ell_a\big(\sum_{k=1}^{i} f_a^k\big)\,(f_a^i - x_a^i) \le 0 \text{ for all feasible flows } \boldsymbol{x}^i \text{ for game } i. \tag{7}$$

The proof is based on the first order optimality conditions and the convexity of $C^i(\cdot)$, see Dafermos and Sparrow [22].

Definition 2.2. *For a given sequence of games σ and a flow $\boldsymbol{f}$ that is produced by* SeqNash, *we define*

$$V^i(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i) := \sum_{a\in A} \ell_a\big(\sum_{k=1}^{i} f_a^k\big)\,(x_a^i - f_a^i) \tag{8}$$

$$V(\boldsymbol{f},\boldsymbol{x},n) := \sum_{i=1}^{n} V_i(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i) \tag{9}$$

where $\boldsymbol{x}^1,\dots,\boldsymbol{x}^n \in \mathcal{F}_n$ is any feasible flow.

Lemma 2.3. *A feasible flow $\boldsymbol{f}$ for a sequence of games σ that is produced by* SeqNash *satisfies:*

$$V(\boldsymbol{f},\boldsymbol{x},n) \ge 0, \quad \text{for all feasible flows } \boldsymbol{x} \text{ for } \sigma.$$

Furthermore,

$$V(\boldsymbol{f},\boldsymbol{x},n) = \sum_{a\in A} V_a(\boldsymbol{f}_a,\boldsymbol{x}_a,n),$$

where $V_a(\boldsymbol{f}_a,\boldsymbol{x}_a,n)$ is defined as

$$V_a(\boldsymbol{f}_a,\boldsymbol{x}_a,n) := \sum_{i=1}^{n} \ell_a\big(\sum_{k=1}^{i} f_a^k\big)\,(x_a^i - f_a^i).$$

Proof. From Lemma 2.1 we know that $V^i(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i)$ is nonnegative for all $i = 1,\dots,n$. Summing over i proves the first claim. The second claim follows by changing the summation order. □

2.2. Characterizing Sequential Nash Equilibria for Atomic Players

In routing games with atomic players, some players may control a significant part of the entire demand. In the follwoing, we characterize the strategy of an atomic player. A strategy distribution or flow for game i is at Nash equilibrium when no player has an incentive to unilaterally change his strategy. It is straightforward to see that a best reply strategy for player ij of game i is to solve the following convex optimization problem.

$$\begin{aligned}
\min \quad & \sum_{a\in A} \ell_a\big(\sum_{j=1}^{i} f_a^j\big) f_a^{ij} & & \\
\text{s.t.} \quad & \sum_{a\in\delta^+(v)} f_a^{ij} - \sum_{a\in\delta^-(v)} f_a^{ij} = \gamma(v) & & \forall\, v\in V,\ ij\in[K_i] \\
& f_a^{ij} \ge 0 & & \forall\, a\in A,\ ij\in[K_i],
\end{aligned} \tag{10}$$

where $\gamma(v)$ is defined as in (1). The following conditions are necessary and sufficient to characterize a Nash equilibrium for game i.

Lemma 2.4. *A feasible flow $\boldsymbol{f}^{ij}$ for player $ij \in [K_i]$ of the game i is at Nash equilibrium if and only if it satisfies:*

$$\sum_{a\in A}\Big(\ell_a\big(\sum_{k=1}^{i} f_a^k\big) + \ell_a'\big(\sum_{k=1}^{i} f_a^k\big) f_a^{ij}\Big)\,(f_a^{ij} - x_a^{ij}) \le 0 \tag{11}$$
$$\text{for all feasible flows } \boldsymbol{x}^{ij} \text{ for game } i.$$

Definition 2.5. *For a given sequence of games σ and a flow $\boldsymbol{f}$ that is produced by* SeqNash, *we define*

$$\begin{aligned}
V^{ij}(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i) &:= \sum_{a\in A}\Big(\ell_a\big(\sum_{k=1}^{i} f_a^k\big) + \ell_a'\big(\sum_{k=1}^{i} f_a^k\big) f_a^{ij}\Big)\,(x_a^{ij} - f_a^{ij})\\
V^{i}(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i,\mathcal{K}_i) &:= \sum_{ij\in[K_i]} V^{ij}(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i)\\
V(\boldsymbol{f},\boldsymbol{x},\mathcal{K},n) &:= \sum_{i=1}^{n} V^{i}(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i,\mathcal{K}_i)
\end{aligned} \tag{12}$$

where $\boldsymbol{x}^1,\dots,\boldsymbol{x}^n \in \mathcal{F}_n$ is any feasible flow.

Lemma 2.6. *A feasible flow $\boldsymbol{f}$ for a sequence of games σ that is produced by* SeqNash *satisfies:*

$$V(\boldsymbol{f},\boldsymbol{x},\mathcal{K},n) \ge 0, \quad \text{for all feasible flows } \boldsymbol{x} \text{ for } \sigma. \tag{13}$$

Furthermore,

$$V(\boldsymbol{f},\boldsymbol{x},\mathcal{K},n) = \sum_{a\in A} V_a(\boldsymbol{f}_a,\boldsymbol{x}_a,\mathcal{K},n),$$

where $V_a(\boldsymbol{f}_a,\boldsymbol{x}_a,\mathcal{K},n)$ is defined as

$$V_a(\boldsymbol{f}_a,\boldsymbol{x}_a,\mathcal{K},n) := \sum_{i=1}^{n}\sum_{ij\in[K_i]}\Big(\ell_a\big(\sum_{k=1}^{i} f_a^k\big) + \ell_a'\big(\sum_{k=1}^{i} f_a^k\big) f_a^{ij}\Big)\,(x_a^{ij} - f_a^{ij}).$$

Proof. From Lemma 2.4 we know that $V^{ij}(\boldsymbol{f}^1,\dots,\boldsymbol{f}^i,\boldsymbol{x}^i)$ is nonnegative for all $ij \in [\mathcal{K}_i]$ and $i = 1,\dots,n$.. Summing over $ij \in [\mathcal{K}_i]$ and $i = 1,\dots,n$ proves the first claim. The second claim follows by changing the summation order. □

For a given sequence σ of games, we denote in the following the deterministic online algorithm that consists of the sequence of Nash equilibria by SeqNash.

2.3. Total Offline Optimum

Finally, the *total offline optimum* is characterized by:

$$\begin{aligned}
\min\quad & C(\boldsymbol{f}) & \\
\text{s.t.}\quad & \sum_{a\in\delta^+(v)} f_a^{ij} - \sum_{a\in\delta^-(v)} f_a^{ij} = \gamma(v) & \forall\, v\in V,\ ij\in[K_i],\ i\in[n]\\
& f_a^{ij} \ge 0 & \forall\, a\in A,\ ij\in[K_i],\ i\in[n]
\end{aligned} \tag{14}$$

where $\gamma(v)$ is defined as in (1). The optimal offline solution for a given sequence σ for the above problem is called Opt. We denote its value for σ by $\text{Opt}(\sigma) = C(\boldsymbol{f})$.

3. Competitive Analysis for Nonatomic Routing

We can classify the presented algorithms SeqNash and Opt as follows: SeqNash is a *deterministic online* algorithms for OnlineSRG; Opt is an *offline* solution for OnlineSRG since here, the entire game sequence is known in advance.

For a solution $\boldsymbol{f}$ produced by an online algorithm Alg for σ, we denote by $\text{Alg}(\sigma) = C(\boldsymbol{f})$ its cost. The online algorithm Alg is called *c-competitive* if the cost of Alg is never larger than c times the cost of an optimal offline solution. The *competitive ratio* of Alg is the infimum over all $c \geq 1$ such that Alg is c-competitive, see Borodin and El-Yaniv [23].

In the following, we use a simple technique to derive upper bounds on the competitive ratio for SeqNash. The idea is to add the variational inequality given in Lemma 2.1 to the cost of the flow $\boldsymbol{f}$ produced by SeqNash. We define for every $a \in A$, for any nonnegative vectors $\boldsymbol{f}_a, \boldsymbol{x}_a \in \mathbb{R}_+^{\mathcal{K}}$ the following values (we assume by convention $0/0 = 0$):

$$\omega(\ell_a; n, \lambda) := \sup_{\boldsymbol{f}_a, \boldsymbol{x}_a \geq 0} \frac{C_a(f_a) - \lambda\, C_a(x_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a, n)}{C_a(f_a)}. \tag{15}$$

Throughout the paper, we define the constraint $\boldsymbol{f}_a \geq 0$ as

$$f_a^{ij} \geq 0, \text{ for all } ij \in [\mathcal{K}] \text{ with } \sum_{ij \in \mathcal{K}} f_a^{ij} = f_a.$$

For a given class $\mathcal{C}$ of nondecreasing latency functions and a nonnegative real number $\lambda \geq 0$, we further define

$$\omega_n(\mathcal{C}; \lambda) := \sup_{\ell_a \in \mathcal{C}} \omega(\ell_a; n, \lambda).$$

Theorem 3.1. *Consider a sequence of n games and separable latency functions drawn from $\mathcal{C}$. If $1 - \omega_n(\mathcal{C}; \lambda) > 0$ holds, then* SeqNash *is*

$$\inf_{\lambda \geq 0} \left[\lambda\, (1 - \omega_n(\mathcal{C}; \lambda)^{-1})\right] - \textit{competitive}$$

for the nonatomic OnlineSRG.

Proof. Let $\boldsymbol{f}$ be the flow generated by SeqNash, and $\boldsymbol{x}$ be any feasible flow for a given sequence of games $\sigma = 1, \ldots, n$.

$$\begin{aligned} C(\boldsymbol{f}) &\leq \sum_{a \in A} \left[C_a(f_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a, n)\right] && (16) \\ &= \sum_{a \in A} \left[C_a(f_a) + \lambda\, C_a(x_a) - \lambda\, C_a(x_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a, n)\right] \\ &\leq \lambda\, C(\boldsymbol{x}) + \omega_n(\mathcal{C}; \lambda)\, C(\boldsymbol{f}). && (17) \end{aligned}$$

Here, (16) follows from the variational inequality stated in Lemma 2.1. The last inequality (17) follows from the definition of $\omega_n(\mathcal{C})$. Taking $\boldsymbol{x}$ as the optimal offline solution yields the claim. □

Using the notation:

$$\vartheta_a^n(\ell_a, \boldsymbol{f}_a) := \ell_a(f_a) f_a - \sum_{i=1}^{n} \ell_a\Big(\sum_{k=1}^{i} f_a^k\Big)\, f_a^i.$$

we can simplify the value $\omega_n(\mathcal{C}; \lambda)$

Lemma 3.2. *The value $\omega(\ell_a; n, \lambda)$ is at most*

$$\sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big) x_a + \vartheta_a^n(\ell_a, \boldsymbol{f}_a)}{\ell_a(f_a) f_a}. \tag{18}$$

Proof. First note that

$$C_a(f_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a) = \vartheta_a^n(\ell_a, \boldsymbol{f}_a) + \sum_{i=1}^{n} \ell_a\big(\sum_{k=1}^{i} f_a^k\big) x_a^i]$$
$$\leq \vartheta_a^n(\ell_a, \boldsymbol{f}_a) + \ell_a(f_a)\, x_a,$$

where the last inequality is valid since latency functions are nondecreasing. Then, using

$$\ell_a(f_a)\, x_a - \lambda\, C_a(x_a) = \big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big) x_a,$$

yields the claim. □

Note that a similar value $\gamma(\mathcal{C})$ without the term $\ell_a(f_a)f_a - \sum_{i=1}^{n} \ell_a(\sum_{k=1}^{i} f_a^k)\, f_a^i$

and the parameter $\lambda \geq 0$ was first defined in Correa, Schulz, and Stier-Moses [11] and also, similarly, by Roughgarden in [9] with the relation $\alpha(\mathcal{C}) = \big(1 - \gamma(\mathcal{C})\big)^{-1}$.

3.1. Affine Linear Latency Functions

In the following, we bound the value $\omega_n(\mathcal{C}, 1)$ for affine linear latency functions. We start with some useful prerequisites.

Lemma 3.3. *For parameters $\kappa_1, \kappa_2 > 0$ and any numbers $x, y \geq 0$ the following inequality is valid:*

$$xy \leq \frac{\kappa_1}{2\,\kappa_2} x^2 + \frac{\kappa_2}{2\,\kappa_1} y^2. \tag{19}$$

The following equation is useful for proving the next Lemma.

$$\sum_{i=1}^{n} \sum_{j=1}^{i} f_a^j\, f_a^i = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_a^j\, f_a^i + \tfrac{1}{2} \sum_{i=1}^{n} (f_a^i)^2. \tag{20}$$

Lemma 3.4. *For affine functions $\ell_a(z) = q_a z + r_a,\ q_a \geq 0, r_a \geq 0,$ we have $\omega_n(\mathcal{C}; 1) \leq \frac{3n-2}{4n}$.*

Proof.

$$\omega(\ell_a; n, 1) = \sup_{x_a, \boldsymbol{f}_a \geq 0} \frac{q_a\,(f_a - x_a)\, x_a + q_a\, \frac{1}{2} f_a^2 - q_a\, \frac{1}{2} \sum_{k=1}^{n} (f_a^k)^2}{q_a\, f_a^2 + r_a\, f_a} \tag{21}$$

$$\leq \sup_{x_a, \boldsymbol{f}_a \geq 0} \frac{(f_a - x_a)\, x_a + \frac{1}{2} f_a^2 - \frac{1}{2} \sum_{k=1}^{n} (f_a^k)^2}{f_a^2} \tag{22}$$

$$\leq \sup_{x_a, f_a \geq 0} \frac{(f_a - x_a)\, x_a + \frac{n-1}{2n} f_a^2}{f_a^2} \tag{23}$$

$$\leq \tfrac{3n-2}{4n}, \tag{24}$$

where (21) follows from (20) and (22) is valid since $r_a \geq 0$. Inequality (24) follows from Lemma 3.3, where we set $x = f_a, y = x_a, \kappa_1 = 1$, and $\kappa_2 = 2$, and (23) follows from Cauchy-Schwarz inequality. □

Corollary 3.5. *If the latency functions of the nonatomic* OnlineSRG *are affine, the online algorithm* SeqNash *is $\frac{4n}{n+2}$-competitive, where n is the number of games.*

Proof. Replacing $\omega_n(\mathcal{C}; 1)$ with $\frac{3n-2}{4n}$ and applying Theorem 3.1 yields the desired result. □

For $n = 1$, we obtain the bound of $\frac{4}{3}$ for nonatomic routing games involving affine linear latency functions that originates in Roughgarden and Tardos [9].

Now, we analyze the case of purely linear latency functions $\ell_a(z) = q_a\, z,\ q_a \geq 0$.

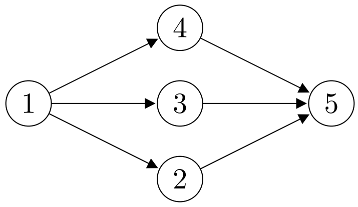

FIGURE 1. Graph construction for the proofs of Proposition 1

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Lemma 3.6. *For linear functions* $\ell_a(z) = q_a z,\ q_a \geq 0,$ *we have*

$$\omega_n(\mathcal{C};\lambda) \leq \frac{n + 2\,\lambda\, n - 2\,\lambda}{4\,\lambda\, n}.$$

Proof. The proof proceeds along the line of the proof of the preceeding lemma.

$$\begin{aligned}\omega(\ell_a; n, \lambda) &\leq \sup_{x_a, \boldsymbol{f}_a \geq 0} \frac{(f_a - \lambda\, x_a)\, x_a + \frac{1}{2} f_a^2 - \frac{1}{2}\sum\limits_{k=1}^{n} (f_a^k)^2}{f_a^2} \\ &\leq \sup_{x_a, f_a \geq 0} \frac{(f_a - \lambda\, x_a)\, x_a + \frac{n-1}{2n}\, f_a^2}{f_a^2} \\ &\leq \frac{1}{4\,\lambda} + \frac{n-1}{2n}.\end{aligned}$$

The last inequality follows from Lemma 3.3, where we set $x = x_a, y = f_a, \kappa_1 = \lambda$, and $\kappa_2 = \frac{1}{2}$. □

Corollary 3.7. *If the latency functions of the nonatomic* ONLINESRG *are linear, the online algorithm* SEQNASH *is* $\frac{4n^2}{(n+1)^2}$*-competitive, where* n *is the number of games.*

Proof. Replacing $\omega_n(\mathcal{C};\lambda)$ with $\frac{n+2\,\lambda\, n-2\,\lambda}{4\,\lambda\, n}$ and applying Theorem 3.1 yields

$$C(\boldsymbol{f}) \leq \frac{4\,\lambda^2\, n}{2\,\lambda\, n - n + 2\,\lambda}\, C(\boldsymbol{x}).$$

Setting $\lambda := \frac{n}{n+1}$ yields

$$C(\boldsymbol{f}) \leq \frac{4\, n^2}{(n+1)^2}\, C(\boldsymbol{x}).$$

□

Note 1. *The value* $\lambda = \frac{n}{n+1}$ *solves the following minimization problem with respect to* λ*:*

$$\min_{\lambda \geq 0} \frac{4\,\lambda^2\, n}{2\,\lambda\, n - n + 2\,\lambda}.$$

3.1.1. *Lower Bounds.* We start with a result that holds for *any* deterministic online algorithm.

Proposition 1. *In case of linear latency functions no deterministic online algorithm for* ONLINESRG *is c-competitive for any* $c < \frac{4}{3}$.

Proof. Consider the network displayed in Figure 1. Each arc a leaving from node 1 has the same latency function $\ell_a(z) = 3\,z$. All the other (those leading into node 5) have the latency function $\ell_a(z) = 0$. Let ALG be an arbitrary deterministic online algorithm. We first present ALG commodity 1 with demand 1 that has to be routed from $s_1 = 1$ to $t_1 = 5$.

Assume the algorithm behaves like the nonatomic SEQNASH. This means that the demand gets evenly divided into three pieces: one third is routed over path $P_1 = (1, 2, 5)$, another over path $P_2 = (1, 3, 5)$, and the later over path $P_3 = (1, 4, 5)$. In this case we reveal commodity 2

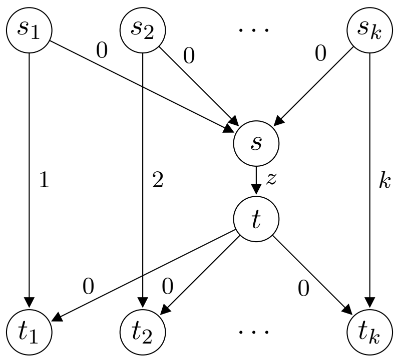

FIGURE 2. Graph construction for the proof of Theorem 3.8.

with demand 1 between 1 and 2. For this commodity there exists a unique path. Therefore, ALG yields for this sequence σ the cost:

$$\text{ALG}(\sigma) = \text{SEQNASH}(\sigma) = 2 \cdot 3 \cdot \tfrac{1}{3} \cdot \tfrac{1}{3} + 3 \cdot \left(\tfrac{1}{3} + 1\right)^2 = 6.$$

An optimal offline solution is to route half of commodity 1 over path P_2 and the other half over path P_3 and commodity 2 along its unique path. Therefore,

$$\text{OPT}(\sigma) = 2 \cdot 3 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + 3 \cdot 1 \cdot 1 = \tfrac{9}{2}.$$

This leads to

$$\frac{\text{ALG}(\sigma)}{\text{OPT}(\sigma)} = \frac{4}{3}.$$

If ALG does not behave like SEQNASH for the first commodity, ALG has to route more than one third of the demand over path P_1, path P_2, or path P_3. If it is path P_1, then we present commodity 2 as above. If its path P_2, then we reveal a commodity 2 with demand 1 between 1 and 3. Otherwise, we present a commodity 2 with demand 1 between 1 and 4. Let α be the demand greater than one third. In all three cases the cost of ALG for the sequence σ is

$$\text{ALG}(\sigma) \geq 2 \cdot 3 \cdot \left(\tfrac{(1-\alpha)}{2}\right)^2 + 3 \cdot \left(\alpha + 1\right)^2 > 6.$$

since $\alpha > \frac{1}{3}$. The optimal cost stays the same as above. Hence,

$$\frac{\text{ALG}(\sigma)}{\text{OPT}(\sigma)} > \frac{4}{3}.$$

□

For SEQNASH we can further lift the lower bound.

Theorem 3.8. *In case of affinely linear latency functions, the online algorithm* SEQNASH *for* ONLINESRG *has a competitive ratio greater than or equal to* $\frac{3n-2}{n}$*, where* n *is the number of games.*

Proof. We consider the network presented in Figure 2 with the latency functions: $\ell_{(s_i,s)}(z) = 0$, $\ell_{(t,t_i)}(z) = 0$, $\ell_{(s_i,t_i)}(z) = i$, $i = 1, \ldots, k$, and $\ell_{(s,t)}(z) = z$. We consecutively release a sequence of games $(1, \ldots, k)$, where in each game j, there is a single player type $j1$. The demand of player type $j1$ is 1 that has to be routed from s_i to t_i, for $i = 1, \ldots, k$. Due to the choice of the affine terms i, SEQNASH routes for every game the corresponding demand over the arc from s to t. Then we release the $(k+1)$-th game with demand d from s to t. Thus, the total cost for the sequence $\sigma = (1, \ldots, k+1)$ for SEQNASH with the new cost function is given by:

$$\text{SEQNASH}(\sigma) = (k+d)^2.$$

The optimal offline algorithm $\textsc{Opt}$ routes the demands of the first k games along the direct arcs from s_i to t_i incurring cost of:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}.$$

The last demand in game $k+1$ is routed from s to t with cost d^2. The total cost for the sequence $\sigma = (1,\dots,k+1)$ for $\textsc{Opt}$ is given by:

$$\textsc{Opt}(\sigma) = \frac{k(k+1)}{2} + d^2.$$

Replacing $k = n-1$ and setting $d = \frac{n}{2}$ yields

$$\frac{\textsc{SeqNash}(\sigma)}{\textsc{Opt}(\sigma)} = \frac{2(k+d)^2}{k(k+1)+2d^2} = \frac{3n-2}{n}, \tag{25}$$

which proves the theorem. □

Note 2. *For $n = 2$, the upper bound given in Corollary 3.5 is tight.*

Corollary 3.9. *For linear latency functions, the online algorithm* $\textsc{SeqNash}$ *for* $\textsc{OnlineSRG}$ *has a competitive ratio greater than or equal to* $\frac{33+5\sqrt{33}}{33+\sqrt{33}}$.

Proof. We consider the network presented in Figure 2 with modified latency functions: $\ell_{(s_i,s)}(z) = 0$, $\ell_{(t,t_i)}(z) = 0$, $\ell_{(s_i,t_i)}(z) = i\,z$, $i = 1,\dots,k$, and $\ell_{(s,t)}(z) = z$. We consecutively release a sequence of games $(1,\dots,k)$, where in each game j, there is a single player type $j1$. The demand of player type $j1$ is 2 that has to be routed from s_i to t_i, for $i = 1,\dots,k$. Due to the choice of the linear terms $i\,z$, $\textsc{SeqNash}$ routes for every game the one unit of the demand over the arc from s to t and the other onit along the direct arc from s_i to t_i. To see this, consider the j-th game. Let the flow of player $j1$ along the middle arc be denoted by x. Then, using the characterization of a Nash flow given in (10), the nonatomic player $j1$ sends flow x^* along the middle arc according to the solution of the following problem

$$\min_{0\le x\le 2} \frac{1}{2}\,j\,x^2 + (j-1)\,x + \frac{1}{2}\,j\,(2-x)^2.$$

The solution to this concave program is given by $x^* = 1$, independently of j.

Then, we release the $(k+1)$-th game with demand d from s to t. Thus, the total cost for the sequence $\sigma = (1,\dots,k+1)$ for $\textsc{SeqNash}$ is given by:

$$\textsc{SeqNash}(\sigma) = \sum_{i=1}^{k} i + (k+d)^2 = \frac{k\,(k+1)}{2} + (k+d)^2.$$

The optimal offline algorithm $\textsc{Opt}$ routes the demands of the first k games along the direct arcs from s_i to t_i incurring cost of:

$$\sum_{i=1}^{k} (i\cdot 2)\cdot 2 = 2\,k\,(k+1).$$

The last demand in game $k+1$ is routed from s to t with cost d^2. The total cost for the sequence $\sigma = (1,\dots,k+1)$ for $\textsc{Opt}$ is given by:

$$\textsc{Opt}(\sigma) = 2\,k\,(k+1) + d^2.$$

Replacing $k = n-1$ and setting $d = \frac{1}{4}\,n + \frac{1}{2} + \frac{1}{4}\sqrt{33\,n^2 - 28\,n + 4}$ yields

$$\frac{\textsc{SeqNash}(\sigma)}{\textsc{Opt}(\sigma)} \ge \lim_{n\to\infty} Z(n) = \frac{33 + 5\sqrt{33}}{33+\sqrt{33}} \approx 1.59,$$

where we define

$$Z(n) := \frac{33\,n^2 - 28\,n + 5\,n\sqrt{33\,n^2 - 28\,n + 4} + 4 - 2\sqrt{33\,n^2 - 28\,n + 4}}{33\,n^2 - 28\,n + n\sqrt{33\,n^2 - 28\,n + 4} + 4 + 2\sqrt{33\,n^2 - 28\,n + 4}}.$$

This proves the claim. □

Note 3. *The parameter d in the previous proof is the optimal solution to the following maximization problem with optimal value $Z(k+1)$:*

$$\max_{d\geq 1} \frac{k\,(k+1)+2\,(k+d)^2}{4\,k\,(k+1)+2\,d^2} = Z(k+1).$$

The table below summarizes the main results for (affine) linear latency functions.

TABLE 1. Competitive Ratio for the online algorithm SEQNASH for affine linear latency functions $\ell_a(x) = q_a\,x + r_a$, $q_a \geq 0$, $r_a \geq 0$. The first row shows known results for nonatomic routing games. The $\frac{4}{3}$ result is due to Roughgarden and Tardos [9].

# games		$\ell_a(0)=0$		$\ell_a(0)$ arbitrary, $\lambda=1$	
	λ	UB	LB	UB	LB
1	1	1	1	$\frac{4}{3}$	$\frac{4}{3}$
2	$\frac{2}{3}$	$1\,\frac{7}{9}$	$\frac{5+2\sqrt{5}}{5+\sqrt{5}}$	2	2
3	$\frac{3}{4}$	$2\,\frac{1}{4}$	$\frac{217+13\sqrt{217}}{217+5\sqrt{217}}$	$2\,\frac{2}{5}$	$2\,\frac{1}{3}$
	.	.	.	.	
	.	.	.		
n	$\frac{n}{n+1}$	$\frac{4\,n^2}{(n+1)^2}$	$Z(n)$	$\frac{4\,n}{n+2}$	$\frac{3n}{n-2}$
∞	1	4	$\frac{33+5\sqrt{33}}{33+\sqrt{33}}$	4	3

3.2. POLYNOMIAL LATENCY FUNCTIONS

In this section, we investigate the case, where we allow for general polynomial latency functions $\ell_a(z) = \sum_{i=0}^{d} a_i\, z^i$ with nonnegative coefficients a_i. We start with a useful observation.

Lemma 3.10. *For polynomial latency functions $\ell_a(z) = \sum_{i=0}^{d} a_i\, z^i$ with nonegative coefficients $a_i \geq 0, i = 0,\dots,d$ we can bound $\sup_{\boldsymbol{f}_a\geq 0} \vartheta_a^n(\ell_a, f_a)$ as follows:*

$$\sup_{\boldsymbol{f}_a\geq 0} \vartheta_a^n(\ell_a, f_a) \leq \sup_{\boldsymbol{f}_a\geq 0} \vartheta_a^\infty(\ell_a, f_a) \leq \frac{d}{d+1}\,\ell_a(f_a)\,f_a,$$

where $\vartheta_a^\infty(\ell_a, f_a) := \lim_{n\to\infty} \vartheta_a^n(\ell_a, f_a)$.

Proof. Recall the definition of $\vartheta_a^n(\ell_a, f_a)$:

$$\vartheta_a^n(\ell_a, f_a) := \ell_a(f_a)\,f_a - \sum_{i=1}^{n} \ell_a(\sum_{k=1}^{i} f_a^k)\, f_a^i.$$

Since polynomials are increasing functions, the following inequalities hold

$$\inf_{\boldsymbol{f}_a\geq 0} \Big[\sum_{i=1}^{n} \ell_a(\sum_{k=1}^{i} f_a^k)\, f_a^i\Big] \geq \inf_{\boldsymbol{f}_a\geq 0} \Big[\sum_{i=1}^{\infty} \ell_a(\sum_{k=1}^{i} f_a^k)\, f_a^i\Big] \geq \int_0^{f_a} l(z)\,dz.$$

Hence, we have

$$\sup_{\boldsymbol{f}_a\geq 0} \vartheta_a^n(\ell_a, f_a) \leq \sup_{\boldsymbol{f}_a\geq 0} \vartheta_a^\infty(\ell_a, f_a) \leq \ell_a(f_a)\,f_a - \int_0^{f_a} l(z)\,dz.$$

Let $\ell_a(z) = \sum_{i=0}^{d} a_i\, z^i$ be a polynomial of degree $d \geq 1$. Then, it follows that

$$\begin{aligned}
\ell_a(f_a)\, f_a - \int_0^{f_a} l(z)\, dz &= \sum_{i=0}^{d} a_i\,(f_a)^{i+1} - \sum_{i=0}^{d} \big(\frac{1}{i+1}\big)\, a_i\,(f_a)^{i+1} \\
&= \sum_{i=0}^{d} \big(\frac{i}{i+1}\big)\, a_i\,(f_a)^{i+1} \leq \frac{d}{d+1} \sum_{i=0}^{d} a_i\,(f_a)^{i+1} \\
&= \frac{d}{d+1}\, \ell_a(f_a)\, f_a.
\end{aligned}$$

□

Using the above lemma, we bound in the following the competitive ratio for SeqNash for quadratic, cubic, and degree 4 polynomials.

Proposition 2. *If the latency functions of the nonatomic* OnlineSRG *are polynomials of degree at most $d \geq 1$, then, the online algorithm* SeqNash *is*

$$\inf_{\lambda \geq 1} \left(\lambda \left(1 - \max_{0 \leq \mu \leq 1} \left[\mu - \lambda\, \mu^{d+1} \right] - \frac{d}{d+1} \right)^{-1} \right) - \textit{competitive}.$$

Proof. By Lemma 18, we have

$$\begin{aligned}
\omega(\ell_a, n; \lambda) &\leq \sup_{\boldsymbol{f}_a, x_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)\, x_a + \vartheta_a^n(\ell_a, \boldsymbol{f}_a, \boldsymbol{x}_a)}{\ell_a(f_a)\, f_a} \\
&\leq \sup_{f_a, x_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)\, x_a + \frac{d}{d+1}\, \ell_a(f_a)\, f_a}{\ell_a(f_a)\, f_a} \qquad (26) \\
&= \sup_{f_a, x_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)\, x_a}{\ell_a(f_a)\, f_a} + \frac{d}{d+1},
\end{aligned}$$

where (26) follows from Lemma 3.10.

We assume $\lambda \geq 1$, which implies that $x_a \leq f_a$. Defining $\mu := \frac{x_a}{f_a}$ (we assume $0/0 = 0$) we have to solve

$$\max_{0 \leq \mu \leq 1} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(\mu\, f_a)\big)\, \mu\, f_a}{\ell_a(f_a)\, f_a}$$

to bound $\omega(\ell_a, n; \lambda)$ from above. Without loss of generality, we can reduce the analysis to monomial latency functions $\ell_a(x) = a_d\, x^d$ of degree at most d. Otherwise, we can subdivide each arc in several arcs with monomial latency functions for every arc. From now on, we only consider the highest degree monomial $\ell_a(x) = a_d\, x^d$, since the value $\omega(\ell_a, n; \lambda)$ is smaller for lower degree polynomials. Thus, we have to solve:

$$\max_{0 \leq \mu \leq 1} \frac{\big(a_d\, f_a^d - \lambda\, a_d\, \mu^d\, f_a^d)\big)\, \mu\, f_a}{a_d\, f_a^{d+1}} = \max_{0 \leq \mu \leq 1} \mu - \lambda\, \mu^{d+1}. \qquad (27)$$

Applying Theorem 3.1 with

$$\omega_n(\mathcal{C}; \lambda) \leq \max_{0 \leq \mu \leq 1} \left[\mu - \lambda\, \mu^{d+1} \right] + \frac{d}{d+1}$$

proves the proposition. □

By optimizing over $\lambda \geq 1$ we get the following bound for polynomial latency functions up to degree two.

Corollary 3.11. *If the latency functions of the nonatomic* OnlineSRG *are polynomials of degree at most $d \leq 2$, then, the online algorithm* SeqNash *is* 19.6*-competitive.*

For general polynomials of degree d, we can prove an exponential bound in the maximum degree on the competitive ratio of SeqNash.

Proposition 3. *For polynnomial latency functions* $\ell_a(z) = \sum_{i=0}^{d} a_i\, z^i$ *with* $a_i \geq 0$ *and* $\lambda := (d+1)^{(d-1)} \geq 1$ *the value* $\omega_n(\mathcal{C};\lambda)$ *is at most* $\frac{d^2+2\,d}{(d+1)^2}$.

Proof. We start with equation (27) given in the proof from Proposition 2.

$$\omega_n(\mathcal{C};\lambda) \leq \max_{0\leq\mu\leq 1} \mu - \lambda\,\mu^{d+1} = \max_{0\leq\mu\leq 1} \mu - (d+1)^{(d-1)}\,\mu^{d+1}.$$

The unique solution is given by $\mu^* = \frac{1}{d+1}$. Evaluating the objective proves the claim:

$$\omega(\ell_a, n;\lambda) \leq \frac{1}{d+1} - (d+1)^{(d-1)}\,(\frac{1}{d+1})^{d+1} + \frac{d}{d+1} = \frac{d^2+2\,d}{(d+1)^2}.$$

□

With this lemma we can prove a constant factor bound on the competitive ratio that depends on the degree d of the considered polynomials.

Theorem 3.12. *If the latency functions of the nonatomic* OnlineSRG *are polynomials with maximum degree d, the online algorithm* SeqNash *is* $(d+1)^{d+1}$ *-competitive.*

Proof. Let the flow $\boldsymbol{f}$ be produced by the online algorithm SeqNash and let $\boldsymbol{x}$ be an arbitrary feasible flow for OnlineSRG. Then, applying Theorem 3.1 yields

$$C(\boldsymbol{f}) \leq \frac{(d+1)^{d-1}}{\big(1-\frac{d^2+2\,d}{(d+1)^2}\big)}\,C(\boldsymbol{x}) = (d+1)^{d+1}\,C(\boldsymbol{x}).$$

Taking $\boldsymbol{x}$ as the optimal offline solution proves the claim. □

3.2.1. *Lower Bounds for Polynomial Latency Functions.* We consider general polynomial latency functions of the form $\ell_a(z) = \sum_{j=1}^{d} a_j z^j$, where all coefficients a_j are nonegative.

We start with a classical result.

Lemma 3.13. *The* $n-th$ *power of the sum of numbers from* 1 *to* k *is a polynomial in* k *given by:*

$$\sum_{i=1}^{k} i^n = \frac{1}{n+1}\sum_{j=0}^{n+1}\binom{n+1}{j}B_j k^{n+1-j},$$

where B_j *are the Bernoulli numbers.*

Consider the network presented in Figure 2 with the following latency functions: $\ell_{(s_i,s)}(z) = 0$, $\ell_{(t,t_i)}(z) = 0$, $\ell_{(s_i,t_i)}(z) = i^d$, $i = 1,\ldots,k$, and $\ell_{(s,t)}(z) = z^d, d \in \mathbb{N}$. We consecutively release games with a single player type $i1$, where a demand of size 1 has to be routed from s_i to t_i, for $i = 1,\ldots,k$. Due to the choice of the affine terms i^d, SeqNash routes every demand over the arc from s to t.

Then we release the $(k+1)$-th game with demand x from s to t. The total cost for the flow generated by SeqNash is given by:

$$\text{SeqNash}(\sigma) = \big(k+x\big)^{d+1}.$$

The optimal offline algorithm Opt routes the demands of the first k games along the direct arcs from s_i to t_i. The last demand is routed from s to t. The total cost for Opt is then given by:

$$\text{Opt}(\sigma) = \sum_{i=1}^{k} i^d + x^{d+1}.$$

From Lemma 3.13, we know that the $d-th$ power of the sum of numbers from 1 to k is a polynomial in k given by:

$$\sum_{i=1}^{k} i^d = \frac{1}{d+1} \sum_{j=0}^{d+1} \binom{d+1}{j} B_j k^{d+1-j},$$

where B_j are the Bernoulli numbers.

Theorem 3.14. *In case of polynomial latency functions, the online algorithm* SeqNash *for* OnlineSRG *has a competitive ratio greater than or equal to* $\frac{d+1}{d+2} 2^{d+1}$*, where* d *is the highest degree of the used polynomials.*

Proof. We have to show that the competitive ratio fulfills:

$$\frac{\textsc{SeqNash}_d(\sigma)}{\textsc{Opt}_d(\sigma)} \geq \frac{d+1}{d+2} 2^{d+1}.$$

We follow the construction of the above discussion,

$$\frac{\textsc{SeqNash}_d(\sigma)}{\textsc{Opt}_d(\sigma)} \geq \lim_{k\to\infty} \frac{\left(k+x\right)^{d+1}}{\sum\limits_{i=1}^{k} i^d + x^{d+1}}.$$

We set $x = k$ which yields:

$$\begin{aligned}\frac{\textsc{SeqNash}_d(\sigma)}{\textsc{Opt}_d(\sigma)} &\geq \lim_{k\to\infty} \frac{\left(2k\right)^{d+1}}{\sum\limits_{i=1}^{k} i^d + k^{d+1}} \\ &= \lim_{k\to\infty} \frac{\left(2k\right)^{d+1}}{\frac{1}{d+1} k^{d+1} + k^{d+1} + \sum\limits_{j=1}^{d+1} \binom{d+1}{j} B_j k^{d+1-j}} = \frac{d+1}{d+2} 2^{d+1},\end{aligned}$$

where the equality follows from Lemma 3.13 and the fact that $B_0 = 1$. ☐

Table 2. Competitive ratio for different polynomial latency functions. Coefficients a_i are assumed to be nonnegative.

Set $\mathcal{C}$ of latency functions	Example	$\omega_\infty(\mathcal{C},\lambda)$	λ	UB	LB
linear functions	$a_1 x + a_0$	$\frac{3}{4}$	1	4	3
quadratic	$\sum_{i=0}^{2} a_i x^i$	0.93	2.18	19.6	7.5
cubic	$\sum_{i=0}^{3} a_i x^i$	$\frac{15}{16}$	64	256	17.32
.	.	.	.	.	.
degree d	$\sum\limits_{i=0}^{d} a_i\, x^i$	$\frac{d^2+2\,d}{(d+1)^2}$	$(d+1)^{(d-1)}$	$(d+1)^{d+1}$	$\frac{d+1}{d+2}\, 2^{d+1}$

4. Competitive Analysis for Atomic Players

Before we state the main result, we need some useful prerequisites. We define for every $a \in A$, for any nonnegative vectors $\boldsymbol{f}_a, \boldsymbol{x}_a \in \mathbb{R}_+^{\mathcal{K}}$ the following values (we assume by convention $0/0 = 0$):

$$\omega(\ell_a, n, \mathcal{K}; \lambda) := \sup_{\boldsymbol{f}_a, \boldsymbol{x}_a \geq 0} \frac{C_a(f_a) - \lambda\, C_a(x_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a)}{C_a(f_a)}. \tag{28}$$

For a given class $\mathcal{C}$ of nondecreasing latency functions and a nonnegative real number $\lambda \geq 0$, we further define

$$\omega_n^{\mathcal{K}}(\mathcal{C}; \lambda) := \sup_{\ell_a \in \mathcal{C}} \omega(\ell_a, n, \mathcal{K}; \lambda).$$

Theorem 4.1. *Consider a sequence of n games involving $\mathcal{K}$ players and separable latency functions drawn from $\mathcal{C}$. If $1 - \omega_n^{\mathcal{K}}(\mathcal{C}; \lambda) > 0$ holds, then* SeqNash *is*

$$\inf_{\lambda \geq 0} \left[\lambda \left(1 - \omega_n^{\mathcal{K}}(\mathcal{C}; \lambda)\right)^{-1} \right] - \textit{competitive}$$

for the atomic OnlineSRG.

Proof. Let $\boldsymbol{f}$ be the flow generated by SeqNash, and $\boldsymbol{x}$ be any feasible flow for a given sequence of games $\sigma = (1, \dots, n)$.

$$\begin{aligned} C(f) &\leq \sum_{a \in A} \left[C_a(f_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}, n) \right] && (29) \\ &= \sum_{a \in A} \left[C_a(f_a) + \lambda\, C_a(x_a) - \lambda\, C_a(x_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}, n) \right] \\ &\leq \lambda\, C(x) + \omega_n^{\mathcal{K}}(\mathcal{C}; \lambda)\, C(f). && (30) \end{aligned}$$

Here, (29) follows from the variational inequality stated in Lemma 2.6. The last inequality (30) follows from the definition of $\omega_n^{\mathcal{K}}(\mathcal{C}; \lambda)$. □

Using the notation:

$$\theta_a^i(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}_i) := \sum_{ij \in [K_i]} (f_a^{ij} x_a^{ij} - f_a^{ij}\, f_a^{ij}),$$

we can simplify the value $\omega_n^{\mathcal{K}}(\mathcal{C}; \lambda)$

Lemma 4.2. *The value $\omega(\ell_a, n, \mathcal{K}; \lambda)$ is at most*

$$\sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big) x_a + \vartheta_a^n(\ell_a, \boldsymbol{f}_a) + \sum_{i=1}^{n} \ell'_a(\sum_{k=1}^{i} f_a^k) \theta_a^i(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}_i)}{\ell_a(f_a) f_a}. \tag{31}$$

Proof. First note that

$$\begin{aligned} C_a(f_a) + V_a(\boldsymbol{f}_a, \boldsymbol{x}_a) &= \vartheta_a^n(\ell_a, \boldsymbol{f}_a) + \sum_{i=1}^{n} \left[\ell'_a(\sum_{k=1}^{i} f_a^k) \theta_a^i(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}_i) + \ell_a\big(\sum_{k=1}^{i} f_a^k\big) x_a^i \right] \\ &\leq \vartheta_a^n(\ell_a, \boldsymbol{f}_a) + \sum_{i=1}^{n} \ell'_a(\sum_{k=1}^{i} f_a^k) \theta_a^i(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}_i) + \ell_a(f_a)\, x_a, \end{aligned}$$

where the last inequality is valid since latency functions are nondecreasing. Then, using

$$\ell_a(f_a)\, x_a - \lambda\, C_a(x_a) = \big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big) x_a,$$

yields the claim. □

Note that for $\lambda = 1$ and $n = 1$ the value $\omega(\ell_a, 1, \mathcal{K}; 1)$ is equal to the value $\beta^K(\ell_a)$ defined by Cominetti, Correa, and Stier-Moses in [16]. For $n > 1$, that is, the sequence σ of games contains more than one game, the main difference between $\beta^K(\ell_a)$ and $\omega(\ell_a, n, \mathcal{K}; \lambda)$ are the values $\lambda \geq 0$ and $\vartheta_a^n(\ell_a, f)$. The value $\vartheta_a^n(\ell_a, f)$ penalizes the efficiency of SeqNash for multiple games. The value λ admits a further degree of freedom to strengthen the analysis.

4.1. Affine Linear Latency Functions

We analyze in the following the value $\omega_n^{\mathcal{K}}(\mathcal{C}; 1)$ for affine linear latency functions.

Lemma 4.3. *For affine linear latency functions $\ell_a(z) = q_a z + r_a,\ q_a \geq 0, r_a \geq 0,$ and $\lambda \geq 1$ the value $\omega_n^{\mathcal{K}}(\mathcal{C}; \lambda)$ is less than or equal to $\frac{4(\mathcal{K}-1)}{5\mathcal{K}+1}$.*

Proof. We start with the bound on $\omega(\ell_a, n, \mathcal{K}; \lambda)$ in equation (31) for affine linear latency functions.

$$
\begin{aligned}
\omega(\ell_a, n, \mathcal{K}; \lambda) &\leq \sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{q_a\big(f_a - \lambda\, x_a\big)x_a + q_a\,(f_a)^2 - q_a \sum_{i=1}^{n} \big(\sum_{k=1}^{i} f_a^k\big)\, f_a^i + q_a \sum_{i=1}^{n} \theta_a^i(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}_i)}{q_a (f_a)^2 + r_a\, f_a} \\
&\leq \sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(f_a - \lambda\, x_a\big)x_a + \frac{1}{2}\,(f_a)^2 - \frac{1}{2} \sum_{i=1}^{n} (f_a^i)^2 + \sum_{i=1}^{n} \theta_a^i(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}_i)}{(f_a)^2} \qquad (32) \\
&\leq \sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(f_a - \lambda\, x_a\big)x_a + \frac{1}{2}\,(f_a)^2 - \frac{1}{2} \sum_{ij \in \mathcal{K}} (f_a^{ij})^2 + \sum_{i=1}^{n} \theta_a^i(\boldsymbol{f}_a, \boldsymbol{x}_a, \mathcal{K}_i)}{(f_a)^2} \qquad (33) \\
&= \sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(f_a - \lambda\, x_a\big)x_a + \frac{1}{2}\,(f_a)^2 + \sum_{ij \in \mathcal{K}} \big(f_a^{ij}\, x_a^{ij} - \frac{3}{2}\,(f_a^{ij})^2\big)}{(f_a)^2},
\end{aligned}
$$

where (32) follows from (20) and $r_a \geq 0$. Note that to obtain the first inequality we have used that $r_a - \lambda\, r_a \leq 0$ since $\lambda \geq 1$. Inequality (33) is valid since the sum of powers is less than the power of the sum. Without loss of generality, we can assume that $f_a^1 := \max_{ij \in [\mathcal{K}]} f_a^{ij}$. Since the individual components x_a^{ij} appear linearly in the expression $f_a^{ij} x_a^{ij}$, we can set $\boldsymbol{x} = (x_a^1, 0, \ldots, 0)$ to bound the above expression from above. Thus, we have to solve:

$$
\omega(\ell_a, n, \mathcal{K}; \lambda) \leq \sup_{0 \leq f_a^1 \leq f_a, x_a^1 \geq 0} \frac{f_a\, x_a^1 - \lambda\,(x_a^1)^2 + \frac{1}{2}\,(f_a)^2 + f_a^1\, x_a^1 - \sum_{ij \in \mathcal{K}} \frac{3}{2}\,(f_a^{ij})^2}{(f_a)^2}.
$$

Because of symmetry in the last sum of the numerator, we can set $f_a^{ij} = \frac{f_a}{\mathcal{K}-1}$.

$$
\omega(\ell_a, n, \mathcal{K}) \leq \sup_{\frac{f_a}{\mathcal{K}} \leq f_a^1 \leq f_a, x_a^1 \geq 0} \frac{f_a x_a^1 - \lambda\,(x_a^1)^2 + \frac{1}{2}\,(f_a)^2 + f_a^1\, x_a^1 - \frac{3}{2}(f_a^1)^2 - \frac{3(f_a - f_a^1)^2}{2(\mathcal{K}-1)}}{(f_a)^2}.
$$

For any choice of f_a, f_a^1, the optimal value for x_a^1 is exactly $x_a^1 = \frac{f_a + f_a^1}{2\,\lambda}$. Inserting the value yields:

$$
\omega(\ell_a, n, \mathcal{K}; \lambda) \leq \sup_{\frac{f_a}{\mathcal{K}} \leq f_a^1 \leq f_a} \frac{(\frac{1}{2} + \frac{1}{4\,\lambda})(f_a)^2 + (\frac{1}{4\,\lambda} - \frac{3}{2})\,(f_a^1)^2 + \frac{1}{2} f_a^1\, f_a - \frac{3(f_a - f_a^1)^2}{2(\mathcal{K}-1)}}{(f_a)^2}.
$$

We replace $f_a^1 = \mu\, f_a$ with $\mu \in [\frac{1}{\mathcal{K}}, 1]$ and solve:

$$
\omega(\ell_a, n, \mathcal{K}; \lambda) \leq \max_{\mu \in [\frac{1}{\mathcal{K}}, 1]} (\tfrac{1}{2} + \tfrac{1}{4\,\lambda}) + (\tfrac{1}{4\,\lambda} - \tfrac{3}{2})\,\mu^2 + \tfrac{1}{2}\,\mu - \tfrac{3(1-\mu)^2}{2(\mathcal{K}-1)}. \qquad (34)
$$

Now we set $\lambda := 1$. Then, the optimal choice is $\mu = \frac{(\mathcal{K}+5)}{5\mathcal{K}+1}$. This leads to:

$$\omega_n^{\mathcal{K}}(\mathcal{C};1) \leq \tfrac{4(\mathcal{K}-1)}{5\mathcal{K}+1}.$$

□

Applying Theorem 4.1 with the above value for $\omega_n^{\mathcal{K}}(\mathcal{C})$ leads to the following result.

Corollary 4.4. *If the latency functions of the atomic* OnlineSRG *are affine, the online algorithm* SeqNash *is* $\frac{5\mathcal{K}+1}{\mathcal{K}+5}$*-competitive, where* $\mathcal{K}$ *is the total number of players.*

Corollary 4.4 gives abound that only depends on the total number of players in the sequence σ of games. This bound states that SeqNash is asymptotically 5-competitive for sequential atomic routing games.

If we optimize over the parameter λ we can derive even better bounds. For ease of presentation we focus on the asymptotic bound, that is, we consider the case where $\mathcal{K} \to \infty$.

Corollary 4.5. *If the latency functions of the atomic* OnlineSRG *are affine, the online algorithm* SeqNash *is* 4.92*-competitive.*

Proof. We start with bounding $\omega_\infty^\infty(\mathcal{C};\lambda)$ using (34):

$$\omega(\ell_a,\infty,\infty;\lambda) \leq \max_{\mu\in[0,1]} \left(\tfrac{1}{2} + \tfrac{1}{4\,\lambda}\right) + \left(\tfrac{1}{4\,\lambda} - \tfrac{3}{2}\right)\mu^2 + \tfrac{1}{2}\,\mu.$$

Then, it follows that

$$\mu^* = \frac{1}{6\,\lambda - 1},$$

and

$$\omega_\infty^\infty(\mathcal{C};1) \leq \frac{4\,\lambda + 13\,\lambda^2 - 1}{4\,\lambda(6\,\lambda-1)}.$$

Note, that we still have $\omega_\infty^\infty(\mathcal{C};1) \leq \frac{1}{5}$ for $\lambda = 1$. Applying Theorem 4.1 with $\lambda = 1.1$ yields the claim. □

In the following, we derive a bound that depends on the number of games.

Corollary 4.6. *If the latency functions of the atomic* OnlineSRG *are affine, the online algorithm* SeqNash *is* $\frac{2(3\mathcal{K}+1)n}{n\mathcal{K}+3n+3\mathcal{K}+1}$*-competitive, where* n *is the number of games and* $\mathcal{K}$ *is the total number of players.*

Proof. We start with equation (32) in Lemma 4.3 to derive another bound on $\omega_n^{\mathcal{K}}(\mathcal{C};\lambda)$.

$$\begin{aligned}
\omega(\ell_a,n,\mathcal{K};\lambda) &\leq \sup_{\boldsymbol{w}_a,\boldsymbol{v}_a\geq 0} \frac{\big(f_a - \lambda\,x_a\big)x_a + \tfrac{1}{2}\,(f_a)^2 - \tfrac{1}{2}\,\sum_{i=1}^n (f_a^i)^2 + \sum_{i=1}^n \theta_a^i(\boldsymbol{f}_a,\boldsymbol{x}_a,\mathcal{K}_i)}{(f_a)^2} \\
&\leq \tfrac{n-1}{2n} + \sup_{\boldsymbol{w}_a,\boldsymbol{v}_a\geq 0} \frac{\big(f_a - \lambda\,x_a\big)x_a + \sum_{i=1}^n \theta_a^i(\boldsymbol{f}_a,\boldsymbol{x}_a,\mathcal{K}_i)}{(f_a)^2}, \qquad (35)
\end{aligned}$$

where (35) follows from Cauchy-Schwarz inequality. Then, the proof proceeds along the lines of the proof of Lemma 4.3 except that we replace the factor $\frac{3}{2}$ by 1.

$$\omega(\ell_a,n,\mathcal{K};\lambda) \leq \tfrac{n-1}{2n} + \max_{\mu\in[\frac{1}{\mathcal{K}},1]} \left(\tfrac{1}{4\,\lambda}\right) + \left(\tfrac{1}{4\,\lambda} - 1\right)\mu^2 + \tfrac{1}{2}\,\mu - \tfrac{(1-\mu)^2}{(\mathcal{K}-1)}. \qquad (36)$$

Setting again $\lambda := 1$ yields

$$\omega(\ell_a,n,\mathcal{K};1) \leq \tfrac{n-1}{2n} + \max_{\mu\in[\frac{1}{\mathcal{K}},1]} \tfrac{1}{4} - \tfrac{3}{4}\mu^2 + \tfrac{1}{2\,\lambda}\mu - \tfrac{(1-\mu)^2}{(\mathcal{K}-1)}.$$

It is easy to see that $\mu = \frac{\mathcal{K}+3}{3\mathcal{K}+1}$ is optimal. Evaluating $\frac{1}{1-\omega_n^{\mathcal{K}}(\mathcal{C})}$ yields the desired bound. □

This bound is asymptotically 6-competitive. It provides, however, an explicit dependency on the number of games involved. For $n = 1$, we obtain a bound of $\frac{3\mathcal{K}+1}{2\mathcal{K}+2}$ for atomic routing games with affine linear latency functions; this bound has previously been established by Cominetti, Correa and Stier-Moses [16]. For $\mathcal{K} \to \infty$ we establish a bound that only depends on the number of games.

Corollary 4.7. *If the latency functions of the atomic* ONLINESRG *are affine and if we allow for infinitely many atomic players, the online algorithm* SEQNASH *is* $\frac{6\,n}{n+3}$*-competitive.*

Corollary 4.8. *If the latency functions of the atomic* ONLINESRG *are affine and we have one atomic player per game, the online algorithm* SEQNASH *is* $\frac{6\,n^2+2\,n}{n^2+6\,n+1}$*-competitive, where* n *is the total number of games.*

Now, we further strengthen the upper bounds by varying λ.

Proposition 4. *If the latency functions of the atomic* ONLINESRG *are affine, the online algorithm* SEQNASH *is*

$$\frac{\left(2\,n+\sqrt{2}\,\sqrt{n(3\,n+1)}\right)n\left(1+3\,n+\sqrt{2}\sqrt{n\,(3\,n+1)}\right)\sqrt{2}}{4\,\sqrt{n\,(3\,n+1)}\,(n+1)^2} - \textit{competitive.}$$

Proof. For $\mathcal{K} \to \infty$ we have $\lim_{\mathcal{K}\to\infty} \frac{(1-\mu)^2}{(\mathcal{K}-1)} = 0$. Hence, (36) reduces to

$$\omega(\ell_a, n, \infty; \lambda) \le \tfrac{n-1}{2n} + \max_{\mu\in[\frac{1}{\mathcal{K}},1]} \tfrac{1}{4\,\lambda} - (\tfrac{1}{4\,\lambda} - 1)\,\mu^2 + \tfrac{1}{2\,\lambda}\mu.$$

The maximization problem can be solved, leading to

$$\max_{\mu\in[\frac{1}{\mathcal{K}},1]} \tfrac{1}{4\,\lambda} - (\tfrac{1}{4\,\lambda} - 1)\,\mu^2 + \tfrac{1}{2\,\lambda}\mu \le \frac{1}{4\,\lambda-1}.$$

Applying Theorem 4.1 yields

$$C(\boldsymbol{f}) \le \frac{2\,\lambda\,n\,(-1+4\,\lambda)}{4\,n\,\lambda - 3\,n + 4\,\lambda - 1}\,C(\boldsymbol{x}).$$

Solving the problem

$$\min_{\lambda\ge 0} \frac{2\,\lambda\,n\,(-1+4\,\lambda)}{4\,n\,\lambda - 3\,n + 4\,\lambda - 1},$$

leads to

$$\lambda^* = \frac{1+3\,n+\sqrt{2\,n+6\,n^2}}{4\,(n+1)}.$$

Inserting this value into the objective, prove the claim. □

Corollary 4.9. *Consider linear latency functions. Then, the price of anarchy is at most* $\frac{1}{8}\,(2+\sqrt{2})\,(1+\sqrt{2})\,\sqrt{2} \approx 1.46$.

Proof. We set $\lambda = \frac{1}{2} + \frac{1}{4}\,\sqrt{2}$, $n = 1$ and, apply Proposition 4. □

This result, however, only holds for purely linear latency functions. The reason why this approach fails for affine linear functions is due to the choice $\lambda < 1$. The value $\omega(\ell_a, n, \infty; \lambda)$ is unbounded for large affine terms r_a if $\lambda < 1$.

Still, this result improves the best known upper bound of $\frac{3}{2}$ on the price of anarchy for atomic games involving linear latency functions.

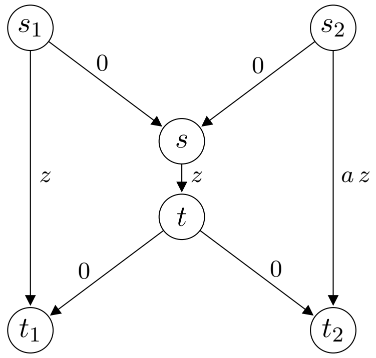

FIGURE 3. Graph construction for the proof of Proposition 6.

4.1.1. *Lower Bounds.* In this section, we provide lower bounds on the competitive ratio for any deterministic online algorithm and SEQNASH. Note that all lower bounds of the nonatomic version of SEQNASH for ONLINESRG carry over to the atomic player case when we allow for infinitely many players in each game i.

We use the network in Fig. 2 to derive a lower bound when we have a single atomic player in each game i.

Proposition 5. *In case of affinely linear latency functions, the online algorithm* SEQNASH *for the atomic* ONLINESRG*, where in each game there is a single atomic player has a competitive ratio greater than or equal to* $\frac{2n-1}{n}$*, where* n *is the number of games.*

Proof. The proof proceeds along the lines of Theorem 3.8 except that we replace the constant costs $\ell_{(s_i,t_i)}(z) = 2\,i$, $i = 1, \dots, k$. This forces the first k atomic players to router their demand along the middle arc (s, t). The reminder of the proof consists of technical calculations that are omitted. □

In the following, we establish a lower bound on the price of anarchy for purely linear latency functions. These bounds demonstrate that in contrast to the nonatomic counterpart the price of anarchy may be larger than 1 for linear latency functions. In the article by Cominetti, Correa and Stier-Moses [16], the authors claim that the price of anarchy can be bounded by 1.17.

Proposition 6. *In case of linear latency functions, the price of anarchy for the atomic network routing game is bounded from below by* $1\,\frac{1}{25}$.

Proof. Consider the network given in Fig 3. Note that all latency functions have $\ell_a(0) = 0$. We assume that a nonatomic player (N) wants to route one unit from node s_1 to node t_1. On the other hand, one atomic player (A) wants to route one unit from s_2 to node t_2. For both players there exist two choices of paths: the direct path (s_1, t_1) and (s_2, t_2) or the path along the shared arc (s, t). If x and y denote the amount of flow for player N, and player A, that is routed along the direct arc (s_1, t_1), and (s_2, t_2), respectively. The response strategies are given by the following two optimizationn problems. For player N we have:

$$\min_{0 \le x \le 1} \frac{1}{2}\,x^2 + \frac{1}{2}\,(1-x)^2 + (1-x)\,(1-y). \tag{37}$$

Note that is assumed that player A sends $1 - y)$ units flow along the middle arc. Hence, $\ell_{(s,t)}(z + (1 - y)) = z + (1 - y)$. The optimal solution to problem (37) is given by

$$x^* = \min\left\{\max\left\{\frac{2-y}{2}, 0\right\}, 1\right\}.$$

For player A we have:

$$\min_{0 \le y \le 1} a\,y^2 + ((1-x) + (1-y))(1-y). \tag{38}$$

The solution is given by

$$y^* = \min\left\{\max\left\{\frac{3-x}{2\,a+2}, 0\right\}, 1\right\}.$$

Plugging both solutions together and assuming $\frac{1}{2} \leq a$ yields:

$$x^* = \frac{4}{4\,a+3}, \text{ and } y^* = \frac{4\,a+1}{4\,a+3}.$$

If we denote the entire flow by $\boldsymbol{f}$, then the cost in equilibrium is given by

$$C(\boldsymbol{f}) = \frac{32\,a^2 + 32\,a + 2}{(4\,a+3)^2}.$$

Now the optimal flow $\boldsymbol{x}^*$ solves:

$$\min_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1}} x^2 + \big((1-x) + (1-y)\big)^2 + a\,y^2. \tag{39}$$

Here, the optimal solutions are given by

$$y^* = \frac{2}{2\,a+1}, \text{ and } x^* = \frac{2\,a}{2\,a+1}.$$

$$C(\boldsymbol{x^*}) = \frac{4\,a}{2\,a+1}.$$

Setting $a := \frac{1}{2}$ yields

$$C(\boldsymbol{f}) = \frac{26}{25}, \text{ and } C(\boldsymbol{x}^*) = 1,$$

proving the claim. □

The table below summarizes the main results presented in this chapter.

TABLE 3. Competitive Ratio for the online algorithm SEQNASH for affine linear latency functions $q_a\,x + r_a$, $q_a, r_a \geq 0$. The first row shows known results for atomic routing games that are due to Cominetti, Correa, and Stier-Moses [16]. UB and LB denote Upper and Lower Bound, respectively.

	arbitrary # of Players		1 player per game	
# games	UB	LB	UB	LB
$\frac{3}{2}$	1	1.343	1	1
2	$2\frac{2}{5}$	2	1.64	$\frac{3}{2}$
3	3	$2\frac{1}{3}$	2.14	$1\,\frac{2}{3}$
.	.	.	.	.
.	.	.	.	.
n	$\min\{\frac{6\,n}{n+3}, 4.92\}$	$\frac{3n}{n-2}$	$\min\{\frac{6\,n^2+2\,n}{n^2+6\,n+1}, 4.92\}$	$\frac{2\,n-1}{n}$
∞	4.92	3	4.92	2

4.2. Polynomial Latency Functions

In this section, we investigate the case, where we allow for general convex latency functions.

Proposition 7. *If $\lambda \geq 0$ and ℓ_a is a convex latency function, then, the following inequality is valid:*

$$\omega(\ell_a; n, \mathcal{K}; \lambda) \leq \sup_{f_a, x_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)\, x_a + \ell_a'(f_a)\, \frac{x_a^2}{4}}{\ell_a(f_a)\, f_a} + \sup_{\boldsymbol{f}_a \geq 0} \frac{\vartheta_a^n(\ell_a, \boldsymbol{f}_a)}{\ell_a(f_a)\, f_a}. \tag{40}$$

Proof. We start with inequality (31) in Lemma 4.2. Using the triangle inequality, we can separate $\vartheta_a^n(\ell_a, \boldsymbol{f}_a)$ from the rest since the supremum over the sum of two functions is less than or equal to the sum of the suprema.

Then, we only have to consider the first supremum that we denote with $\omega^1(\ell_a; n, \mathcal{K}; \lambda)$:

$$\omega^1(\ell_a; n, \mathcal{K}; \lambda) \leq \sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)x_a + \sum_{i=1}^{n} \ell_a'(\sum_{k=1}^{i} f_a^k)\Big(\sum_{ij \in [K_i]} (f_a^{ij}\, x_a^{ij} - f_a^{ij}\, f_a^{ij})\Big)}{\ell_a(f_a) f_a}.$$

First, we bound the last difference in the nominator:

$$f_a^{ij}\, x_a^{ij} - f_a^{ij}\, f_a^{ij} \leq \frac{1}{4}\, (x_a^{ij})^2.$$

This yields:

$$\begin{aligned}
\omega^1(\ell_a; n, \mathcal{K}; \lambda) &\leq \sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)x_a + \sum_{i=1}^{n} \ell_a'(\sum_{k=1}^{i} f_a^k)\Big(\sum_{ij \in [K_i]} \frac{(x_a^{ij})^2}{4}\Big)}{\ell_a(f_a) f_a} \\
&\leq \sup_{\boldsymbol{x}_a, \boldsymbol{f}_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)x_a + \sum_{i=1}^{n} \ell_a'(f_a)\Big(\sum_{ij \in [K_i]} \frac{(x_a^{ij})^2}{4}\Big)}{\ell_a(f_a) f_a},
\end{aligned}$$

where the last inequality follows from the convexity of ℓ_a, which implies that ℓ_a' is nondecreasing. Finally, using

$$\frac{1}{4} \sum_{i=1}^{n} \sum_{ij \in [K_i]} (x_a^{ij})^2 \leq \frac{1}{4} \sum_{i=1}^{n} (x_a^i)^2 \leq \frac{1}{4}\, x_a^2$$

proves the proposition. □

Corollary 4.10. *If $\lambda \geq 0$ and ℓ_a is a convex latency function, then, the following inequality is valid:*

$$\omega(\ell_a; 1, \infty; \lambda) \leq \sup_{f_a, x_a \geq 0} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)\, x_a + \ell_a'(f_a)\Big(\frac{x_a^2}{4}\Big)}{\ell_a(f_a)\, f_a}.$$

Proof. The inequality is derived by using $\vartheta_a^1(\ell_a, \boldsymbol{f}_a) = 0$. □

Proposition 8. *If $\lambda \geq 1$ and $\ell_a(f_a)$ is a convex function, then the value $\omega(\ell_a; n, \mathcal{K}; \lambda)$ is at most:*

$$\omega(\ell_a; n, \mathcal{K}; \lambda) \leq \sup_{0 \leq x_a \leq f_a} \frac{\big(\ell_a(f_a) - \lambda\, \ell_a(x_a)\big)\, x_a + \ell_a'(f)\, \frac{(x_a)^2}{4}}{\ell_a(f_a)\, f_a} + \sup_{\boldsymbol{f}_a \geq 0} \frac{\vartheta_a^n(\ell_a, \boldsymbol{f}_a)}{\ell_a(f_a)\, f_a}. \tag{41}$$

Proof. We start with the characterization of $\omega(\ell_a; \infty, \infty; \lambda)$ given in Proposition 7: consider the function $h(x_a)$ defined as the numerator of the first supremum in equation (40). To prove that the solution satisfies $x_a \leq f_a$, we show that $h'(x_a) \leq 0$ if $x_a \geq f_a$. Using that $h'(x_a) = \ell_a(f_a) - \lambda\,\ell_a(x_a) - \lambda\,x_a\,\ell'_a(x_a) + \frac{x_a}{2}\,\ell'_a(f_a)$, the derivative is negative if and only if

$$\ell_a(f_a) + \frac{x_a}{2}\,\ell'_a(f_a) \leq \lambda\left(\ell_a(x_a) + x_a\,\ell'_a(x_a)\right).$$

By assumption $\ell_a(f_a)\,f_a$ is convex, hence,

$$\ell_a(f_a) + \ell'_a(f_a) \leq \ell_a(x_a) + \ell'_a(x_a)$$

for $x_a \geq f_a$. Since furthermore $\lambda \geq 1$, the proof is complete. □

Proposition 9. *Let $\mathcal{C}$ be a family of continuous, nondecreasing and convex latency functions ℓ_a. Furthermore, assume that $\lambda \geq 1$ and $\ell_a(\kappa\,f_a) \geq s(\kappa)\,\ell_a(f_a)$ for all $\kappa \in [0,1]$, where $s : [0,1] \to [0,1]$ is a differentiable function with $s(1) = 1$. Then,*

$$\omega(\ell_a; \infty, \infty; \lambda) \leq \max_{0 \leq u \leq 1} u\left(1 - \lambda\,s(u) + s'(1)\,\frac{u}{4}\right) + \frac{d}{d+1} \tag{42}$$

$$\omega(\ell_a; 1, \infty; \lambda) \leq \max_{0 \leq u \leq 1} u\left(1 - \lambda\,s(u) + s'(1)\,\frac{u}{4}\right). \tag{43}$$

Proof. We start with the characterization of $\omega(\ell_a; \infty, \infty; \lambda)$ given in Proposition 7:

$$\omega(\ell_a; \infty, \infty; \lambda) \leq \sup_{f_a, x_a \geq 0} \frac{\left(\ell_a(f_a) - \lambda\,\ell_a(x_a)\right)x_a + \ell'_a(f_a)\left(\frac{x_a^2}{4}\right)}{\ell_a(f_a)\,f_a} + \sup_{\boldsymbol{f}_a \geq 0} \frac{\vartheta_a^\infty(\ell_a, \boldsymbol{f}_a)}{\ell_a(f_a)\,f_a}.$$

Recall from Lemma 3.10 that

$$\sup_{\boldsymbol{f}_a \geq 0} \frac{\vartheta_a^\infty(\ell_a, f_a)}{\ell_a(f_a)\,f_a} \leq \frac{d}{d+1}, \quad \text{and} \quad \vartheta_a^1(\ell_a, f_a) = 0.$$

Hence, from now on we only consider $\omega(\ell_a; 1, \infty; \lambda)$.

The reminder of the proof is based on a result obtained by Cominetti, Correa, and Stier-Moses [16].

For $z \geq z'$, we can bound $\ell'_a(z)$:

$$\ell_a(z') = \ell_a(\tfrac{z'}{z}\,z) \geq s(\tfrac{z'}{z})\,\ell_a(z)\,z. \tag{44}$$

Furthermore,

$$\ell'_a(f) = \lim_{\epsilon \to 0} \frac{\ell_a(f_a + \epsilon) - \ell_a(f_a)}{\epsilon} \leq \ell_a(f_a) \lim_{\epsilon \to 0} \frac{1 - s\left(\frac{f_a}{f_a + \epsilon}\right)}{\epsilon} = \ell_a(f_a)\,\frac{s'(f_a)}{f_a}.$$

Thus, we conclude

$$\begin{aligned}
\omega(\ell_a; 1, \infty; \lambda) &\leq \sup_{0 \leq x_a \leq f_a} \frac{x_a\,\ell_a(f_a)\left(1 - \frac{\lambda\,\ell_a(x_a)}{\ell_a(f_a)} + \frac{s'(1)\,x_a}{4\,f_a}\right)}{\ell_a(f_a)\,f_a} \\
&\leq \sup_{0 \leq x_a \leq f_a} \frac{x_a\left(1 - \lambda\,s(\frac{x_a}{f_a}) + \frac{s'(1)\,x_a}{4\,f_a}\right)}{f_a},
\end{aligned}$$

where we used (44) for the second inequality.

Defining $0 \leq u := \frac{x_a}{f_a} \leq 1$ yields

$$\begin{aligned}
\omega(\ell_a; 1, \infty; \lambda) &\leq \max_{0 \leq u \leq 1} u\left(1 - \lambda\,s(u) + s'(1)\,\frac{u}{4}\right) \\
\omega(\ell_a; \infty, \infty; \lambda) &\leq \max_{0 \leq u \leq 1} u\left(1 - \lambda\,s(u) + s'(1)\,\frac{u}{4}\right) + \frac{d}{d+1}.
\end{aligned}$$

□

Corollary 4.11. *If $\mathcal{C}$ only contains polynomials of degree $d \geq 1$, the competitive ratio of* SeqNash *is at most*

$$\inf_{\lambda \geq 1} \left[\lambda \left(1 - \max_{0 \leq u \leq 1} u \left(1 - \lambda\, u^d + d\, \frac{u}{4}\right) + \frac{d}{d+1} \right)^{-1} \right]. \tag{45}$$

Proof. All assumptions of Proposition 9 are satisfied with $s(f) = f^d$. Therefore, $s'(1) = d$ and

$$\omega(\ell_a; \infty, \infty; \lambda) \leq \max_{0 \leq u \leq 1} u \left(1 - \lambda\, u^d + d\, \frac{u}{4}\right) + \frac{d}{d+1}. \tag{46}$$

Applying Theorem 4.1 yields the claim. □

Using Corollary 4.11, we can determine bounds on the competitive ratio for SeqNash for general polynomials.

Corollary 4.12. *If $\mathcal{C}$ only contains polynomials of degree $d \geq 1$, the price of anarchy is at most*

$$\inf_{\lambda \geq 1} \left[\lambda \left(1 - \max_{0 \leq u \leq 1} u \left(1 - \lambda\, u^d + d\, \frac{u}{4}\right) \right)^{-1} \right]. \tag{47}$$

Proof. All assumptions of Proposition 9 are satisfied with $s(f) = f^d$. Therefore, $s'(1) = d$ and

$$\omega(\ell_a; 1, \infty; \lambda) \leq \max_{0 \leq u \leq 1} u \left(1 - \lambda\, u^d + d\, \frac{u}{4}\right). \tag{48}$$

Applying Theorem 4.1 yields the claim. □

In the following we present price of anarchy results for network games with atomic players, that is, we assume $n = 1$. We present results for squared, cubic, and degree four, and five polynomials. Note that all results up to degree two improve known bounds or establish the first known bounds for polynomials of degree $d \geq 4$. The results itself have been obtained by optimizing the expression in (47) over the parameter $\lambda \geq 1$.

Table 4. Price of Anarchy for different polynomial latency functions. Coefficients a_i are assumed to be nonnegative.

Set $\mathcal{C}$ of allowable latency functions	Example	$\omega_1^{\infty}(\mathcal{C}, \lambda)$	λ	Price of Anarchy $\alpha^{\infty}(\mathcal{C})$ arbitrary # of players
linear functions	$a_1 x + a_0$	$\frac{1}{3}$	1	1.5
quadratic functions	$\sum_{i=0}^{2} a_i x^i$	0.58	1.08	2.55
cubic functions	$\sum_{i=0}^{3} a_i x^i$	$\frac{2}{3}$	1.69	5.06
polynomials $d \leq 4$	$\sum_{i=0}^{4} a_i x^i$	$\frac{2}{3}$	3.8	11.3
polynomials $d \leq 5$	$\sum_{i=0}^{5} a_i x^i$	$\frac{2}{3}$	9.69	29.07

Theorem 4.13. *If $\mathcal{C}$ only contains polynomials with degree $d \geq 1$, the price of anarchy is at most $\left(1 + \frac{d}{4}\right)^{d+1}$.*

Proof. We start the proof by bounding the value $\omega(\ell_a; 1, \infty; \lambda)$ from above. Recall from Equation (48) that

$$\omega(\ell_a; 1, \infty; \lambda) \leq \max_{0 \leq u \leq 1} u \left(1 - \lambda\, u^d + d\, \frac{u}{4}\right).$$

Setting $u = 1$ in the last term yields

$$\omega(\ell_a; 1, \infty; \lambda) \leq \max_{0 \leq u \leq 1} u \left(1 - \lambda\, u^d + \frac{d}{4}\right).$$

This problem is a standard concave program on a compact interval. Hence, it admits a solution. For $d \geq 1$ the objective is strictly concave implying that there exists a unique optimal solution. The necessary and sufficient optimality condition for a global optimum that satisfies $u \in (0,1)$ is given by

$$1 + \frac{d}{4} - (d+1)\,\lambda\, u^d = 0.$$

Hence, the optimal solution is given by

$$u^* = \min\left\{\max\left\{\left(\frac{4+d}{4\,\lambda\,(d+1)}\right)^{\frac{1}{d}}, 0\right\}, 1\right\}.$$

We assume $1 \leq \lambda < \infty$ which implies $0 < u^* = \frac{4+d}{4\,\lambda\,(d+1)}^{\frac{1}{d}} < 1$. Inserting this solution into the objective leads to

$$\omega(\ell_a; 1, \infty; \lambda) \leq \left(\frac{4+d}{4\,\lambda\,(d+1)}\right)^{\frac{1}{d}} \left(\frac{4\,d + d^2}{4\,(d+1)}\right).$$

We construct a function $1 \leq \lambda(d) < \infty$ such that for all $d \geq 1$ the following equation holds

$$\left(\frac{4+d}{4\,\lambda(d)\,(d+1)}\right)^{\frac{1}{d}} \left(\frac{4\,d + d^2}{4\,(d+1)}\right) = \frac{d}{d+1}.$$

Solving the above equation with respect to $\lambda(d)$ yields

$$\lambda^*(d) = \frac{(4+d)^{d+1}}{(d+1)\,4^{d+1}}.$$

Thus, by construction we have

$$\omega(\ell_a; 1, \infty; \lambda^*(d)) \leq \frac{d}{d+1}.$$

Applying Theorem 4.1 with $\lambda := \lambda^*(d)$ and $\omega(\ell_a; 1, \infty; \lambda^*(d)) \leq \frac{d}{d+1}$ leads to

$$C(\boldsymbol{f}) \leq \frac{\lambda^*(d)}{1 - \frac{d}{d+1}}\, C(\boldsymbol{x}) = (d+1)\,\lambda^*(d)\, C(\boldsymbol{x}) = \big(1 + \frac{d}{4}\big)^{d+1} C(\boldsymbol{x}).$$

□

Note that a similar technique as in the preceeding proof can be applied to strengthen the bounds on the price of anarchy. The idea is to construct a function $\lambda(d)$ such that

$$\max_{0 \leq u \leq 1} u\,\big(1 - \lambda(d)\, u^d + d\,\frac{u}{4}\big) = \frac{2}{3}$$

holds for all $d \geq 1$. Then, the price of anarchy can be bounded by $3\,\lambda(d)$. The function $\lambda(d)$ behaves asymptotically like $\Theta\big(\exp(\frac{2}{5}\,\log(d))\big)$.

The techniques used in Theorem 4.13 carry over to the general case of $n \geq 1$, that is, we consider an arbitrary number of games.

Theorem 4.14. *If the latency functions of the atomic* OnlineSRG *are polynomials with maximum degree $d \geq 1$, the online algorithm* SeqNash *is*

$$\big(1 + \frac{5}{4}\,d + \frac{1}{4}\,d^2\big)^{d+1} \quad -\textit{competitive}.$$

Proof. Let the flow $\boldsymbol{f}$ be produced by the online algorithm SeqNash and let $\boldsymbol{x}$ be an arbitrary feasible flow for the atomic OnlineSRG.

From Equation (46) in Proposition 4.11 we have the relation

$$\omega(\ell_a; \infty, \infty; \lambda) \leq \max_{0 \leq u \leq 1} u\,\big(1 - \lambda\, u^d + d\,\frac{u}{4}\big) + \frac{d}{d+1}.$$

Now, we follow along the lines of the proof of the preceeding theorem.

$$\max_{0\leq u\leq 1} u\left(1-\lambda\, u^d + d\,\frac{u}{4}\right) \leq \left(\frac{4+d}{4\,\lambda\,(d+1)}\right)^{\frac{1}{d}} \left(\frac{4\,d+d^2}{4\,(d+1)}\right).$$

We construct a function $\lambda(d)$ such that

$$\left(\frac{4+d}{4\,\lambda(d)\,(d+1)}\right)^{\frac{1}{d}} \left(\frac{4\,d+d^2}{4\,(d+1)}\right) = \frac{d}{(d+1)^2}$$

holds for all $d \geq 1$. Solving the above equation with respect to $\lambda(d)$ yields

$$\lambda^*(d) = \frac{(4+d)^{d+1}\,(d+1)^{d-1}}{4^{d+1}}.$$

Hence, by construction, we have

$$\omega(\ell_a;\infty,\infty;\lambda^*(d)) \leq \frac{d}{(d+1)^2} + \frac{d}{d+1} = \frac{d^2+2\,d}{(d+1)^2}.$$

Applying Theorem 4.1 with $\lambda := \lambda^*(d)$ and $\omega(\ell_a;\infty,\infty;\lambda^*(d)) \leq \frac{d^2+2\,d}{(d+1)^2}$ leads to

$$\begin{aligned} C(\boldsymbol{f}) &\leq \frac{\lambda^*(d)}{1-\frac{d^2+2\,d}{(d+1)^2}}\, C(\boldsymbol{x}) = (d+1)^2\,\lambda^*(d)\, C(\boldsymbol{x}) \\ &= \left(1+\frac{d}{4}\right)^{d+1} (d+1)^{d+1}\, C(\boldsymbol{x}) = \left(1+\frac{5}{4}\,d+\frac{1}{4}\,d^2\right)^{d+1} C(\boldsymbol{x}). \end{aligned}$$

□

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