Group Theoretical Mode Interactions with Different Symmetries

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Abstract: In two-parameter systems two symmetry breaking bifurcation points of different types coalesce generically within one point. This causes secondary bifurcation points to exist. The aim of this paper is to understand this phenomenon with group theory and the innerconnectivity of irreducible representations of supergroup and subgroups. Colored pictures of examples are included.

1. Introduction

Mode interaction is a typical bifurcation phenomenon for dynamical systems

$$\dot{x} = f(x, \lambda, \alpha), \quad x \in X := \mathbb{R}^n \tag{1}$$

depending on two real parameters λ and α. In connection with equilibria x_0 one distinguishes steady–state/steady–state (double zero eigenvalue of the Jacobian $D_x f(x_0, \lambda_0, \alpha_0)$), steady–state/Hopf (a zero and a pair of imaginary eigenvalues) and Hopf/Hopf mode interaction (two pairs of imaginary eigenvalues).

The situation is more complex if symmetries are present. Assuming the equivariance condition

$$f(g\,x, \lambda, \alpha) = g\,f(x, \lambda, \alpha) \qquad \forall (x, \lambda, \alpha) \in \mathbb{R}^{n+2}, \quad \forall\, g \in G, \tag{2}$$

involving a compact Lie group G, (critical) eigenvalues are in general no more simple. Moreover, one has to distinguish the different irreducible representations of G acting on the eigenspaces, hence the eigenvalues have to be classified according to their symmetry type.

Analytical results about mode interaction for specific groups and irreducible representations are obtained by singularity theory and normal forms (Golubitsky, Stewart, Schaeffer [8]). But each group has to be considered different. In contrast to these theories we present a general method which is valid for arbitrary groups and is related to numerical computations. We will focus on steady–state/steady–state mode interaction with *different* symmetry types of the zero–eigenvalues.

We are interested in analytical results that such mode interactions cause steady–state *secondary* bifurcation. The well known principle *double eigenvalues lead to secondary bifurcation* (see Bauer, Keller, Reiss [1] and Shearer [11]) can be generalized

(Werner [14]). We will recall these results and try to make them more transparent (sec. 3.).

The main aim of this paper is the investigation of the purely group theoretical interaction conditions (Def. 3.6) with group theoretical methods only (sec. 4.). The innerconnectivity between irreducible representations of supergroups and subgroups as introduced in Gatermann [6] helps to understand the mode interaction. It turns out that sufficient conditions for secondary bifurcation resulting from mode interaction can be easily visualized by *bifurcation graphs* as introduced in Dellnitz, Werner [2], Gatermann, Hohmann [4], see Fig. 2, Fig. 3, and Fig. 4. In this case the bifurcation scenario resulting from mode interaction can be described as follows. Branches with the symmetry of subgroups H and K respectively of G are emanating from the primary bifurcation points coalescing at the mode interaction point and causing secondary bifurcation on the H-branch with bifurcation symmetry $L := H \cap K$.

The interaction results will be thoroughly discussed for the dihedral group $G := D_6$, where many new mode interaction results can be derived from our interaction conditions. There are 5 nontrivial irreducible representations, and we have to investigate 20 different mode interactions. In 15 cases the group theoretical mode interaction conditions turned out to be true (sec. 5.).

This paper will be completed by numerical results for the 6–cell brusselator and a hexagonal lattice dome (see Healey [9]) with many colorful bifurcation diagrams which have been computed using SYMCON ([7]).

2. Group theoretical notations

In this section we try to summarize some important notions and standard theory. For a deeper understanding we refer to Fässler, Stiefel [3], Serre [10] and Golubitsky, Stewart, Schaeffer [8].

2.1 Linear representations

In the following G will be a compact Lie **group**. The basic notion is that of a **linear representation** δ of G acting on a real (or complex) vector space X by a group homomorphism $g \mapsto \delta(g) \in Gl(X)$. The dimension of X is the **dimension of the representation** δ.

In the following we consider *real* vector spaces X only, where for $X = \mathbb{R}^n$ all representation matrices $\delta(g)$ are assumed to be orthogonal. Hence we are concerned with *real, orthogonal, finite dimensional representations* of G. These representations will be denoted by the letters δ, D, δ_k, d_k and in the next section by ϑ, η, ϱ or γ.

The basic representation we will mostly refer to is called D by which G is acting orthogonally on the state space $X = \mathbb{R}^n$ of our dynamical system (1) via the equivariance condition (2) which may be written more precisely as

$$f(D(g)x, \lambda, \alpha) = D(g)f(x, \lambda, \alpha) \qquad \forall (x, \lambda, \alpha) \in \mathbb{R}^{n+2}, \quad \forall\, g \in G\,. \tag{3}$$

In the following we recall some notations for arbitrary linear representations $\delta : G \to Gl(X)$, giving examples for our special representation D.

A point $x \in X$ is **G–invariant** iff $\delta(g)x = x$ for all $g \in G$. A subspace U of X is **G–invariant** iff $\delta(g)u \in U$ for all $g \in G$. An example of a G–invariant space is the kernel of $D_x f$ of the dynamical system (1) at a G-invariant critical point.

A G–invariant subspace U of X induces a representation δ_U of G on U by defining $\delta_U(g)$ to be the restriction of $\delta(g)$ to U.

A G–invariant subspace U of X is **G–irreducible** iff there does not exist any nontrivial, proper G–invariant subspace of U.

A real representation δ of G on a real finite dimensional vector space X is an **irreducible representation** iff X is G–irreducible.

It follows that any G–irreducible subspace U of X induces an irreducible representation δ_U on U.

Two representations d_i on real vector spaces $V_i, i = 1, 2$ are **equivalent** iff there is an isomorphism T from V_1 to V_2 such that

$$d_1(g) = T^{-1} d_2(g) T \text{ for all } g \in G.$$

Two G–invariant subspaces $U_i, i = 1, 2$ of X are **G–isomorphic** iff their induced representations are equivalent.

For common groups G all irreducible representations of G (up to equivalence) are well known. Finite groups have a finite number of equivalence classes of irreducible representations. The irreducible representations of our group G will be denoted by δ_k or later by ϑ, η.

In [13] three different types of irreducible representations are distinguished: real, complex and quaternonian ones. In [8] the real type is called absolutely irreducible. Any one-dimensional representation d is irreducible and for each $g \in G$ either $d(g) = 1$ or $d(g) = -1$. The **trivial representation** is called δ_1 and is defined by $\delta_1(g) \equiv 1$.

For a given real representation $\delta : G \to Gl(X)$, each G–invariant subspace U of X (including X itself) is a direct sum of G–irreducible subspaces.

There is only a finite number of *not* G–isomorphic, G–irreducible subspaces U of X. Hence there exists a finite number (say N) pairwisely not equivalent irreducible representations δ_k of G of **dimension** $n_k, k = 1, \ldots, N$, such that each G–irreducible subspace U of X is of *type* δ_k for a certain k, in the sense that the induced representation d_U is equivalent to δ_k.

The number m_k of pairwisely orthogonal irreducible subspaces of the same type δ_k is called the **multiplicity** of δ_k in X or in δ. This gives the canonical decomposition

$$\delta = \sum_{k=1}^{N} m_k \delta_k, \quad m_k =: m(\delta, \delta_k). \tag{4}$$

Some irreducible representations may be not present in δ. Then $m_k = 0$ in (4).

The sum of all G–irreducible subspaces of the same type δ_k is the **isotypic component** X_k of δ_k in X,

$$X = \bigoplus_{k=1}^{N} X_k \,.$$

In the equivariance condition (3) $D : G \to Gl(X)$ is an orthogonal representation. Also D has a canonical decomposition (4).

Matrices $A \in \mathbb{R}^{n,n}$ having the **symmetry of** δ ($\delta(g)A = A\delta(g), \forall\, g \in G$) are of special interest. The Jacobians $D_x f$ at G–invariant points represent an important class of matrices having the symmetry of D. They may be thought of as block diagonal,

$$\exists \quad M \in \mathbb{R}^{n,n} \quad \text{independent of } A : \quad M^T A M = \mathrm{diag}(B_{\delta_\mathrm{k}}).$$

The columns of M form a symmetry adapted basis of X. Each block corresponds to an irreducible representation. For multiple representations of real type there is even a finer structure ($B_{\delta_\mathrm{k}} = \mathrm{diag}(A^i_{\delta_\mathrm{k}})$). The same block $A^i_{\delta_\mathrm{k}} = A_{\delta_\mathrm{k}}$ appears n_k times.

Each representation $\delta : G \to Gl(X)$ may also be considered as a restricted representation $[\delta\!\downarrow\! H] : H \to Gl(X), [\delta\!\downarrow\! H](t) := \delta(t), \forall t \in H$ for any subgroup H of G. We will also consider matrices having the symmetry of $D\!\downarrow\! H$.

2.2 Group theoretical notations in bifurcation analysis

In bifurcation theory some definitions turn out to be useful which depend on the representation D. Throughout this paper we will assume that G and $D(G)$ are isomorphic and thus $g \in G$ and $D(g)$ may be identified.

The **isotropy subgroup** G_x of $x \in X$ is

$$G_x := \{g \in G : g\,x = x\}.$$

G_x can be considered as the symmetry of x. Not necessarely each subgroup H of G is an isotropy subgroup in the sense that there is an $x \in X$ with $H = G_x$.
The **isotropy subgroup** G_U of a *subspace* $U \subset X$ is defined by

$$G_U := \{g \in G : g\,u = u \quad \text{for all } u \in U\}.$$

For $x \in U$ we have $G_U \subset G_x$, not necessarely $G_U = G_x$. But we have $G_U = G_x$ for at least on $x \in U$, hence G_U is indeed an isotropy subgroup.

The **fixed point space** $X^H \subset X$ of a subgroup H of G is

$$X^H := \{x \in X : hx = x \text{ for all } h \in H\},$$

the set of all x having at least the symmetry of H.

X^H equals the first isotypic component X_1 if $X = \oplus_{k=1}^m X_k$ denotes the decomposition with respect to H.

The **normalizer** $N_G(H)$ of a subgroup H of G is defined by

$$N_G(H) := \{g \in G : gHg^{-1} = H\}.$$

A subgroup H of G is a **normal subgroup** of G if $N_G(H) = G$.

Two subgroups H and H' of G are **conjugate** iff $H' = gHg^{-1}$, for some $g \in G$.
If H is the isotropy group of x then $H' = H_{gx}$ for a **conjugate** vector gx.

Lemma 2.1 *Let U be a G–invariant subspace of X and let $K = G_U$ be the symmetry of U. Then K is a normal subgroup of G.*

Proof: Because K is the isotropy subgroup of U the fixed point space U^K is U itself. For $g \in G$ and $k \in K$ we define $k' := g^{-1}kg$. For K being normal in G we have to show that $k' \in K$. For each $u \in U$ we have $gu \in U$ and thus $k'u = g^{-1}kgu = u$. Because G operates faithful on X and $K = G_U$ one derives $k' \in K$. ∎

Definition 2.2 *A subgroup H of G is called a* **symmetry** *in a subspace U of X iff there is a nontrivial $u \in U$ with $H = G_u$.*
A symmetry H in U is called **maximal** *iff there is no larger symmetry than H in U.*

A maximal symmetry in U is always an isotropy subgroup G_u for a suitable $u \in U$.

A special case for a maximal symmetry in U occurs if there is a symmetry H in U such that $\dim U \cap X^H = 1$. Such a symmetry is called a **bifurcation symmetry** or a **bifurcation subgroup** in U.
A maximal, (respectively a bifurcation–) symmetry H in an irreducible subspace U of X of type δ_k depends only on δ_k and is called a **maximal**, respectively a **bifurcation symmetry** or a **bifurcation subgroup for** δ_k.
Bifurcation symmetries are important in connection with the *equivariant branching lemma* (see [8] and below).

Concerning bifurcation symmetries we recall (see Dellnitz–Werner [2]).

Theorem 2.3 *a) If $\dim \delta_k = 1$ then there exists a unique bifurcation symmetry H for δ_k being a normal subgroup of G and satisfying $G/H \cong \mathbb{Z}_2$.*

b) If a bifurcation symmetry for δ_k is normal, then $\dim \delta_k = 1$.

c) Let $\dim\ \delta_k > 1$. Then every bifurcation symmetry H for δ_k is not normal and satisfies

$$N_G(H) = H \quad \text{or} \quad N_G(H)/H \cong \mathbb{Z}_2.$$

There exists at least one bifurcation symmetry H' for δ_k different from H, but conjugate to H (that means: $H' = gHg^{-1}$ for some $g \in G$.)

All subgroups H' being conjugate to H are bifurcation symmetries for δ_k.

If $N_G(H) = H$ we call H **asymmetric** in G. If $N_G(H)/H = \mathbb{Z}_2$ then H is called **symmetric**.

Remark: For many finite groups (dihedral groups for instance) every maximal symmetry for δ_k is a bifurcation symmetry if δ_k is of *real type*.
In the context of mode interaction the question arises under which circumstances a bifurcation symmetry for δ_k is still a bifurcation symmetry for $\delta_k + \delta_i$ with non-isomorphic irreducible representations δ_k and δ_i.

3. Mode interaction analysis

The aim of this section is to present the interaction results in [14] in a transparent way. We will choose a slightly different approach than in [14] to make the simple geometrical principle behind these results more clearly.
Our starting interest is in steady state bifurcation of the dynamical system (1) depending on two parameters λ and α and being G–equivariant (3).

Let $x_0 \in X^G$ be a G–symmetric equilibrium ($x_0 \in X^G$) for $\lambda = \lambda_0$ and $\alpha = \alpha_0$. Steady-state bifurcation occurs if the kernel N^0 of the Jacobian $f_x^0 := D_x f(x_0, \lambda_0, \alpha_0)$ is nontrivial or if $\mu = 0$ is a critical eigenvalue of f_x^0. The equivariance implies that N_0 is G–invariant. If one of the two parameters are fixed — say α — generically N_0 is G–irreducible of some *absolutely irreducible type* ϑ, see [8]. Bifurcation of branches with isotropy group H in (x_0, λ_0) can be guaranteed for any bifurcation symmetry H for ϑ, if a certain eigenvalue crossing condition is satisfied (*equivariant branching lemma*).

Mode interaction we are interested in, is defined by

$$N_0 = U + V,$$

where U and V are G–irreducible subspaces of X of *different* nontrivial types ϑ and η. For two parameter problems this mode interaction will occur generically.

3.1 ϑ–η mode interaction

To formulate the interaction results we give the following definition which slightly differs from the comparable Def.5.1 in [14].

Definition 3.4 *The G–symmetric equilibrium x_0 for $\lambda = \lambda_0, \alpha = \alpha_0$ is called a G–* **semisimple** *steady–state bifurcation point of* **symmetry type** ϑ *if there is precisely one G–irreducible subspace U of type ϑ in the kernel N_0.*
*If $N_0 = U$, x_0 is called a G–***simple bifurcation point of symmetry type** ϑ.

If there is at least one irreducible subspace U of N_0 of type ϑ, x_0 is called a **potential** *steady state bifurcation point of symmetry type ϑ.*

We have chosen slightly weaker requirements than in Def.5.1. in [14] by neglecting eigenvalue crossing (transversality) conditions and possible steady–state/Hopf mode interactions.

Now the mode interaction above can be redefined:

Definition 3.5 *A point $(x_0, \lambda_0, \alpha_0)$ is called ϑ–η–mode interaction point if x_0 is a G–semisimple bifurcation point of nontrivial symmetry types ϑ* **and** *η ($N_0 = U + V$) and if no other irreducible representations of G are present in the kernel.*

Since the irreducible representations ϑ and η are nontrivial, we can draw the first conclusion: there exists a two-dimensional manifold

$$\mathcal{C}^G = \{(x_G(\lambda, \alpha), \lambda, \alpha) : |\lambda - \lambda_0| < \varepsilon, |\alpha - \alpha_0| < \varepsilon\} \subset X^G \times \mathbb{R}^2, \quad \varepsilon > 0 \tag{5}$$

of G–symmetric equilibria of $\dot{x} = f(x, \lambda, \alpha)$ (for fixed α we have G–branches of equilibria we will refer to).

3.2 Analysis

Being interested in symmetry breaking bifurcation from $\mathcal{C}^G$, the first mode interaction condition enters:

Assume that there exists a bifurcation symmetry H in U *and* in $N_0 = U + V$. **(IC$_1$)**

Restricting the search for equilibria to those with at least the symmetry H, the Lyapunov Schmidt method ends up with a *scalar* bifurcation equation

$$g(u, \lambda, \alpha) = 0, \quad u \in \mathbb{R}, \quad g(u, \lambda, \alpha) \in \mathbb{R}, \quad g(0, \lambda, \alpha) \equiv 0, \quad g_u(0, \lambda_0, \alpha_0) = 0. \tag{6}$$

More precisely, locally (in neighborhoods of $(x_0, \lambda_0, \alpha_0)$ and $(0, \lambda_0, \alpha_0)$) there is a unique correspondence between solutions of $f(x, \lambda, \alpha) = 0$ with $x \in X^H$ and solutions of the scalar bifurcation equation $g(u, \lambda, \alpha) = 0$. Here $u = 0$ corresponds with G–symmetric equilibria in $\mathcal{C}^G$, and $u \neq 0$ with equilibria $x \in X^H \setminus X^G$.

The transversality conditions which guarantee structural stable symmetry breaking bifurcation can now be formulated in terms of g and even simpler in terms of h, where

$$g(u, \lambda, \alpha) = u\, h(u, \lambda, \alpha).$$

Treating λ as the primary bifurcation– and α as a control or imperfection parameter, this transversality condition is

$$c_\lambda := g^0_{u\lambda} = h^0_\lambda \neq 0. \tag{7}$$

c_λ can be expressed by the original function f, see (6.13) in [14], but observe the different notations.

Under the assumption (7) the Implicit Function Theorem yields a two–dimensional manifold $\{(u, \lambda(u, \alpha), \alpha) : |u| < \varepsilon, |\alpha - \alpha_0| < \varepsilon\}$ of nontrivial solutions of $g = 0$ which corresponds to a two–dimensional manifold

$$\mathcal{C}^H = \{(x_H(u, \alpha), \lambda(u, \alpha), \alpha) : |u| < \varepsilon, |\alpha - \alpha_0| < \varepsilon\} \subset X^H \times \mathbb{R}^2, \quad \varepsilon > 0 \tag{8}$$

of H–symmetric equilibria of $\dot{x} = f(x, \lambda, \alpha)$ (for fixed α we have the H–branches we will refer to).

Moreover, by applying the Implicit Function Theorem to $g_u(0, \lambda, \alpha) = h(0, \lambda, \alpha) = 0$ using (7) again, the existence of a unique branch of symmetry–breaking bifurcation points (from G to H)

$$\mathcal{C}_\vartheta = \{(x_\vartheta(\alpha), \lambda_\vartheta(\alpha), \alpha) : |\alpha - \alpha_0| < \varepsilon\} \subset X^G \times \mathrm{I\!R}^2, \quad \varepsilon > 0 \tag{9}$$

given by

$$g_u(0, \lambda_\vartheta(\alpha), \alpha) \equiv 0$$

is proved.
It can be shown that $\mathcal{C}_\vartheta$ consists of G–semisimple bifurcation points of type ϑ.

In the next step we present conditions under which there is also a branch

$$\mathcal{C}_\eta = \{(x_\eta(\alpha), \lambda_\eta(\alpha), \alpha) : |\alpha - \alpha_0| < \varepsilon\} \subset X^G \times \mathrm{I\!R}^2, \quad \varepsilon > 0 \tag{10}$$

of G–semisimple bifurcation points of type η together with a branch

$$\mathcal{C}_\varrho = \{(x_\varrho(\tau), \lambda_\varrho(\tau), \alpha(\tau)) : |\tau| < \varepsilon)\} \subset \mathcal{C}^H, \quad \varepsilon > 0 \tag{11}$$

of *secondary* potential bifurcation points of H–symmetry type ϱ resulting from the mode interaction. Note that $\varrho : H \to Gl(W)$ is an irreducible representation of H, but ϑ and η are irreducible representations of the supergroup G. The relations between ϑ, η and ϱ will become important.

3.3 Test function

To this end the concept of *test functions* being used in the numerical analysis of bifurcation problems (as presented in [14]) is helpful:

Our condition is that there is a scalar smooth (test) function $t(x, \lambda, \alpha)$ defined for $x \in X^H$ such that $t(x, \lambda, \alpha) = 0$ is an equivalent condition for x being a potential bifurcation point of H–symmetry type ϱ. Especially, for $x \in X^G$ we assume that $t(x, \lambda, \alpha) = 0$ is equivalent for x being a potential bifurcation point of G–symmetry type η.

This condition can be completely formulated in terms of representation theory, see sec. 3.4.

After applying the Lyapunov Schmidt method again we analyse the primary η–bifurcation and the secondary ϱ–bifurcation by studying the system of two scalar equations

$$g(u, \lambda, \alpha) = 0, \quad \tau(u, \lambda, \alpha) = 0\,. \tag{12}$$

Here $\tau(u, \lambda, \alpha)$ is the Lyapunov-Schmidt analougue to $t(x, \lambda, \alpha)$.

Now trivial solutions $u = 0$ of (12), and hence solutions of $\tau(0, \lambda, \alpha) = 0$ (remember that $g(u, \lambda, \alpha) \equiv 0$) correspond to bifurcation points of type η on the G–manifolds $\mathcal{C}^G$.
Nontrivial solutions $u \neq 0$ of (12) correspond to bifurcation points of type ϱ on the H–manifold $\mathcal{C}^H$.

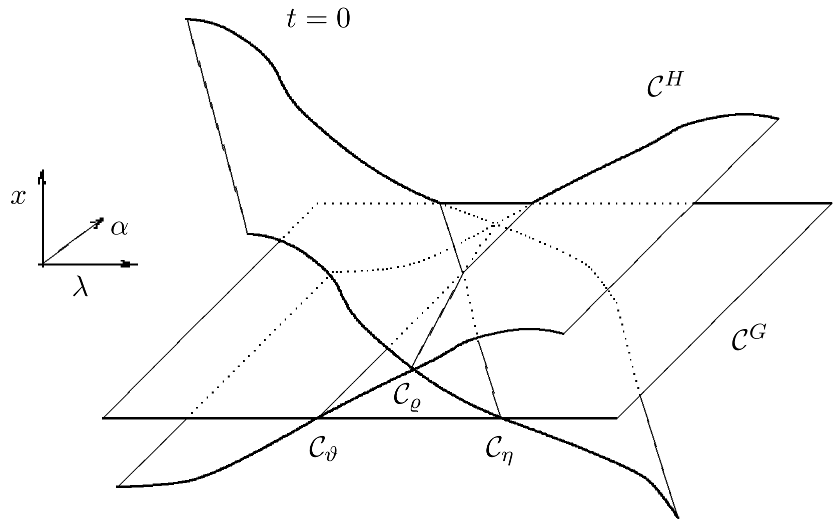

Figure 1: Manifolds of G– and H–invariant solutions and level surface zero of test function t. Intersections are (primary ($\mathcal{C}_\vartheta, \mathcal{C}_\eta$) and secondary ($\mathcal{C}_\varrho$)) symmetry breaking bifurcation points.

Using Implicit Function techniques, it is an easy task to find out that the following two conditions have to be required:

$$d_\lambda := t_\lambda^0 := t_\lambda(0, \lambda_0, \alpha_0) \neq 0 \tag{13}$$

and

$$D(c,d) := \mathrm{Det}\begin{pmatrix} c_\lambda & c_\alpha \\ d_\lambda & d_\alpha \end{pmatrix} \neq 0, \tag{14}$$

where c_α and d_α are partial derivatives w.r.t. α in analogy to (7) and (13).

The first condition (13) ensures the existence of the branch $\mathcal{C}_\eta$ in (10), while the second condition (14) yields the branch $\mathcal{C}_\varrho$ in (11). Condition (14) expresses the fact that the λ–speed of the ϑ and the η–bifurcation points w.r.t. α is different (see also [14], (7.19)). This interpretation allows the conclusion that before and after mode interaction ($\alpha \neq \alpha_0$), the ϑ– and the η–bifurcation points are seperated and hence are G–simple.

The two one–dimensional manifolds $\mathcal{C}_\varrho$ and $\mathcal{C}_\eta$ are just the intersections of a codimension-1– manifold defined by $t(x, \lambda, \alpha) = 0$ with the two–dimensional manifolds $\mathcal{C}^H$ and $\mathcal{C}^G$ respectively. In Fig. 1 the situation is shown where before and after the mode interaction the branch $\mathcal{C}_\varrho$ exists. But it is also possible that the branch exists only before mode interaction or only after mode interaction (see Fig. 7).

It should be remarked that further generic conditions can be formulated to guarantee that the potential bifurcation points of H–symmetry type ϱ are H–simple bifurcation points.

3.4 Group theoretical mode interaction conditions

What are the group theoretical conditions for the assumption in sec. 3.3?
We recall that U and V in $N_0 = U + V$ are irreducible subspaces of X of *different* (absolutely irreducible) types ϑ and η respectively and that H is a bifurcation symmetry for ϑ.

The following **interaction conditions (IC)** imply the assumption in sec. 3.3, the existence of a test function. This depends on the way how ϑ and η decompose being now considered as (not necessarely irreducible) representations of H.

Let ϱ_1 be the trivial and ϱ any nontrivial irreducible representation of H. For the following the multiplicities $m_1(\eta{\downarrow}H) := m(\eta{\downarrow}H, \varrho_1)$ and

$$m_\varrho(\vartheta{\downarrow}H) := m(\vartheta{\downarrow}H, \varrho), \quad m_\varrho(\eta{\downarrow}H) := m(\eta{\downarrow}H, \varrho)$$

enter (compare with (4)). Since H is a bifurcation subgroup for ϑ we have $m_1(\vartheta{\downarrow}H) = 1$ or equivalently

$$\dim\,(U \cap X^H) = 1.$$

Definition 3.6 *Let ϑ and η be irreducible representations of G, let H be a bifurcation symmetry of ϑ and let ϱ be a nontrivial irreducible representation of H. We say that the* **interaction condition IC**$(\vartheta, \eta, H, \varrho)$ *holds iff*

- $m_1(\vartheta{\downarrow}H) = 1$ *and* $m_1(\eta{\downarrow}H) = 0$
  (equivalently H is a bifurcation symmetry in $U + V$ and in U as required in (IC$_1$)). **(IC)**
- $m_\varrho(\eta{\downarrow}H) = 1$ *and* $m_\varrho(\vartheta{\downarrow}H) = 0$.

We will use the following notions:

IC(ϑ, η, H) *holds iff there exists ϱ such that $IC(\vartheta, \eta, H, \varrho)$ is true.*

ϑ interacts with η via H, if $IC(\vartheta, \eta, H)$ is true.

We say that the nontrivial irreducible representation ϱ of H **satisfies the interaction condition** in Def. 3.6.

If ϑ interacts with η via H, then it is in general not true that η interacts with ϑ via a bifurcation symmetry K for η (if such K exists).

We sketch the proof in [14] that (IC) implies the validity of the assumptions in sec. 3.3 in the case of an absolutely irreducible ϱ. We present briefly the idea based on block diagonalization w.r.t. to a symmetry adapted basis. The Jacobian $J := f_x(x, \lambda, \alpha)$ evaluated at an H–invariant point has the symmetry of $D{\downarrow}H$ ($D(h)J = JD(h)\,\forall h \in H$). Evaluated at an G–invariant point the Jacobian has the symmetry of G. In section 2.1 it was mentioned that such matrices may be transformed to block diagonal structure. Because we consider two groups H and G, we consider two block diagonal structures.

Even worse the Jacobian evaluated at an G–invariant point has the symmetry of G **and** of H. Thus the relation between the G–block structure and the rougher H–block structure has to be taken into account. On the G–manifold $\mathcal{C}^G$, the transformed Jacobians have (among others) the G–blocks A_ϑ and A_η. Zero-determinants of A_ϑ and A_η characterize bifurcation points of type ϑ and η, respectively. In the mode interaction point simultaneously $\det A_\vartheta = \det A_\eta = 0$ with rank deficiency one.

On the H-manifold $\mathcal{C}^H$ we are interested in the H–blocks A_1 and A_ϱ of the transformed Jacobian where A_1 and A_ϱ correspond to the trivial representation and the absolutely irreducible representation ϱ of H, respectively satisfying (IC). The test function in sec. 3.3 may be defined as $t(x, \lambda, \alpha) = \det A_\varrho$. For points with isotropy H it characterizes bifurcation points of type ϱ. In the mode interaction point (which has isotropy G !) the blocks A_1 and A_ϱ become simultaneously singular. The interaction condition assures that the rank deficiency one of A_η corresponds to a rank deficiency one of A_ϱ.

3.5 The mode interaction theorem

For completeness we now state the mode interaction theorem in [14], Th.7.1, in our language:

Theorem 3.7 *Assume that*

- *there is a steady state ϑ–η–mode interaction at $(x_0, \lambda_0, \alpha_0)$ with nontrivial absolutely irreducible representations ϑ and η and corresponding G–irreducible subspaces U, V in $N_0 = U + V$, see Def. 3.5,*
- *there is a subgroup H of G and a nontrivial absolutely irreducible representation ϱ of H satisfying the interaction condition $IC(\vartheta, \eta, H, \varrho)$ in Def. 3.6.*
- *the transversality conditions (7), (13) and (14), $c_\lambda \neq 0$, $d_\lambda \neq 0$ and $D(c, d) \neq 0$, hold.*

Then there exist two–dimensional manifolds $\mathcal{C}^G$ (5) and $\mathcal{C}^H$ (8) of G–symmetric and H–symmetric equilibria intersecting in the one–dimensional manifold $\mathcal{C}_\vartheta$ (9) of G–semisimple bifurcation points of type ϑ.

Moreover, there are two one–dimensional manifolds $\mathcal{C}_\eta \subset \mathcal{C}^G$ (10) and $\mathcal{C}_\varrho \subset \mathcal{C}^H$ (11) of G–semisimple bifurcation points of type η and of potential bifurcation points of H–symmetry type ϱ.
The G–semisimple bifurcation points of type ϑ and η are G–simple for $\alpha \neq \alpha_0$.

All these manifolds intersect in the mode interaction point $(x_0, \lambda_0, \alpha_0)$.

Remark 3.8 *A similar result for steady state (ϑ) – Hopf (η) mode interaction holds. Then there appear secondary Hopf–bifurcations of symmetry type ϱ on the H–branches. For this the condition that η and ϱ have to be absolutely irreducible can be dropped.*
The main point in the proof is to express a Hopf bifurcation by a scalar equation $t(x, \lambda, \alpha) = 0$. See [14].

There remain several open questions concerning Hopf/steady-state and Hopf/Hopf mode interaction with different symmetry types.

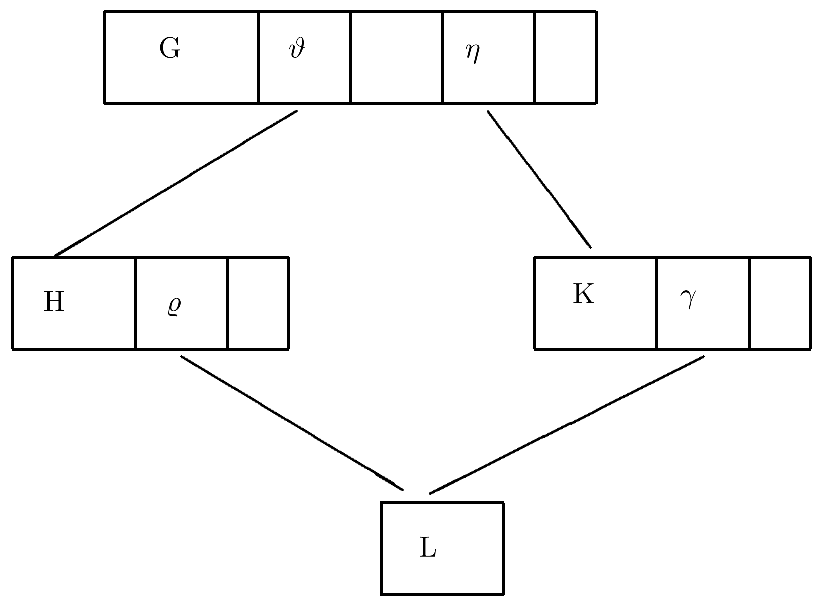

Figure 2: Situation of Theorem 4.12: ϑ and η act symmetrically.

4. Investigation of interaction conditions (IC)

The aim of this section is to give sufficient and necessary conditions for (IC) to hold. To this end we play around with the irreducible representation ϑ of G, a bifurcation symmetry H for ϑ, an irreducible representation ϱ of H and a bifurcation symmetry L (subgroup of H) for ϱ, and analogously with the irreducible representation η of G, a bifurcation symmetry K for η, an irreducible representation γ of K with bifurcation symmetry L' (a subgroup of K) for γ.

4.1 Symmetric and asymmetric interaction

The irreducible representations ϑ and η cannot be interchanged in Th. 3.7. But in several cases the theorem holds twice. It is valid for both orderings of ϑ and η.

Definition 4.9 *We say that ϑ and η* **interact symmetrically** *iff there exist bifurcation symmetries H for ϑ, and K for η and nontrivial irreducible representations ϱ (of H) and γ (of K) such that $IC(\vartheta, \eta, H, \varrho)$* **and** *$IC(\eta, \vartheta, K, \gamma)$ hold with $L := H \cap K$ being a bifurcation symmetry for ϱ and γ as well.*

Fig.2 shows part of a bifurcation graph where the situation of Def.4.9 is fulfilled. In the case of symmetric mode interaction we expect a H–branch (ϑ–bifurcation) and a K–branch (η–bifurcation) bifurcating from the primary G–branch with possible $H \cap K$–branches bifurcating from the H–branch and from the K–branch in a secondary ϱ or γ–bifurcation.

Fig. 3 shows a second possibility where only $IC(\vartheta, \eta, H)$ holds, but not $IC(\eta, \vartheta, K)$.

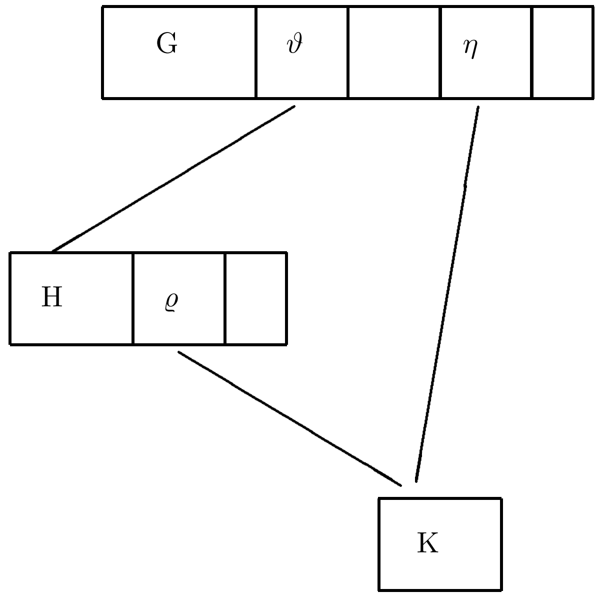

Figure 3: Situation of Theorem 4.11.

4.2 Sufficient interaction conditions

Let H be a bifurcation symmetry for ϑ. It is clear that $IC(\vartheta, \eta, H)$ does not hold if $m_1(\eta{\downarrow}H) \geq 1$ holds.

For this whole subsection we assume that H is a bifurcation symmetry in $U + V$ and that K is a bifurcation symmetry in V (w.r.t. G) and we set $L := H \cap K$.

We will see (Theorem 4.10) that one of the following three equivalent conditions will be sufficient for $IC(\vartheta, \eta, H)$ to hold:

- $$m_1(\vartheta{\downarrow}L) = 1 = m_1(\eta{\downarrow}L). \tag{15}$$
- $$\dim\,(U \cap X^L) = 1 = \dim\,(V \cap X^L). \tag{16}$$
- $$U \cap X^L = U \cap X^H, \quad V \cap X^L = V \cap X^K. \tag{17}$$

(17) says the following: looking for elements in U (or in V) with less symmetry L than H (or K) we do not find any new ones.

Theorem 4.10 *Assume that H is a bifurcation symmetry in $U + V$ (not only in U). Let one of the three equivalent conditions (15)–(17) keep true.*
Then there exists an irreducible representation ϱ of H such that $IC(\vartheta, \eta, H, \varrho)$ holds true with bifurcation symmetry $L = H \cap K$ for ϱ.

PROOF: Let v_K span the one–dimensional subspace $V^K := V \cap X^K$ of V. Let $V^\varrho \subset V$ be the H–irreducible space of type ϱ containing V^K.
Because of (15), v_K has still the maximal bifurcation symmetry L for ϱ w.r.t. H. ϱ is nontrivial (because of $X^H \cap V = \{0\}$) and $m_\varrho(\eta{\downarrow}H) \geq 1$. We can exclude $m_\varrho(\eta{\downarrow}H) > 1$, since the dimension of $V \cap X^L$ is one.

It remains to show that $m_\varrho(\vartheta{\downarrow}H) = 0$. If not, we have a nontrivial $u_1 \in U$ with the symmetry of L, but not of H. Since H is a bifurcation symmetry in U there is an $u_2 \in U$ having the symmetry H and being independent of u_1. Hence the two–dimensional space spanned by u_1 and u_2 is contained in X^L contradicting $\dim(U \cap X^L) = 1$. ∎

The following Theorems 4.11, 4.12, and 4.14 can be derived from Theorem 4.10. First we make the further assumption that K is a **proper** subgroup of H.
Then $L = K$ and K is a bifurcation symmetry for η, the irreducible representation of G and for ϱ, the irreducible representation of H, as well. The bifurcation graph of G acting on X contains the subgraph shown in Fig.3.
The right hand side of (15) is trivially fulfilled. The left hand side is responsible for $m_\varrho(\vartheta{\downarrow}H) = 0$.

Now we can state

Theorem 4.11 *Assume that H is a bifurcation symmetry in $U + V$ (not only in U). Assume there is a bifurcation symmetry K in V (for η, w.r.t. G) being a* **proper** *subgroup of H. Then the following holds.*

a) There exists a nontrivial irreducible representation ϱ of H such that $m_\varrho(\eta{\downarrow}H) = 1$ and $L := K$ is a bifurcation symmetry for ϱ (w.r.t. H).

b) If the dimension of ϑ is ≤ 2, then (15) and hence (by Th. 4.10) $IC(\vartheta, \eta, H, \varrho)$ holds.

c) $IC(\eta, \vartheta, K)$ is not valid.

PROOF: a) as in the proof of Th.4.10.

b) If $\dim \vartheta = 1$, then $\dim U = 1$ and therefore $\vartheta{\downarrow}H = \varrho_1$ and $m_\varrho(\vartheta{\downarrow}H) = 0$.

Let $\dim \vartheta = 2$. Suppose that $m_\varrho := m(\vartheta{\downarrow}H, \varrho) > 0$. Then $m_\varrho = 1$ and $K = G_U$. This implies that K is a normal subgroup of G (by Lemma 2.1).
If K is a normal bifurcation subgroup for η, then $\dim \eta = 1$ and $G/K = \mathbb{Z}_2$ (Th.2.3). This is a contradiction to K being a proper subgroup of H, since H is a (non normal) subgroup of G due to Th.2.3.

c) Since $K \subset H$ and $m_1(\vartheta{\downarrow}H) = 1$, it follows that $m_1(\vartheta{\downarrow}K) \geq 1$. ∎

Looking at bifurcation graphs, the assumptions of Theorem 4.11 can be easily checked, see Fig.3, where we refer to H and K as bifurcation subgroups of *level 1* (for ϑ and η, irreducible representations of G) and to K as bifurcation subgroup of *level 2* (for ϱ, irreducible representation of H).

The interaction leads to a K–branch bifurcating from a H–branch which bifurcates from a primary G–branch. It is possible that the K–branch is connected via the secondary ϱ-bifurcation with the G–branch by a primary η-bifurcation.
See the examples in sec.5.

Theorem 4.12 *Let ϑ and η be (inequivalent) one-dimensional irreducible representations of G. Then ϑ and η interact symmetrically.*

PROOF: There are unique bifurcation subgroups H and K for ϑ and η, with $G/H \cong \mathbb{Z}_2 \cong G/K$, but $H \neq K$ (Th.2.3). Then $\eta \downarrow H$ is still a one–dimensional, nontrivial irreducible representation of H which we call ϱ while $\vartheta \downarrow H$ is the trivial representation of H. ∎

A typical example is the Kleinian group $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with three nontrivial one-dimensional irreducible representations and pairwise symmetric mode interactions (here $L := H \cap K$ is the trivial group).

Remark 4.13 *It is possible to reduce Th.4.12 to the last (well known) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$– mode interaction case by considering L–symmetric equilibria only, $L = H \cap K$.*

We can generalize Th.4.12 in the spirit of Th.4.10:

Theorem 4.14 *Assume that both, H and K, are bifurcation symmetries in $U + V$, (w.r.t. G). Set $L := H \cap K$ and assume as above that one of the equivalent conditions (15)-(17) holds. Then ϑ and η interact symmetrically.*

Remark 4.15 *Considering L–symmetric equilibria only, there is a double eigenvalue at the mode interaction point. Hence Th.4.14 expresses the well known principle that double eigenvalues lead to secondary bifurcations, mentioned in the Introduction.*

4.3 A negative criterion

The following theorem gives a negative criterion for the interaction condition

Theorem 4.16 *Let $H = \mathbb{Z}_2$ and $dim\ \vartheta \geq 2$. Then $IC(\vartheta, \eta, H)$ cannot be satisfied whatever η is.*

PROOF: H has only two irreducible representations, the trivial ϱ_1 and the nontrivial ϱ. Necessarely $m_\varrho := m_\varrho(\vartheta \downarrow H) = dim\, \vartheta - 1 \geq 1$ in contradiction to Def.3.6. ∎

4.4 Conjugate interaction condition

For the following definition see also [6].

Definition 4.17 *Let H be any proper subgroup of G and $g \in G \setminus H$. Set $H' := gHg^{-1}$ (H' is another subgroup of G conjugate to H).*
Let ϱ be any representation of H acting on U.
Then

$$\varrho^g : H' \to Gl(U),\, h' \mapsto \varrho(g^{-1}h'g)$$

defines a linear representation of H' on U which is called a representation **conjugate** *to ϱ.*

It is easy to see that ϱ^g is an irreducible representation of H' iff ϱ is an irreducible representation of H.
By use of Th. 2.3 one gets immediately

Theorem 4.18 *The interaction conditions $IC(\vartheta, \eta, H, \varrho)$ and $IC(\vartheta, \eta, H', \varrho^g)$ are equivalent.*

The preceding theorems, Th. 4.11, 4.12, 4.14, 4.16, 4.18 may help to decide in a concrete case whether $IC(\vartheta, \eta, H)$ holds or not.

The Theorem 4.18 has the consequence that in special cases the branch of secondary bifurcation points exists either before ($\alpha < \alpha_0$) or after ($\alpha > \alpha_0$) the mode interaction for $\alpha = \alpha_0$.

Theorem 4.19 *Let H be a bifurcation symmetry for ϑ and a symmetric subgroup of G. Let $IC(\vartheta, \eta, H, \varrho)$ be satisfied with an absolutely irreducible ϱ and assume that for $g \in N_G(H)/H$ the conjugate representation ϱ^g equals ϱ. Assume that in $(x_0, \lambda_0, \alpha_0)$ two branches $\mathcal{C}_\vartheta, \mathcal{C}_\eta$ of primary bifurcation points intersect. Let as in Theorem 3.7 the transversality conditions (7), (13), (14) keep true. Then there exists a branch of secondary bifurcation points*

$$\mathcal{C}_\varrho = \{(x_\varrho(\tau), \lambda_\varrho(\tau), \alpha(\tau)) : |\tau| < \varepsilon)\} \subset \mathcal{C}^H, \quad \varepsilon > 0,$$

where either $\alpha(\tau) \geq \alpha_0 \,\forall\, \tau$ or $\alpha(\tau) \leq \alpha_0 \,\forall\, \tau$.

Proof: The existence of a branch $\mathcal{C}_\varrho$ follows from Th. 3.7. It remains to show that either $\alpha(\tau) \geq \alpha_0 \,\forall\, \tau$ or $\alpha(\tau) \leq \alpha_0 \,\forall\, \tau$. Because H is symmetric, the solutions (x, λ, α) of $f(x, \lambda, \alpha) = 0$ with H-invariant $x \in X^H$ have a special property. For x with isotropy $H = G_x$ and for $g \in N_G(H) - H$ we have $gx \neq x$. A solution (x, λ, α) with $H = G_x$ corresponds to another solution (gx, λ, α). Only for $x \in X^H$ with isotropy $G = G_x$ the two points coalesce.
For $x \in X^H$ define the test function $t(x, \lambda, \alpha) = \det A_\varrho(x, \lambda, \alpha)$ where $A_\varrho(x, \lambda, \alpha)$ corresponds to the ϱ-block of the Jacobian $f_x(x, \lambda, \alpha)$ transformed with respect to H. Since $f_x(gx, \lambda, \alpha) = gf_x(x, \lambda, \alpha)g^{-1}$ we have $t(gx, \lambda, \alpha) = \det A_{\varrho^g}(x, \lambda, \alpha)$ for ϱ^g defined in Definition 4.17. By assumption $\varrho^g = \varrho$ and thus

$$t(gx, \lambda, a) = t(x, \lambda, \alpha) \quad \text{for} \quad g \in N_G(H)/H\,.$$

The test function is invariant w.r.t. $N_G(H)$. Restricting to $x \in X^H$ the mode interaction point $(x_0, \lambda_0, \alpha_0)$ is a turning point of the system $f(x, \lambda, \alpha) = 0, \quad t(x, \lambda, \alpha) = 0.$ ∎

Note the restriction for the branch of secondary bifurcation points in contrast to Th. 3.7 and equation (11). An example is given in Fig. 7.

If H is an asymmetrical subgroup of G we expect that generically the branch of secondary bifurcation points exists before and after the mode interaction.

4.5 Symmetric and asymmetric bifurcation subgroups of level 2

We distinguished 2 different types of mode interaction (see Fig. 2 and Fig. 3).

The mode interaction results in sec. 4.1-4.3 do not depend on H or L being symmetric subgroups or asymmetric subgroups. But indeed not all combinations are possible.

Lemma 4.20 *Let H be a bifurcation symmetry for ϑ and for $\vartheta + \eta$. Let $IC(\vartheta, \eta, H, \varrho)$ holds and let $K = L$ be a bifurcation symmetry for η and ϱ. If L is asymmetrical in G then L is asymmetrical in H as well.*

PROOF: The assumptions imply $N_G(L) = L$. Since $N_H(L) \subseteq N_G(L)$ we have $N_H(L) = L$ and thus L is asymmetrical in H. ∎

As an example we present now a complete discussion of the dihedral group $G = D_6$.

5. An example: $G = D_6$

Let

$$G = D_6 = \{I, R, R^2, R^3, R^4, R^5, S_1, S_1', S_1'', S_2, S_2', S_2''\},$$

be the dihedral group D_6. In [4] the bifurcation subgroups are shown in a bifurcation graph. Because we will refer to the notations, the figure is included (Fig. 4), compare also with Fig. 11 in DELLNITZ–WERNER [2]. ϑ^5 and ϑ^6 are the two-dimensional irreducible representations of D_6 where

$$\vartheta^5(R) = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \quad \text{and} \quad \vartheta^6(R) = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}.$$

Table 1 shows 15 situations in which the mode interaction condition (IC) holds and gives the theorems which applies to this situation.
In Table 1, $\varrho^2, \varrho^3, \varrho^4$ are certain irreducible representations of various H, not all of which we define precisely. For $H = D_3^1$ (or D_3^2), ϱ^3 denotes the two-dimensional irreducible representation. For $H = K_6^1(K_6^2, K_6^3)$ ϱ^3 and ϱ^4 denote two nontrivial one-dimensional irreducible representations.

$\vartheta \ddots \eta$	ϑ^2	ϑ^3	ϑ^4	ϑ^5	ϑ^6
ϑ^2		$\diamond$ $H = C_6$ $\varrho = \varrho^2$ $K = D_3^1$ $L = C_3$	$\diamond$ $H = C_6$ $\varrho = \varrho^2$ $K = D_3^2$ $L = C_3$	C $H = C_6$ $\varrho = \varrho^3$	C $H = C_6$ $\varrho = \varrho^4$
ϑ^3	$\diamond$ $H = D_3^1$ $\varrho = \varrho^2$ $K = C_6$ $L = C_3$		$\diamond$ $H = D_3^1$ $\varrho = \varrho^2$ $K = D_3^2$ $L = C_3$	$\lhd$ $H{=}D_3^1$ $\varrho{=}\varrho^3$ $K{=}L$ $={Z_2^0}(Z_2^2, Z_2^4)$	$\Diamond$ $H{=}D_3^1$ $\varrho{=}\varrho^3$ $K{=}K_6^1(K_6^2, K_6^3)$ $L{=}Z_2^1(Z_2^2, Z_2^4)$
ϑ^4	$\diamond$ $H = D_3^2$ $\varrho = \varrho^2$ $K = C_6$ $L = C_3$	$\diamond$ $H = D_3^2$ $\varrho = \varrho^2$ $K = D_3^1$ $L = C_3$		$\lhd$ $H{=}D_3^2$ $\varrho{=}\varrho^3$ $K{=}L$ $={Z_2^1}(Z_2^3, Z_2^5)$	$\Diamond$ $H{=}D_3^2$ $\varrho{=}\varrho^3$ $K{=}K_6^1(K_6^2, K_2^3)$ $L{=}Z_2^1(Z_2^3, Z_2^5)$
ϑ^5	No	No	No		No
ϑ^6	O	$\Diamond$ M $H{=}K_6^1(K_6^2, K_6^3)$ $\varrho{=}\varrho^4(\varrho^3, \varrho^3)$ $K{=}D_3^1$ $L{=}Z_2^0(Z_2^4, Z_2^2)$	$\Diamond$ M $H{=}K_6^1(K_6^2, K_6^3)$ $\varrho{=}\varrho^3(\varrho^4, \varrho^4)$ $K{=}D_3^2$ $L{=}Z_2^3(Z_2^1, Z_2^5)$	$\lhd$ M $H{=}K_6^1(K_6^2, K_6^3)$ $\varrho{=}\varrho^3, \varrho^4$ $K{=}L$ $={Z_2^0}, Z_2^3$ $(Z_2^1, Z_2^4; Z_2^2, Z_2^5)$	

$\diamond$ – Theorem 4.12: ϑ and η have dimension 1 and act symmetrically.

$\Diamond$ – Theorem 4.14: H, K are bifurcation symmetries in $\vartheta + \eta$ and (15) holds.

$\lhd$ – Theorem 4.11: H is a bifurcation symmetry in $\vartheta + \eta$. K is a subgroup of H.

C – IC holds, but ϱ is of complex type.

M – Theorem 4.18: conjugate interaction conditions are satisfied.

No – Theorem 4.16: IC cannot be satisfied.

O – IC does not hold

Table 1: Mode Interaction condition for D_6

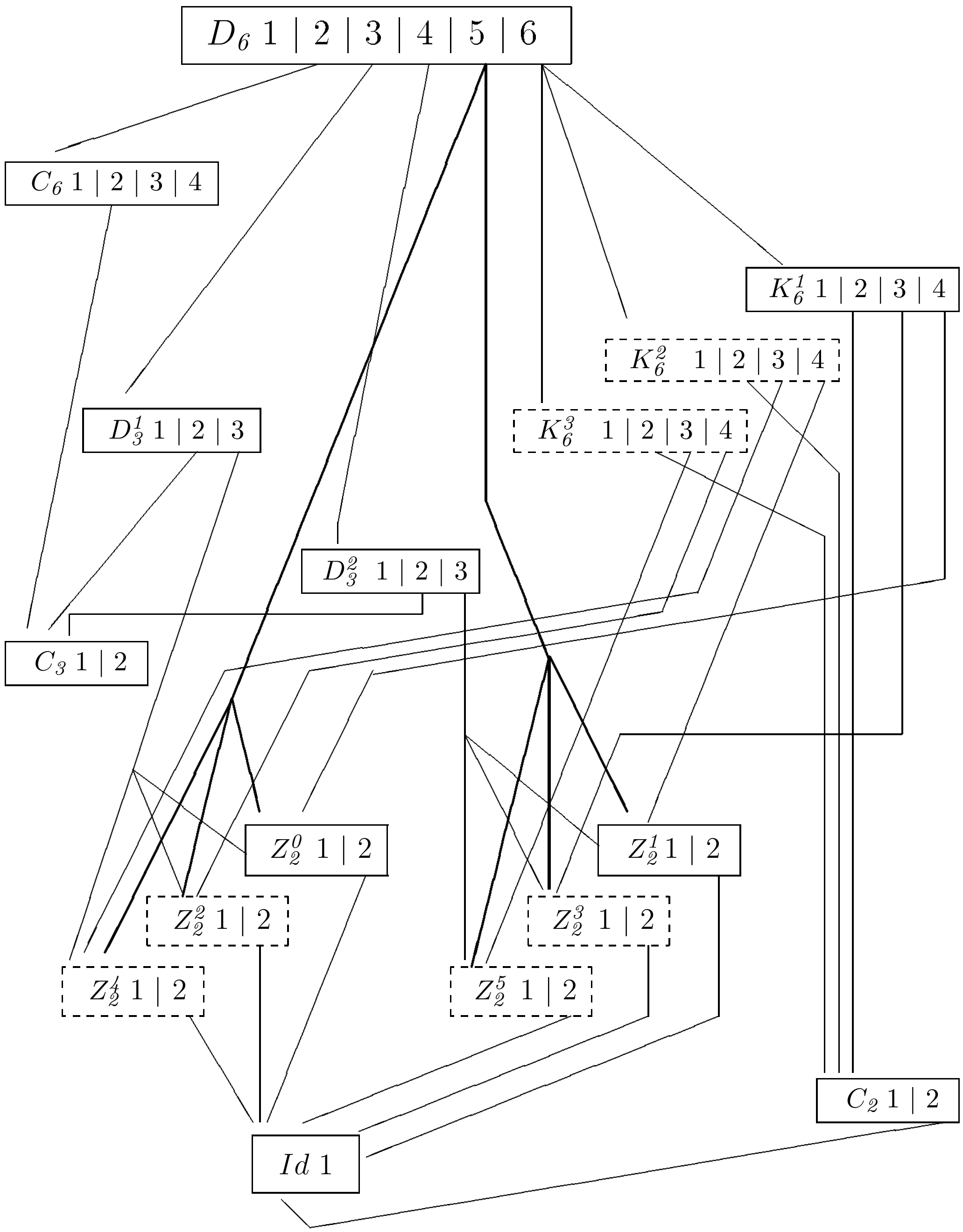

Figure 4: Bifurcation graph for D_6.

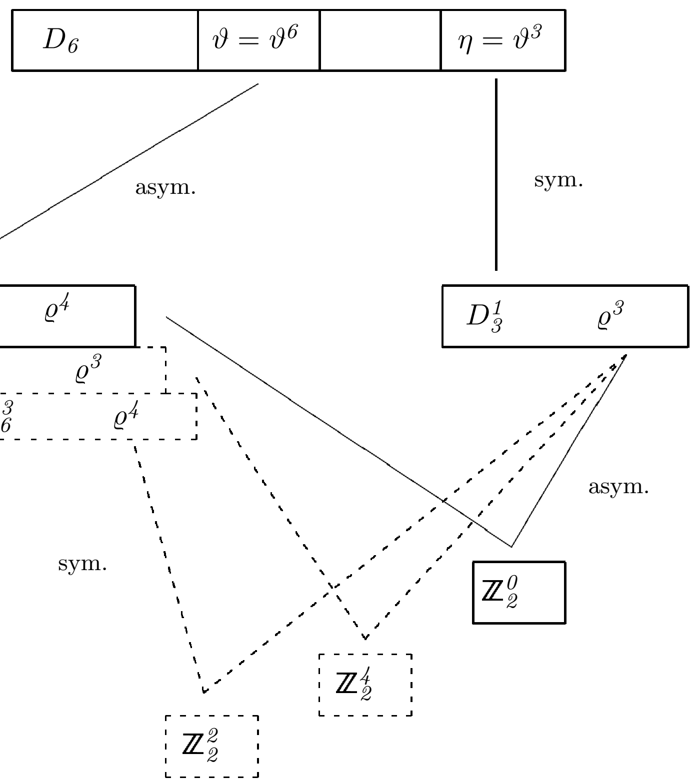

Figure 5: $\vartheta = \vartheta^6$ and $\eta = \vartheta^3$ interact symmetrically (see Fig. 2).

In Fig. 5 and Fig. 6 two parts of the bifurcation graph (Fig. 4) are extracted giving examples for Fig. 2 and 3, respectively. Fig. 5 presents an example where ϑ and η act symmetrically. Also Theorem 4.18 applies and conjugate interaction conditions are satisfied.

These results were verified numerically. We investigate a D_6–brusselator with 6 cells (see [4]) and choose $A = 1.0$ and $\alpha := B$ as second parameter. The hexagonal lattice dome introduced by Healey [9] (see also [5] and [12]) is the second example where we are taking the stiffness of the 6 inner rods as the second parameter α.

In the colored pictures one situation before and after the mode interaction is shown, i.e. bifurcation diagrams for $\alpha > \alpha_0$ and for $\alpha < \alpha_0$. For clarity only non-conjugate solutions are given.

An example of $(\vartheta^3 - \vartheta^6)$ mode interaction is given in Fig. 7. In Fig. 5 it is shown that ϑ^3 and ϑ^6 act symmetrically. As D_3^1 is symmetric in D_6 the branch of D_3^1–invariant secondary bifurcation points exist only on one side of the mode interaction point (see Theorem 4.19).

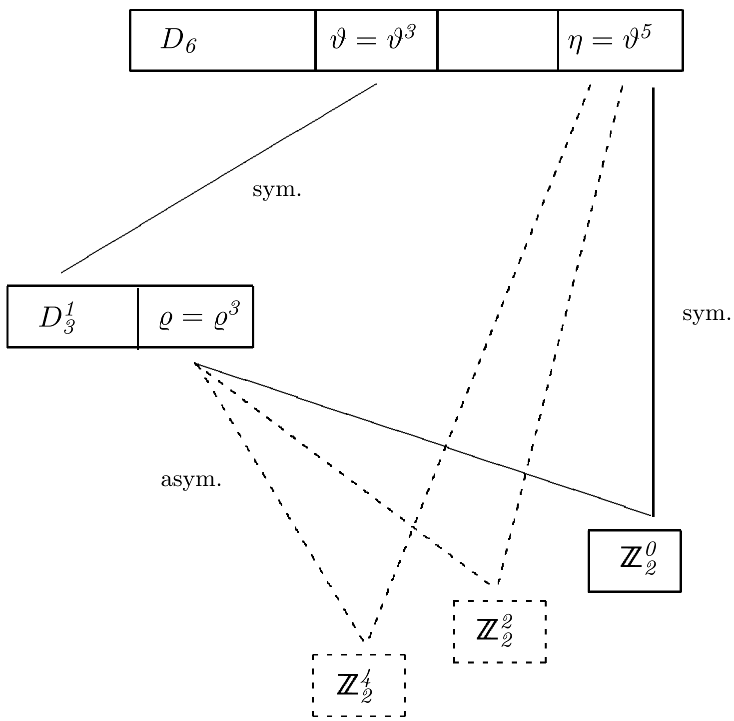

Figure 6: Mode interaction $D_6, \vartheta = \vartheta^3, \eta = \vartheta^5$ (see also Fig. 3).

The second example (Fig. 8) shows $(\vartheta^3 - \vartheta^5)$ mode interaction. This situation is explained in Fig. 6. The mode interaction causes a branch of D_3^1–invariant secondary bifurcation points. Theorem 4.19 applies again. The secondary bifurcation points exist only on one side of the mode interaction. (Observe that no secondary bifurcation occurs for $\alpha = 0.219$.)

For $(\vartheta^5 - \vartheta^6)$ mode interaction we present two examples: Fig. 9 and Fig. 10. The interaction condition holds for two different irreducible representations of the Kleinian group giving two branches of secondary bifurcation points. As the Kleinian group is asymmetrical in D_6 the branches exist before and after the mode interaction. Since there are three conjugate Kleinian groups K_6^i in D_6, the mode interaction causes 6 branches of secondary bifurcation points.

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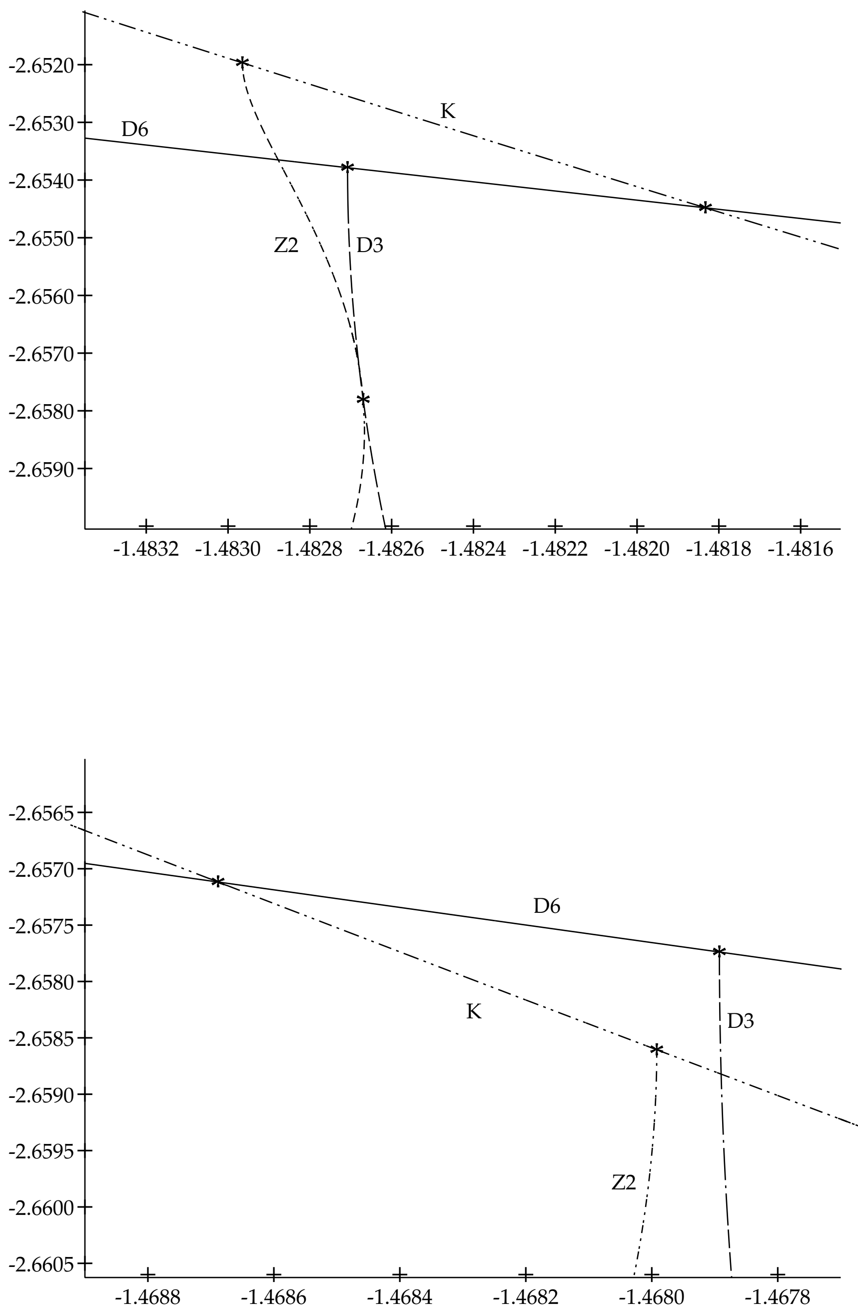


Figure 7: Symmetric $\vartheta^3 - \vartheta^6$ mode interaction for hexagonal lattice dome – bifurcation diagram x_9 versus λ at auxiliary parameter $\alpha = 0.665$ and $\alpha = 0.650$.

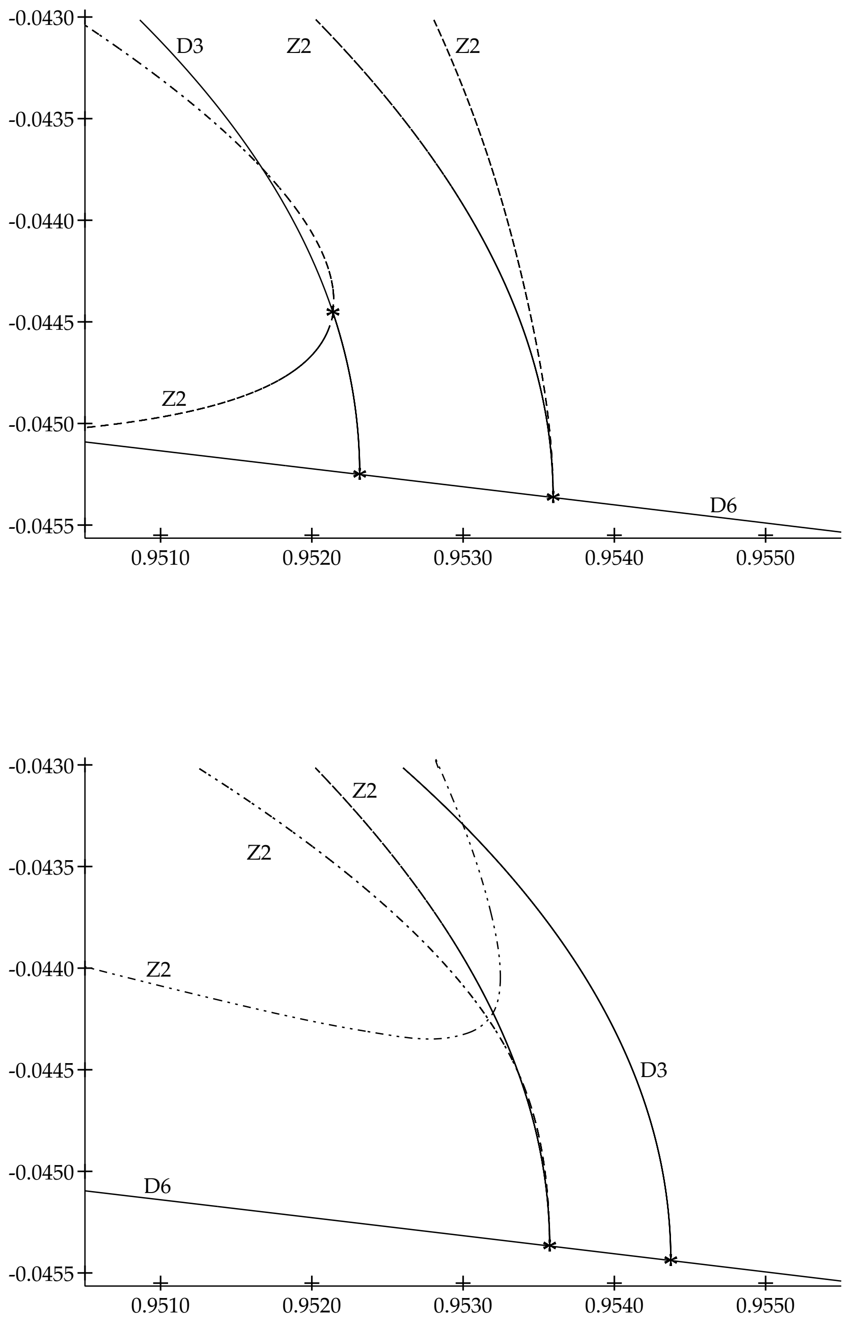
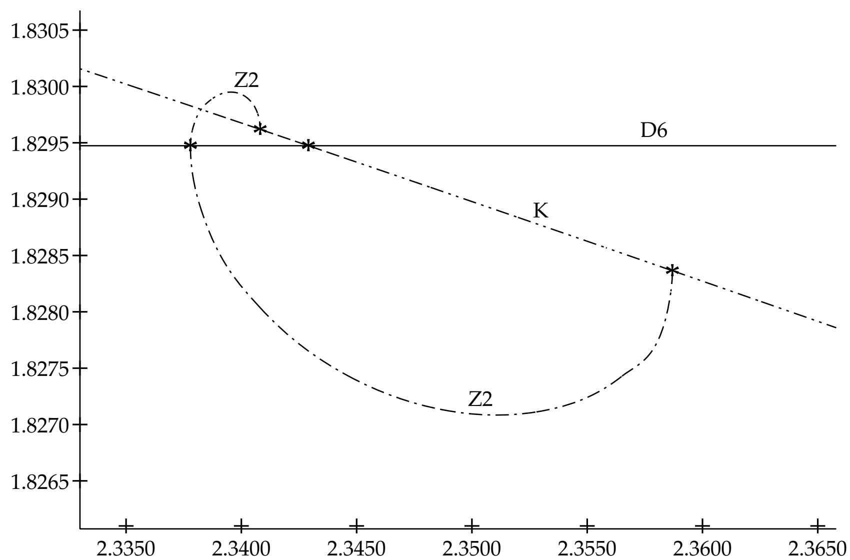


Figure 8: $\vartheta^3 - \vartheta^5$ mode interaction for hexagonal lattice dome – bifurcation diagram x_4 versus λ at auxiliary parameter $\alpha = 0.217$ and $\alpha = 0.219$.



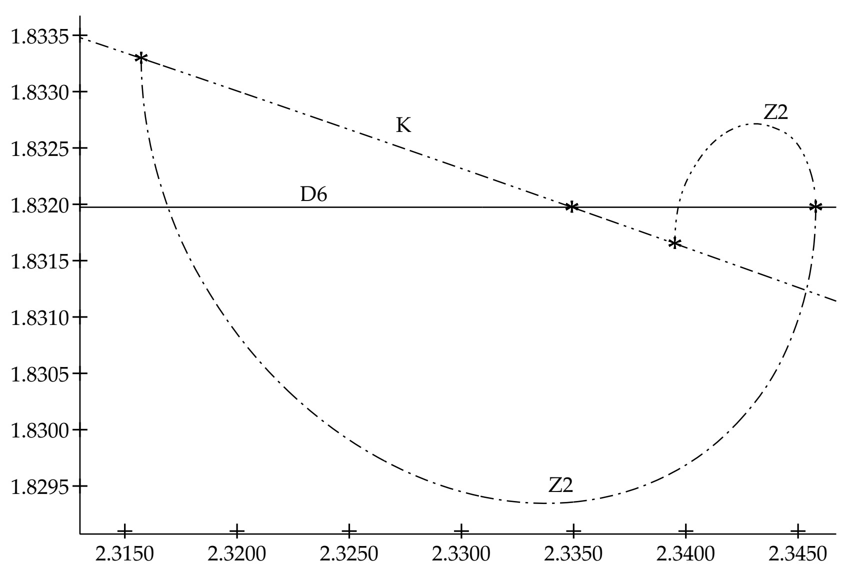


Figure 9: $\vartheta^6 - \vartheta^5$ mode interaction for D_6–brusselator – bifurcation diagram x_2 versus λ at auxiliary parameter $\alpha = 1.8295$ and $\alpha = 1.8320$.

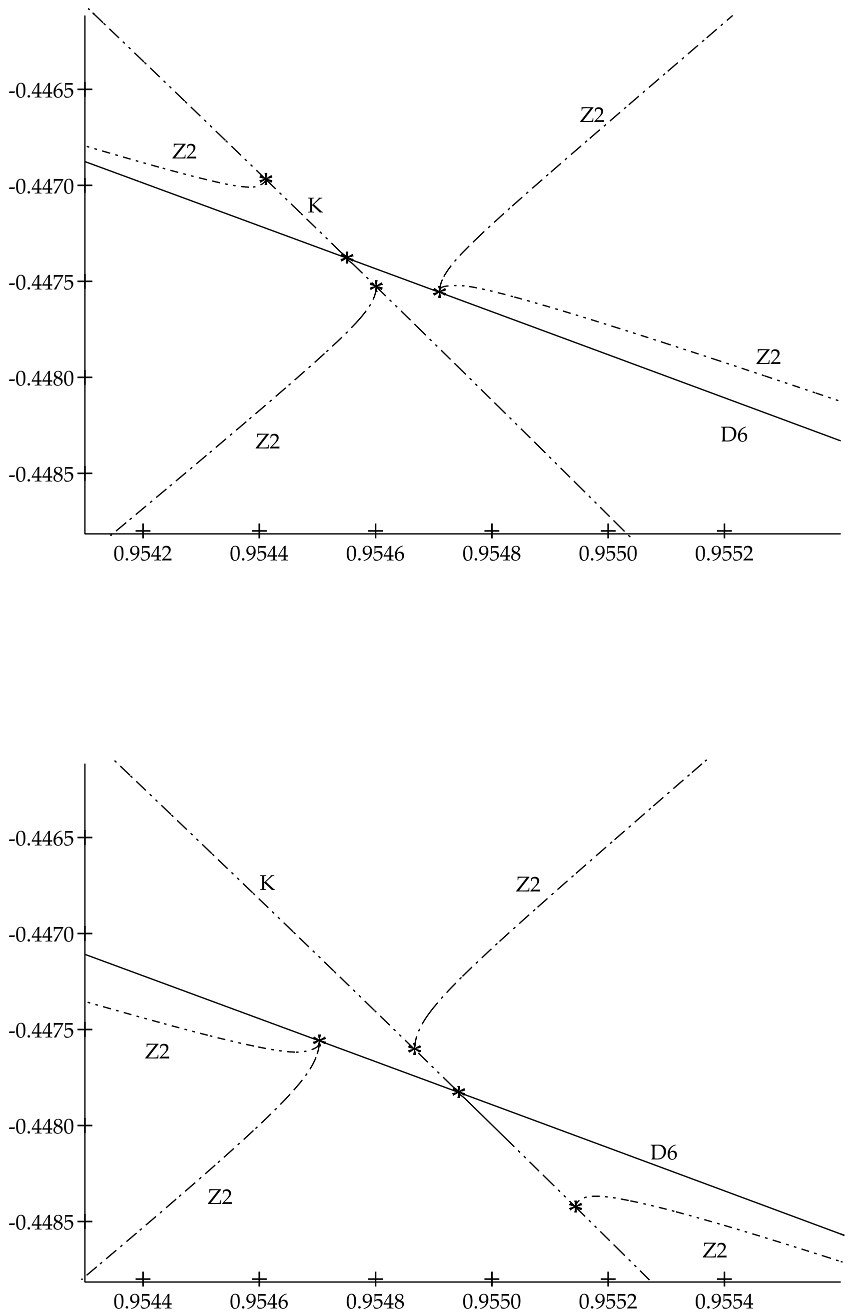


Figure 10: $\vartheta^6 - \vartheta^5$ mode interaction for hexagonal lattice dome – bifurcation diagram x_{18} versus λ at auxiliary parameter $\alpha = 0.1472$ and $\alpha = 0.1475$.